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$APQB$ is a parallelogram. The n^{th} side PQ will therefore always be parallel to the first AB .

Now let lines be drawn from a fixed point O the same in magnitude, direction, and sense as $AB, BC, CD, \dots$. Let these lines be $Oa, Ob, Oc, \dots$, so that the points $a, b, c, \dots$ are ranged on a circle S_1 . Let $a', b', c', \dots$ be the middle points of $ab, bc, cd, \dots$; then these points are also ranged on a circle S_2 , since $Oa', Ob', \dots$ are parallel to and equal to the halves of $AC, BD, \dots$ respectively. Hence also the intersections of tangents at a and b, b and c, c and $d, \dots$ lie on a fixed circle S_3 , which is the inverse of S_2 with respect to S_1 .

If now PQ is parallel to AB , the n^{th} point of the series $ab\gamma \dots$ will coincide with a , so that the tangents will form a polygon of $(n-1)$ sides inscribed in S_3 and circumscribed to S_1 . If this happens for any one position of a , it will happen for every position.

Since each point of the series $B, C, D, E, \dots$ traces a circle, it follows that the mean centre of the first r points $a, b, c, \dots$ traces a circle of $\frac{1}{r}^{\text{th}}$ the radius; and since P is a fixed point, the mean centre of the $n-1$ points $ab\gamma \dots$ is also fixed.* For instance, the line AD is parallel to and treble of the line joining O to the mean centre of $ab\gamma$, and A, O are fixed points.

The locus of the mean centre of r consecutive vertices of the series $a, b, c, \dots$ is a circle, but not, as sometimes stated, that of any r vertices. I find the degree of this locus to be $2s$, where s is the number of gaps made in the ring by leaving out the other $n-r-1$ vertices; there is an s -ple point at each of the focoids. For instance, the middle point of $a\gamma$ traces a bicircular quartic.

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NOTES ON ELEMENTARY DYNAMICS. IV.

THE PARALLELOGRAM OF VELOCITIES, AND RELATIVE VELOCITY.

THE aim of this note is to point out a certain looseness of method and haziness of conception which seems to me to exist in the treatment of the above subjects in our current text-books, and to suggest improvements.

In a deservedly popular text-book of the Elements of Dynamics, we find the following accurate, and nearly† complete enunciation of the "Parallelogram of Velocities":

* This is Weill's Theorem. Compare Casey's *Sequel*, p. 164.

† It could be made quite complete by substituting "drawn from the point" for the last four words, which as they stand do not indicate the *sense*

If a moving point possess simultaneously velocities which are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, they are equivalent to a velocity which is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.

But the inquisitive learner will ask: What is meant by a body possessing simultaneously *two* velocities?

If we recur to the definition of velocity we shall find that it gives us no help here. We may get some light as to what is really meant, by considering the *proof* usually given, or by observing the use made of the proposition in its applications.

The usual proof involves the idea of a body *moving relatively to another*, and in fact the meaning tacitly attached to the phrase under consideration is, that the body has a motion relative to a moving base, and that the actual motion of the body is said to consist of two simultaneous velocities, viz. that of the base and that of the body relative to the base.

But as the "Parallelogram of Velocities" is usually given before Relative Motion has been discussed, and is employed in the discussion of Relative Motion, the learner's mind is often puzzled, so that he fails to grasp the force of the theorem, and its proof.

Next consider the *applications* of the "Parallelogram of Velocities." The most important of these is perhaps that in the proof of the "Parallelogram of Forces." That theorem depends on the principle that the accelerations produced by two forces acting simultaneously on the same body are *independent*, i.e. that they co-exist without affecting one another.

But here again we must ask: What is meant by two accelerations *co-existing*?

I do not see that we can avoid taking co-existing accelerations in the same sense as we took co-existing velocities, that is, considering one acceleration as that of a moving base, and the other that of a point or body *relatively* to the base—unless indeed we cut the knot by turning the "Parallelogram of Velocities" or accelerations into a *definition* of what is meant by component velocities or accelerations—in which case of course a *proof* of the "Parallelogram of Velocities" would be illogical.

