On Certain Conics connected with a Plane Unicursal Quartic.

By R. A. ROBERTS, M.A.

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1. The object of the present paper is to demonstrate the existence of conics which are related to a unicursal quartic in such a manner that an infinite number of closed polygons can be simultaneously circumscribed about them and inscribed in the quartic. I have found that certain conics touching two double tangents and the curve twice, and certain other conics touching the curve four times, are connected with the quartic in this manner, if a further condition is satisfied. This condition depends on the number n of the sides of the polygon, and will give a finite number of conics for each value of n .

2. Suppose a conic U referred to two tangents x , y , and z , their chord of contact, to be written in the form $z^2 - 4xy = 0$, then any tangent will be $\mu^2x - \mu z + y = 0$, and, if μ_1, μ_2 are the parameters of the tangents passing through the point x, y, z , we may write

$$\rho x = 1, \quad \rho y = \mu_1 \mu_2, \quad \rho z = \mu_1 + \mu_2, \dots\dots\dots(1);$$

and hence we may consider μ_1, μ_2 as a system of coordinates determining the position of any point on the plane. This system has been used by Darboux ("Sur une classe remarquable de courbes et de surfaces algébriques," p. 183) in investigating some properties of polygons circumscribed about a conic and inscribed in another curve. Now, if we write $f(\mu_1), f(\mu_2)$, where f is an arbitrary function, for μ_1 and μ_2 respectively, we have a transformation by which a given curve will be transformed into another, and, in particular, any tangent to U will be transformed into the product of as many tangents to U as the equation $f(\mu) = \text{a constant}$ has roots.

3. Let us consider now the quadratic transformation

$$f(\mu) = \frac{a + b\mu + c\mu^2}{a' + b'\mu + c'\mu^2} \dots\dots\dots(2),$$

then it is evident from (1) that the coordinates of any point on the transformed curve are expressed as functions of the second degree of those of the point on the given curve; and, hence, the transformed curve will be, in general, of double the degree of that of the given curve, and will have the same deficiency. Now, it is to be ob-

served that, if we substitute a homographic transformation for μ , this is only equivalent to a change of the axes of coordinates. It will follow hence that the transformation (2) is reducible to a transformation in which we substitute their squares for μ_1, μ_2 , respectively. In this case, if α, β, γ is a point on the given curve, and x, y, z the corresponding point on the transformed curve, we may write

$$\rho x = \alpha^2, \quad \rho y = \beta^2, \quad \rho z = \gamma^2 - 2\alpha\beta \dots\dots\dots(3).$$

4. If we suppose the point α, β, γ to lie on a general conic V , α, β, γ can be expressed as quadratic functions of a parameter, and x, y, z will then lie on a unicursal quartic. Since the conics U and V and the point xy involve twelve constants, and a unicursal quartic involves eleven, it follows that, when the quartic is given, the conic U involves but one indeterminate constant, and, therefore, satisfies four conditions. It is easy to find what these conditions are; for from (3) we see that x, y , which are tangents to U , are also double tangents to the quartic, and, since we have

$$\rho^2 (z^2 - 4xy) = \rho^2 U = \gamma^2 (\gamma^2 - 4\alpha\beta),$$

it follows that U has double contact with the curve. Now, we have seen that to a polygon circumscribed about U corresponds a polygon also circumscribed about U , and, if the former polygon be inscribed in V , the latter polygon will be inscribed in the quartic; and we know that, in order that this should be possible, it is only necessary that U and V should be connected by a single relation. This relation, which depends upon the number of sides of the polygon, will evidently serve to determine a finite number of conics, such as U , connected with the quartic.

