

Heat Losses in the Conductors¹ of Alternating-Current Machines

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WITH the increase in the capacity of alternating-current generators, it has become more and more important to determine the exact relation between the heat developed in any conductor and the currents that it and other neighboring conductors carry. Solutions for the distribution of alternating current within conductors that are embedded in open rectangular slots have been obtained for a number of cases. The methods of attack and the results obtained have heretofore involved trigonometric and hyperbolic functions of real angles. For this reason the work has been unnecessarily complicated, and its scope considerably cramped. Had the investigators used hyperbolic functions of *complex* angles, they would have accomplished more with much less effort. These hyperbolic functions of *complex* angles are such a powerful tool in the solution of this current distribution problem that the author feels fully justified in presenting this discussion of their application, even though many of the results have been obtained before.

At the outset it should be observed that certain assumed ideal conditions are necessary in order to bring the problem within the range of our mathematical ability.² Briefly these conditions are (1) that an element of current in the slot produces a uniform parallel magnetic field above itself and none below it; (2) that the current density along any line parallel to the bottom of the slot is constant; (3) that the

resistivity of the conductor is uniform throughout, even though more heat is developed in some portions than in others; (4) that voltages in the end connections due to leakage flux are the same for every element of the conductor.

1. The first assumption can be shown to be exactly true in the hypothetical case of an infinitely long rectangular conductor placed in an infinitely deep rectangular slot of the same width which is cut into an infinite medium of infinite permeability.

2. The second assumption is probably sufficiently accurate, except when conductors that carry different currents are placed side by side in the same slot, a condition which we will not consider.

3. The heat conductivity of copper is so high that the temperature of any one conductor is probably nearly uniform, except in the case of extremely deep conductors, and the resistivity would thus not vary appreciably from point to point. In the case of multiple layer coils, however, there may be such a difference in the amount of heat developed in successive layers that it may be desirable to use different resistivities for different layers. In order to carry out this refinement in the calculations, a considerable practical knowledge of the principles of heat radiation in this problem would be necessary, and the final result would probably be obtained by making successive approximations.

4. The leakage flux about the end connections is of course much less than that about the embedded portion of the coils, and is distributed in a totally different manner. Any rational consideration of the effect of this flux would lead to considerable complication in the mathematical analysis. Thus for the sake of simplicity—a feeble reason, no doubt—it is not considered. It is probable, however, that in a great many instances this coil-end leakage is of relatively minor importance.

The best test of the value of the mathematical deductions based on the foregoing assumptions lies in experimental research. The little evidence that we now have seems to confirm the theory within reasonable limits of error, and it is hoped that a more extended investigation, which is being carried on, will prove conclusive.

To the author, it seems nearer the physical reality to say that the increased heat loss in a conductor carrying alternating rather than direct current is caused by a redistribution of current over the cross-

1. An abstract of this paper was presented before the Research Division of the Electrical Engineering Department at the Massachusetts Institute of Technology, April 13, 1920. The author's attention was first directed to this method of analysis by Dr. A. E. Kennelly to whom great credit is due especially on account of the extensive tables he has published from which numerical computations may readily be made: "Tables of Complex Hyperbolic and Circular Functions," A. E. Kennelly.

See also:

"Eddy Currents in Large Slot-Wound Conductors," A. B. Field, Annual Convention, A. I. E. E., 1905.

"Current Distribution in Armature Conductors," W. V. Lyon, *Electrical World*, July 12, 1919.

Since this present analysis was made, Mr. R. E. Gilman has presented at the Annual Convention, A. I. E. E., 1920, a paper entitled "Eddy Current Losses in Armature Conductors." This paper of Mr. Gilman's, although he uses real quantities exclusively, is more complete in one respect than the one here presented in that he considers the case of a laminated conductor with a finite number of strands.

2. Both A. B. Field and R. E. Gilman assume these ideal conditions in the derivation of their formulas.

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section of the conductor.³ This redistribution is due to the electromotive forces set up by the magnetic flux *within* the conductor itself. Flux that is wholly without the conductor links all elements of it equally, produces the same voltage in each element, and thus does not affect the current distribution.