I am not at all certain that the latter method of treatment

in which the diagonal is to be taken. In this connection I should like to suggest that the type of enunciation of such theorems in which a lettered diagram (actual or imaginary) is referred to, is preferable for clearness and brevity combined: e.g. *If a moving point possess simultaneously two velocities represented in magnitude and direction by OA, OB, two lines drawn from a point O, then the resultant velocity is represented in direction and magnitude on the same scale by the diagonal OC of the completed parallelogram OACB.* The late Professor Casey adopted this type of enunciation in his *Euclid*.

would not be the best in an elementary book, if the changes necessarily accompanying it were made. It would then cease to be illogical to use the "Parallelogram of Velocities" in the investigation of Relative Velocity. But for the present I shall take the former alternative, which involves the discussion of Relative Velocity first. It might perhaps be argued that *all* motion being *relative*, it is unphilosophical to make any distinction: but in an elementary course we must I think be content to start with the idea of an immovable base, such as we instinctively regard the Earth to be. At anyrate the object of this note is not to revolutionize Elementary Dynamics, but if possible to clarify to some extent the conceptions which are actually current.

What knowledge of Relative Velocity is necessary in order to prove the "Parallelogram of Velocities"? Since we cannot strictly speak of the motion of a point relatively to a *point* (unless a base of reference for directions is supposed in addition), it would seem that we must first consider Displacements of Bodies. For simplicity let us confine ourselves to two dimensions, a plane body moving in a plane.

The following proposition is required, and easily proved:

When a plane body A is displaced in its plane so that a line in it remains parallel to a certain fixed line, then every line in A remains parallel to a fixed line, and the displacement of every point in A is the same in magnitude and direction.

Such a displacement of *A* is called a *translation* or a *displacement without rotation*.

A similar proposition holds good for the displacement of a body *A* relative to any body *B*, which may or may not be in motion; only instead of *fixed line* we must say *line fixed in B*, and instead of *displacement* we must say *relative displacement*.

We can now prove the "Parallelogram of Displacements without Rotation" or "of Steps" (Clifford).

Suppose a point *P* to move from a point *H* to a point *K* in a body *B*. Suppose then the body moves without rotation to *B'* so that *HK* comes to *LM*. On the whole the point *P* has been displaced from *H* to *M*. But it is obvious that the same

result is attained whether the two motions are simultaneous or not. Hence at once we get the "Parallelogram of Steps."

Note that the theorem is true whatever be the intermediate motion of *B* or *P* during the displacement, only the initial and final positions being had in view. For this reason it might be well, following Clerk Maxwell in *Matter and Motion*, to refrain at first from joining the points in displacement-diagrams.

The "Parallelogram of Velocities" may now be deduced. Let

P move relatively to B with a uniform velocity v , so that its displacement after t seconds is vt , and let B move translationally with a uniform velocity u , so that its displacement after t seconds is ut . In the same diagram as before let B' represent the position of B after t seconds. Then we have

$$\frac{ML}{HL} = \frac{vt}{ut} = \frac{v}{u} = \text{constant}.$$

Thus the angle MHL is constant, and $\frac{HM}{HL}$ is constant (Euc. VI. 6).

$$\therefore MH \propto t.$$

Thus the resultant motion of P is uniform and rectilinear, and is represented by HM in direction and magnitude on the same scale on which the velocity of B is represented by HL .

So far we have referred only to *uniform rectilinear* velocities. The proof as it stands does not apply to those which are not uniform and rectilinear. This remark applies to all proofs of this kind, such as are usually given in the text-books, and yet we find the difficulty ignored or slurred over, the result being tacitly assumed to hold for varying velocity.

To extend the proposition to varying velocities we might proceed thus: We note that a point can only then be said to have a definite velocity when by taking a small enough interval of time we find the motion up to that instant as nearly as we please uniform and rectilinear. Now when this is the case, the construction given for uniform velocities will fit our case as nearly as we please to have it, if we make the interval of time small enough. We may for clearness suppose the diagram correspondingly magnified in scale, so that it always remains finite.

A more formal proof will be found in Clifford's *Dynamic*, pp. 59, 60.

Some may object to what precedes as hypercritical. The reply is that what is given in elementary text-books ought at least to have a definite and unmistakeable *meaning*. I think it is quite an open question how far young students should be troubled with formal proofs of fundamental propositions, as such proofs must of necessity involve really difficult abstract ideas. I think clear explanations and illustrations (the illustrations preferably coming first) are at anyrate more useful. But the proofs that are given ought at least to be real proofs so far as they go, and the terms employed ought to be clearly defined where any doubt can exist. I think there is room for improvement in this respect in nearly all the current text-books.

In a further note I propose to discuss two or three problems which have been set of late years, involving relative motion.

R. F. MUIRHEAD.