5. We can verify from (3) that a tangent to U meets the quartic in a pair of points whose parameters are connected by a relation of the same nature as that which connects the pair of points where the tangent meets V . Let a tangent to U be written $\mu^2 x - \mu z + y = 0$, then, for the four points where it meets the quartic, we have

$$\mu^2 \alpha^2 + \beta^2 - \mu (\gamma^2 - 2\alpha\beta) = 0,$$

which breaks up into the two quadratics

$$\mu \alpha + \beta \pm \sqrt{\mu} \gamma = 0;$$

but these are evidently the equations which we should get to determine the points where the tangents $\mu x \mp \sqrt{\mu} z + y = 0$ meet V . If θ is the parameter involved in α, β, γ , this relation connecting either

pair of points may be written in a form involving elliptic integrals as

follows :
$$\frac{d\mathfrak{S}_1}{\sqrt{(\gamma_1^2 - 4\alpha_1\beta_1)}} \pm \frac{d\mathfrak{S}_2}{\sqrt{(\gamma_2^2 - 4\alpha_2\beta_2)}} = 0.$$

6. Since a unicursal quartic has four double tangents, there are six pairs of such lines, and thus we have six distinct systems of conics of the kind found above. To investigate the conditions corresponding to a given number of sides of the polygon, I write the equation of the quartic in the symmetrical form

$$\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{u} = 0 \dots\dots\dots(4),$$

where x, y, z, u are the four double tangents, and are connected by the identical relation

$$ax + by + cz + du = 0.$$

When the curve is written in this form (4), it is easy to see that

$$\mathfrak{S}^2 x + \mathfrak{S}(x + y - z - u - 2\sqrt{zu}) + y = 0 \dots\dots\dots(5)$$

represents a conic touching the lines z, u and having double contact with the curve, and belongs to the system we have been considering. It may be observed that we can show, from (5), that the chord of contact with the curve is $y - \mathfrak{S}^2 x = 0$, which therefore passes through the fixed point xy ; also, that the chord of contact with the double

tangents is $\mathfrak{S}^2 x + \mathfrak{S}(x + y - z - u) + y = 0$,

which touches the conic

$$(x + y - z - u)^2 = 4xy.$$

Putting now $\mathfrak{S}x + \frac{y}{\mathfrak{S}} + x + y - z - u = v$,

the equation (5) becomes $v^2 - 4zu = 0$,

and this coincides with the conic U . By means, then, of the substitutions $z^2 = z', \quad u^2 = u', \quad v^2 - 2zu = v' \dots\dots\dots(6),$

[see (3),] we must be able to transform the quartic

$$\sqrt{x'} + \sqrt{y'} + \sqrt{z'} + \sqrt{u'} = 0$$

back into the conic V . From (6), we get

$$v^2 - 2zu = \mathfrak{S}x' + \frac{y'}{\mathfrak{S}} + x' + y' - (z^2 + u^2),$$

and

$$\sqrt{x'} + \sqrt{y'} = z - u;$$

hence

$$v^2 = \mathfrak{S}x' + \frac{y'}{\mathfrak{S}} + x' + y' - (\sqrt{x'} + \sqrt{y'})^2,$$

and

$$v = \sqrt{(\mathfrak{S}x')} - \sqrt{\left(\frac{y'}{\mathfrak{S}}\right)}.$$

We thus find

$$v + \frac{(z-u)}{\sqrt{\mathfrak{S}}} = m\sqrt{x'}, \quad v - (z-u)\sqrt{\mathfrak{S}} = -m\sqrt{y'};$$

but we have

$$ax' + by' + cz' + du' = 0;$$

hence we get, finally,

$$\begin{aligned} V \equiv & \left(\frac{a+b\mathfrak{S}^2}{\mathfrak{S}} + cm^2 \right) z^2 + \left(\frac{a+b\mathfrak{S}^2}{\mathfrak{S}} + dm^2 \right) u^2 \\ & + (a+b)v^2 + \frac{2(a-b\mathfrak{S})}{\sqrt{\mathfrak{S}}} v(2-u) - \frac{2(a+b\mathfrak{S}^2)}{\mathfrak{S}} zu = 0, \end{aligned}$$

where $m = \frac{1+\mathfrak{S}}{\sqrt{\mathfrak{S}}}$. Forming now the discriminant Δ of $V + \lambda U$, we get Δ proportional to

$$\begin{aligned} 4\lambda^3 + 4\lambda^2(a+b\mathfrak{S})\left(\frac{1+\mathfrak{S}}{\mathfrak{S}}\right) - \lambda\frac{(1+\mathfrak{S})^2}{\mathfrak{S}^2}\{cd(1+\mathfrak{S})^2 + (c+d)(a+b\mathfrak{S}^2) - 4ab\mathfrak{S}\} \\ - \frac{(1+\mathfrak{S})^4}{\mathfrak{S}^2}(abc+bcd+cda+dab). \end{aligned}$$