Consider any conductor lying in the midst of others, as shown in cross-section in Fig. A. Further, consider any element of this conductor of depth $d x$ situated x centimeters from the bottom of the conductor. With a solid conductor the length of this element should be taken only as the length of the armature core. The allowance that need be made for ventilating ducts, and more particularly for end turns, can only be determined by considerable experimental research. With finely laminated conductors, the current density is constrained to be the same in, at least, the length of a half turn, and the length of the element is then that of a half turn.

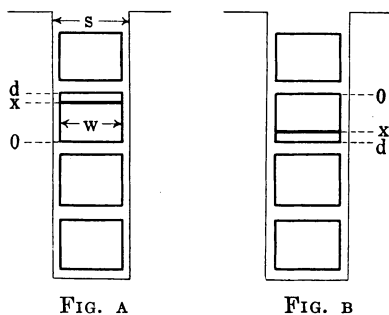


FIG. A

FIG. B

The net voltage acting on the element is the sum of the resistance drop and the voltage drop due to flux linkages. The former is $l_1 \rho c$; where l_1 is the length of the core with solid conductors, or a half turn with finely laminated ones, ρ is the resistivity and c is the current density—all in c. g. s. units. The voltage drop due to flux linkages may be divided into two parts; that due to flux set up by the current in the conductor itself, and that due to flux set up by current in other conductors in the slot *below* the one we are considering. Any current above this conductor produces flux that links all elements of it *alike*, and induces the *same* voltage in each. The manner in which a *given* current is distributed in any conductor below the one in question does not affect the amount of flux linking the element considered. Furthermore any flux due to these latter currents that passes across the slot above the conductor we are considering links all of its elements alike, and, since it produces the same voltage in each, need not be considered. Thus the total flux linking the element in question that need be taken into account is:

$$\phi = 4 \pi i_b \frac{d-x}{s} l_2 + \int_x^d \left\{ 4 \pi \int_0^x w c \partial x \right\} \frac{l_2 \partial x}{s}$$

3. Others prefer to say that the increased heating is due to the eddy currents produced by the magnetic flux within the conductor. Either description is permissible.

The first term is the flux, linking the element due to a total current of i_b below the conductor of which the element is a part. The second term is the flux from x to d due to the current within the conductor itself from zero to x . l_2 is the length of the armature core, s is the width of the slot, and w that of the conductor.

The total voltage drop that need be considered is:

$$e = l_1 \rho c + \frac{\partial}{\partial t} \left\{ 4 \pi i_b \frac{d-x}{s} l_2 + \int_x^d \left\{ 4 \pi \int_0^x w c \partial x \right\} \frac{l_2 \partial x}{s} \right\} \quad (\text{A})$$

In a solid conductor this voltage drop is the same for every element, and $\frac{\partial e}{\partial x}$ will thus be zero. Also l_1 and l_2 are equal. Differentiating e and dividing by l_2 gives:

$$\rho \frac{\partial c}{\partial x} - \frac{4 \pi}{s} \frac{\partial i_b}{\partial t} - \frac{4 \pi}{s} \int_0^x w \frac{\partial c}{\partial t} \partial x = 0 \quad (\text{B})$$

With finely laminated conductors whose laminations are joined at the beginning and end of each half turn, the same result is obtained except that l_1 and l_2 are not equal. Then:

$$\frac{l_1}{l_2} \rho \frac{\partial c}{\partial x} - \frac{4 \pi}{s} \frac{\partial i_b}{\partial t} - \frac{4 \pi}{s} \int_0^x w \frac{\partial c}{\partial t} \partial x = 0$$

With finely laminated conductors whose end turns are untwisted and in which the laminations are continuous throughout a whole turn or a whole coil, the voltage in a half turn of any element is as given above. (Equation A). The sum of these half-turn voltages between the points at which the laminations are joined together must be the same for each lamination. This sum will consist of a number, n , of resistance drops, and an equal number of voltage drops due to flux linkages. The resultant drop is

$$e = n l_1 \rho c + \frac{\partial}{\partial t} \sum_1^n 4 \pi i_b \frac{d-x}{s} l_2 + n \frac{\partial}{\partial t} \int_x^d \left\{ 4 \pi \int_0^x w c \partial x \right\} \frac{l_2 \partial x}{s} = 0$$

The current i_b , below the conductor of which the half-turn element is a part, is not the same for the different half-turn elements. Nevertheless, it is a simple matter to compute the average value of this quantity, *viz.*, $1/n \sum_1^n i_b$ for any arrangement of conductors. It is, of course, not necessary that the component i_b 's should be in phase with each other. This computation is subsequently shown in detail. We will represent this average value of i_b by i_0 . If the entire voltage for this element be divided by $n l_2$ and then differentiated with respect to x , we have:

$$\frac{l_1}{l_2} \rho \frac{\partial c}{\partial x} - \frac{4 \pi}{s} \frac{\partial i_0}{\partial t} - \frac{4 \pi}{s} \int_0^x w \frac{\partial c}{\partial t} \partial x = 0$$

In the case of finely laminated conductors that are twisted in the end connections so that the top lamination of one half turn becomes the bottom lamination of the next half turn, a similar equation may be derived. When the end turn is twisted in passing from one coil side to the next, the flux within the conductor linking the half-turn element of one side has already been given. (Equation A). The flux linking the next half turn of this element is, (Fig. B):

$$\rho = 4 \pi i_b \frac{x l_2}{s} + \int_0^x \left\{ 4 \pi \int_x^d w c \, dx \right\} \frac{l_2 \, dx}{s}$$

The first term is the flux from zero to x due to a current of i_b below this conductor. The second term is the flux from zero to x due to current from x to d within the conductor itself. The second term may be re-written in this way:

$$\int_0^x 4 \pi \left\{ \int_0^d w c \, dx - \int_0^x w c \, dx \right\} \frac{l_2 \, dx}{s}$$

The voltage drop in this half-turn element becomes:

$$e = l_1 \rho c + \frac{\partial}{\partial t} \left\{ 4 \pi i_b \frac{x l_2}{s} + \int_0^x 4 \pi \int_0^d w c \, dx \cdot \frac{l_2 \, dx}{s} - \int_0^x 4 \pi \int_0^x w c \, dx \cdot \frac{l_2 \, dx}{s} \right\}$$

$\int_0^d w c \, dx$ is the entire current in the conductor of which the element is a part. Represent this current by i . Differentiate e with respect to x and divide by l_2 . We have:

$$\frac{l_1}{l_2} \rho c + \frac{4 \pi}{s} \frac{\partial i_b}{\partial t} + \frac{4 \pi}{s} \frac{\partial i}{\partial t} - \frac{4 \pi}{s} \int_0^x w \frac{\partial c}{\partial t} \, dx = 0 \quad (C)$$

Thus in every case that we shall discuss, the following equation may be written:

$$\rho \frac{\partial c}{\partial x} - \frac{\partial}{\partial t} \frac{4 \pi i_0}{s} - \frac{i}{\partial t} \frac{4 \pi}{s} \int_0^x w c \, dx = 0 \quad (4)$$

ρ is the resistivity in the case of solid conductors, and is the resistivity multiplied by the ratio of the length of a half turn to the length of the core in the case of infinitely laminated conductors. c is the instantaneous

4. An equation of this sort applies only to the conductor in which the current density is a continuous function with respect to x , such as is the case with solid or finely laminated conductors. When the laminations have appreciable depth the current density changes abruptly as we pass from one lamination to the next. The effect of this is that the vector constant i_0 is different in successive laminations. A comparison of equations (B) and (C) shows that twisting the end connections reverses the effect of the current, i_b , below the conductor we are considering.

current density in amperes per square centimeter along a line x centimeters from the bottom of the conductor. s is the width of the slot and w , that of the conductor. The first term is the differential of the resistance drop per centimeter in any element. The second term is the differential of the voltage per centimeter due to the flux produced by other currents in the same slot. As we shall see, i_0 is determined by the arrangement of the conductors and the currents they carry. The third term is the differential of the voltage per centimeter due to the flux produced by current within the conductor itself below the element considered. If the currents vary sinusoidally with

the time, the differential operator $\frac{\partial}{\partial t}$ may be sym-

bolically represented by $j \omega$; $j = \sqrt{-1}$, $\omega = 2 \pi f$ where f = frequency.