We may simplify this expression by putting

$$\lambda = \frac{(1+\mathfrak{S})}{\mathfrak{S}} k,$$

when Δ is found to vary as

$$\begin{aligned} 4k^3 + 4(a+b\mathfrak{S})k^2 - \{cd+bc+bd\}\mathfrak{S}^2 + 2(cd-2ab)\mathfrak{S} + cd+ac+ad\}k \\ - \mathfrak{S}(1+\mathfrak{S})(abc+bcd+cda+dab) \dots\dots\dots(7). \end{aligned}$$

Now, when the discriminant of $V + \lambda U$ is written in the form

$$A\lambda^3 + B\lambda^2 + C\lambda + D,$$

the condition that it should be possible to circumscribe a triangle about U which shall be inscribed in V is $D^2 - 4AC = 0$. Hence, in this case, we get from (7), to determine $\mathfrak{S}$,

$$(b+c)(b+d)\mathfrak{S}^2 + 2(cd-ab)\mathfrak{S} + (a+c)(a+d) = 0 \dots\dots(8).$$

There are, therefore, two conics of each of the six systems which touch the sides of an infinite number of triangles inscribed in the quartic. For the quadrilateral, we have

$$B^3 - 4ABC + 8DA^3 = 0,$$

and we get then from (7), to determine $\mathfrak{S}$,

$$(a + b\mathfrak{S})^3 + (a + b\mathfrak{S})\{(cd + bc + bd)\mathfrak{S}^3 + 2(cd - 2ab)\mathfrak{S} + cd + ac + ad\} \\ - 2\mathfrak{S}(1 + \mathfrak{S})(abc + bcd + cda + dab) = 0 \dots \dots \dots (9).$$

From the general conditions given by Professor Cayley, we find that for n , an odd number, $= 2m + 1$, the degree of the equation for $\mathfrak{S}$ is $2 \cdot 4 \cdot 6 \dots 2m$; and for n , an even number, $= 2m$, is $1 \cdot 3 \cdot 5 \dots 2m - 1$.

7. In a paper published by me in the *Proceedings*, "On Certain Results obtained by means of the Arguments of Points on a Plane Curve," Vol. xv., p. 4, I have shown, in § 17, that the tangents to a certain system of conics touching a bicircular quartic four times, meet the curve in pairs of points whose parameters are connected by

$$\text{the relations} \quad u_1 + u_3 = \sigma, \quad u_2 + u_4 = w - \sigma \dots \dots \dots (10),$$

where σ depends on the undetermined constant in the equation of the conic, and w is an absolute constant of the curve. These relations (8) will enable us to determine conics of the system which touch the sides of an infinite number of polygons of an even number of sides inscribed in the curve. As, for instance, for a quadrilateral, if a, b, c, d are the arguments of the vertices, we have

$$a + b = \sigma, \quad b + c = w - \sigma, \quad c + d = \sigma, \quad d + a = w - \sigma,$$

whence, to determine σ , we get $4\sigma = 2w$. The argument u , which for the binodal quartic is an elliptic integral, reduces for the trinodal quartic to a circular function or a logarithm. I proceed to investigate this result independently for the trinodal quartic. Let α, β, γ be quadratic functions of a parameter $\mathfrak{S}$, then the equations

$$\rho x = \alpha\gamma, \quad \rho y = \beta\gamma, \quad \rho z = \gamma^3 + \alpha\beta \dots \dots \dots (11)$$

represent a unicursal quartic which the conic $z^2 - 4xy \equiv U$, say, touches four times. Seeking, then, the points where the tangent to U ,

$$\mu^2 x - \mu z + y = 0,$$

meets the curve, we get, from (11),

$$\mu^2 \alpha\gamma + \beta\gamma - \mu(\gamma^3 + \alpha\beta) = 0,$$

which breaks up into the factors

$$(\gamma - \mu\alpha)(\mu\gamma - \beta) = 0 \dots\dots\dots(12),$$

showing that the parameters of the corresponding pairs of points belong to systems in involution. If we suppose the roots of γ to be

$$0, \infty, \text{ and put } \alpha = \mathfrak{S}^2 + p\mathfrak{S} + q, \beta = \mathfrak{S}^2 + p'\mathfrak{S} + q',$$

$$\text{we get, from } \gamma - \mu\alpha = 0, \quad \mathfrak{S}_1\mathfrak{S}_2 = q,$$

$$\text{and from } \mu\gamma - \beta = 0, \quad \mathfrak{S}_3\mathfrak{S}_4 = q'.$$