The equation may now be written in the complex or vector form:

$$\frac{\partial c}{\partial x} - j \frac{4 \pi \omega}{\rho s} I_0 - j \frac{4 \pi \omega}{\rho s} \int_0^x w c \, dx = 0 \quad (1)$$

Differentiating a second time gives:

$$\frac{\partial^2 c}{\partial x^2} = j \frac{4 \pi \omega}{\rho s} w c$$

$$\frac{\partial^2 c}{\partial x^2} = \alpha^2 c \quad (2)$$

or:

$$\left. \begin{aligned} \text{where} \quad \alpha^2 &= j \frac{8 \pi^2 w f}{\rho s} \\ \text{or} \quad \alpha &= \sqrt{\frac{8 \pi w f}{\rho s}} / 45^\circ \end{aligned} \right\}$$

The solution of equation (2) may be written in the form

$$c = A \cosh \alpha x + B \sinh \alpha x \quad (3)$$

This is a vector equation for the root-mean-square current density at a point x centimeters from the bottom of the conductor. The constants of integration A and B are vector quantities and are usually determined by the current in the conductor considered and by the vector I_0 in equation (1). Substitution of equation (3) in equation (1) shows that:

$$B = \alpha / w I_0$$

If the depth of the conductor is d centimeters, the current in it is

$$I_1 = \int_0^d w c \, dx$$

from which it follows that:

$$A = \frac{\alpha}{w} \left\{ \frac{I_1}{\sinh \alpha d} - I_0 \tanh \frac{\alpha d}{2} \right\}$$

Therefore, the general solution for the vector current density may be written:

$$c = \frac{1}{w d} \left\{ \frac{I_1 \alpha d \cosh \alpha x}{\sinh \alpha d} - I_0 \alpha d \tanh \frac{\alpha d}{2} \cosh \alpha x + I_0 \alpha d \sinh \alpha d \right\} \quad (4)$$

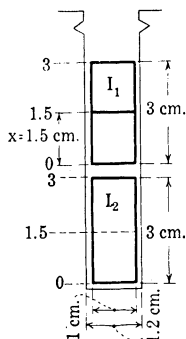


FIG. 1

A numerical calculation is helpful. Consider the case of two solid rectangular conductors situated as shown in Fig. 1. The 60-cycle currents are each 1000 amperes but the lower current leads the upper by 60 degrees as would be the case in some of the slots of a three-phase fractional pitch winding. As we shall presently see $I_0 = I_2$ in this case.

$\rho = 2100$ c. g. s. units of resistance

$w = 1$ cm.

$s = 1.2$ cm.

$d = 3$ cm.

$f = 60$ cycles

$$\alpha d = 3 \sqrt{\frac{8 \pi^2 \times 1 \times 60}{2100 \times 1.2}} / 45^\circ$$

$$= 4.11 / 45^\circ$$

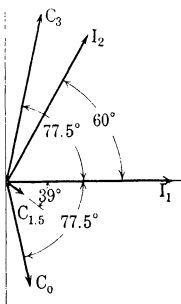


FIG. 2—CURRENT DENSITIES IN UPPER CONDUCTOR

Referred to I_1 the current densities are:

$$C_0 = 1620 / -77.5^\circ \quad \frac{\text{amperes}}{\text{sq. cm.}}$$

$$C_{1.5} = 305 / -39.^\circ \quad \frac{\text{amperes}}{\text{sq. cm.}}$$

$$C_3 = 2490 / 77.5^\circ \quad \frac{\text{amperes}}{\text{sq. cm.}}$$

$$\alpha x = 2.06 / 45^\circ$$

$$\sinh \alpha d = 9.12 / 166.^\circ 4$$

$$\tanh \frac{\alpha d}{2} = 1.11 / 1^\circ. 41$$

$$\sinh \alpha x = 2.24 / 84.^\circ 1$$

$$\cosh \alpha x = 2.03 / 82.^\circ 7$$

If we choose I_1 along the horizontal, i. e., with zero phase:

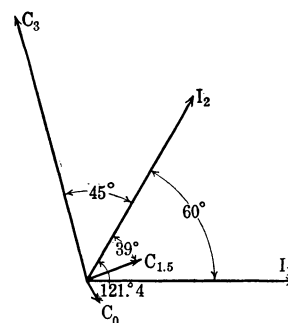
$$I_1 = 1000 / 0^\circ, I_0 = 1000 / 60^\circ$$

$$c_{1.5} = \frac{1}{1 \times 3} \left\{ (1000 / 0^\circ \times 4.11 / 45^\circ \times 2.03 / 82.^\circ 4 \div 9.12 / 166.^\circ 4) - (1000 / 60^\circ \times 4.11 / 45^\circ \times 1.11 / 1^\circ. 41 \times 2.03 / 82.^\circ 7) + (1000 / 60^\circ \times 4.11 / 45^\circ \times 2.24 / 84.^\circ 1) \right\} = \frac{1}{1 \times 3} \left\{ 915 / -39.^\circ 0 - 923 / 189.^\circ 1 + 923 / 189.^\circ 1 \right\}$$

$$c_{1.5} = 305 / -39.^\circ 0$$

$$c_0 = 1620 / -77.^\circ 5$$

$$c_3 = 2490 / 77.^\circ 5$$

FIG. 2A—CURRENT DENSITIES IN LOWER CONDUCTOR
Referred to I_2 the current densities are:

$$C_0 = 150 / -121.^\circ 4 \quad \frac{\text{amperes}}{\text{sq. cm.}}$$

$$C_{1.5} = 305 / 39^\circ \quad \frac{\text{amperes}}{\text{sq. cm.}}$$

$$C_3 = 1380 / 45^\circ \quad \frac{\text{amperes}}{\text{sq. cm.}}$$

Fig. 2 is the vector diagram showing the current densities at the bottom, middle and top of the conductor, together with the total current in it, I_1 , and the current below it, I_2 .

The current density at the center, where αx equals

$$\frac{\alpha d}{2}, \text{ is:}$$

$$c = \frac{I_1}{w d} \frac{\frac{\alpha d}{2}}{\sinh \frac{\alpha d}{2}}$$

It depends only upon the average current density and the angular depth, αd , and is in no way affected by the position of the conductor in the slot. It is the same for solid and finely laminated conductors.

The current density is nowhere greater than at the top of the conductor, where it is:

$$c_d = \frac{1}{w d} \left\{ I_1 \alpha d \coth \alpha d + \frac{I_0}{2} \alpha d 2 \tanh \frac{\alpha d}{2} \right\} \quad (5)$$

The ratio of this maximum current density to the average is:

$$\frac{c_d}{c_{av}} = \alpha d \coth \alpha d + \frac{I_0}{2 I_1} \alpha d 2 \tanh \frac{\alpha d}{2} \quad (6)$$

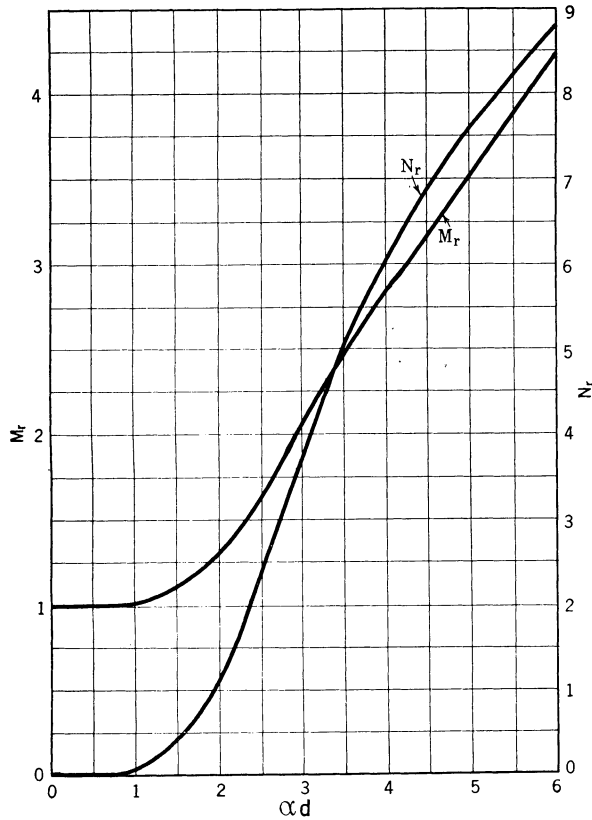


FIG. 3

The complex quantities $\alpha d \coth \alpha d$ and $\alpha d 2 \tanh \frac{\alpha d}{2}$ occur in all of the expressions which we shall