Now, if any line $lx + my + nz = 0$, meet the curve, we have from (11),

$$\mathfrak{S}(la + m\beta) + n(\mathfrak{S}^2 + a\beta) = 0,$$

from which we see that the four parameters are connected by the relation $\mathfrak{S}_1\mathfrak{S}_2\mathfrak{S}_3\mathfrak{S}_4 = qq'$. We see thus that qq' is a constant belonging to the curve and independent of the particular conic, and may therefore be put equal to unity. Hence, for a polygon of $2n$ sides, we put

$$\mathfrak{S}_1\mathfrak{S}_2 = q, \quad \mathfrak{S}_2\mathfrak{S}_3 = \frac{1}{q}, \dots \mathfrak{S}_{2n}\mathfrak{S}_1 = \frac{1}{q},$$

from which we get $q^{2n} - 1 = 0$. The factor $q^2 - 1$ is irrelevant, and, as q and $\frac{1}{q}$ lead to the same conic, there would appear to be $n-1$ solutions. We must, however, diminish this number by the solutions corresponding to a lesser number of sides of the polygon, if there are primes contained in n . Thus, for $n = 2$, there is but one conic, and for $n = 3$, there are two conics. Also for $n = 4$ and 5, there are two and four conics respectively.

We can easily show that there are three distinct systems of conics of the kind which we have been just considering. For, from (11), we see that xy is a node whose parameters are given by $\gamma = 0$, and, from (12), γ belongs to both the involutions. There are, therefore, three systems corresponding to the three pairs of nodes.

8. The conics of the two different kinds mentioned in §1 are the only systems I have been able to discover that touch the sides of an infinite number of polygons inscribed in the general unicursal quartic. For unicursal quartics, however, satisfying certain invariant conditions, other conics can be found. For instance, if the quartic can be written in the form

$$x^3y^3 - z^3uv = 0 \dots\dots\dots(13),$$

the conic touching the lines x, y, z, u, v will touch the sides of an in-

finite number of quadrilaterals inscribed in the curve (see p. 186, Darboux, "Sur une classe remarquable de courbes et de surfaces algébriques").

If the trinodal quartic

$$y^2z^2 + z^2x^2 + x^2y^2 + 2xyz(ax + by + cz) = 0$$

is capable of being written in the form (13), it is easily seen that we must have $c = 1 + ab$, or either of two similar relations.

9. The investigation in § 6 will include the case of a quartic with a triple point. For the equation (4) will represent such a quartic, if $a + b + c + d = 0$, the triple point being $x = y = z = u$. The equation (8) then gives $(\S + 1)^2 = 0$, and is irrelevant, showing that there are no conics which touch the sides of an infinite number of triangles inscribed in a quartic with a triple point.

On the Theory of Screws in Elliptic Space. (Supplementary Note.)

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At the January Meeting of the Society, I read a paper "On the Theory of Screws in Elliptic Space," which has since appeared in the *Proceedings*. My object was "to show that the *Ausdehnungslehre* supplies all the necessary materials for a calculus of screws in Elliptic Space." When I wrote that paper, I did not see how the same methods could (except in one obvious and unsatisfactory way) be extended to other kinds of space. In a paper on biquaternions, which is to appear in the *American Journal of Mathematics*, I have developed Clifford's calculus, in such a way as to make the methods and formulæ apply simultaneously to the three kinds of uniform space. While writing that paper, I saw how Grassmann's methods may be extended so as to give the metric formulæ for all kinds of space. This extension I explain in the present note.

The fundamental idea of the extension in question is that the symbol K no longer denotes the *Ergänzung* (complement) of a figure, but its polar with respect to the absolute. Besides this extension of the *Ausdehnungslehre*, I give a few kinematical investigations.