develop, and thus it will be simpler to represent them by single letters. Hereafter

$$\alpha d \coth \alpha d = M = M_r + j M_x$$

$$\alpha d 2 \tanh \frac{\alpha d}{2} = N = N_r + j N_x$$

In Figs. 3 and 4 the abscissas are the numerical values of αd and the ordinates are M_r, M_x, N_r, N_x . The real portions of M and N , viz., M_r and N_r , appear in the expressions for resistance, and the imaginary portions, M_x and N_x , in the expressions for reactance.

The voltage drop per centimeter in the conductor considered due to its own resistance and to all of the leakage flux below its topmost layer is:

$$\rho c_d = \frac{\rho}{w d} \left\{ I_1 M + \frac{I_0}{2} N \right\} \quad (7)$$

The flux within the conductor due to its own current and all of the current, I_b , below it is:

$$\varphi = \int_0^d dx \frac{4 \pi}{s} \int_0^x w c dx + - \frac{4 \pi d}{s} I_b$$

On integration the expression for the flux may be written:

$$\varphi = \frac{1}{j \omega} \frac{\rho}{w d} \left\{ \left(\frac{I_1}{2} + I_0 \right) N + (I_b - I_0) \alpha^2 d^2 \right\}$$

The voltage drop per centimeter produced by this flux in each conductor below the one considered is $j \omega \varphi$, which may now be written:

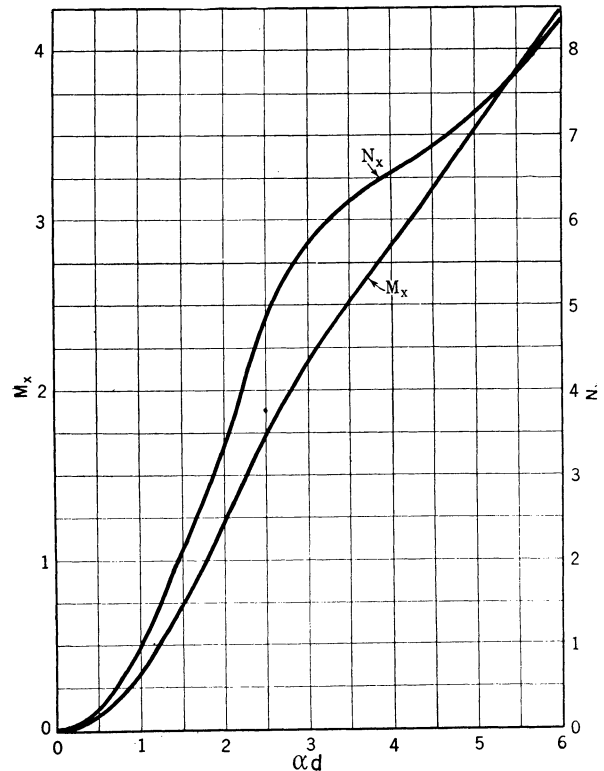


FIG. 4

$$E = \frac{\varphi}{w d} \left\{ \left(\frac{I_1}{2} + I_0 \right) N + (I_b - I_0) \alpha^2 d^2 \right\} \quad (8)$$

The relations expressed in equations (7) and (8) are very important since their proper combination will give the leakage impedance drop due to resistance and leakage flux within the conductors themselves for any arrangement of solid or infinitely laminated conductors. The reactance due to slot-leakage flux which does not pass through the conductors can be calculated by well-known methods and need not be considered here.

If the equations (7) and (8) are each multiplied by

the length of the core, l , they will apply to the embedded portion of a solid conductor, the true resistance of which is R , and to the half turn of an infinitely laminated one, the true resistance of which is also R . Thus (7) and (8) may be written:

$$l \rho c_d = R \left\{ I_1 M + \frac{I_0}{2} N \right\} \quad (7a)$$

$$l E = R \left\{ \left(\frac{I_1}{2} + I_0 \right) N + (I_b - I_0) \alpha^2 d^2 \right\} \quad (8a)$$

The *additional* voltage produced in all conductors below the one in question by flux within it due to its own current is

$$l E' = R \left\{ \left(\frac{I_1}{2} + I_0 \right) N - I_0 \alpha^2 d^2 \right\} \quad (8b)$$

Having found the vector impedance drop in the conductors of one phase by properly applying equations (7a) and (8a), the effective resistance and reactance of that phase are determined by dividing this drop by the vector current. The real portion of the result is the effective resistance and the imaginary portion, the effective reactance. This method, however, gives no indication of the distribution of the copper loss among the several conductors of the phase. This may be of considerable importance, especially in the case of solid conductors when the heat developed in the topmost conductor in a slot may be several times that developed in the bottom conductor.

The current through a conductor is non-uniformly distributed on account of flux *within* the conductor. This flux is due only to the current in the conductor itself and to current below it in the slot. Any current above the conductor in question has no effect on the current distribution within it. The heat generated within a conductor depends only upon the manner in which the current is distributed. The current density is completely determined by equation (1) and the total current in the conductor itself. For any particular value of I_0 (equation 1) the heat developed by a given current in the conductor does not depend upon whether I_0 is some particular current or some combination of currents. Thus if it is possible to find the heat loss when I_0 is some particular current, we will have obtained a general expression for the loss in terms of the current in the conductor, I_1 , and the constant, I_0 , in equation (1). The particular case which we will consider is that of a solid conductor carrying a current of I_1 with a total current of I_b below it. The special form of equation (1) is then,

$$\frac{\partial c}{\partial x} - j \frac{4 \pi \omega}{\rho s} I_b - j \frac{4 \pi \omega}{\rho s} \int_0^x w c \partial x = 0 \quad (1a)$$

Thus for this arrangement, $I_0 = I_b$.

The heat loss in the conductor is equal to the total

power supplied both to this conductor and to all of those below it less the power supplied to those below it when the conductor is removed from the slot. The addition of the conductor in question increases the power supplied to the lower conductors without increasing the heat loss in them on account of its mutual inductive effect upon them *i. e.*, as some would say, on account of the eddy currents which are produced in it by the total current I_b , below it. The power supplied to the upper conductor is, symbolically, since $I_0 = I_b$

$$I_1 R \left\{ I_1 M + \frac{I_b}{2} N \right\} \quad (\text{See 7a})$$

This indicates the product of the numerical values of the current, I_1 , the voltage applied to the conductor and the cosine of the phase angle between them. Flux above the conductor produces a quadrature voltage and thus does not affect the power. The *additional* power supplied to the conductors below this one is symbolically

$$I_b R \left\{ \left(\frac{I_1}{2} + I_b \right) N - I_b \alpha^2 d^2 \right\} \quad (\text{See 8b})$$

The actual heat loss in the conductor is thus symbolically:

$$I_1 R \left\{ I_1 M + \frac{I_b}{2} N \right\} + I_b R \left\{ \left(\frac{I_1}{2} + I_b \right) N - I_b \alpha^2 d^2 \right\}$$

This reduces to:

$$R \{ I_1^2 M_r + (I_b^2 + I_1 I_b \cos \theta) N_r \} \quad (9a)$$

where I_1 and I_b are the numerical values of the currents and θ is the phase angle between them. Therefore the general expression for the heat loss in any conductor, solid or infinitely laminated, is:

$$R \{ I_1 M_r + (I_0^2 + I_1 I_0 \cos \theta) N_r \} \quad (9)$$

where I_1 is the numerical value of the current in the conductor, I_0 is the numerical value of the vector constant in the differential equation (1), and θ is the phase angle between I_1 and I_0 . The ratio of alternating-current to direct-current resistance is thus:

$$K = M_r + \left(\left| \frac{I_0}{I_1} \right|^2 + \left| \frac{I_0}{I_1} \right| \cos \theta \right) N_r \quad (10)$$

The vertical lines $||$ indicate that the division is one of numerical values and not of vector values. The first term M_r accounts for the natural non-uniformity of current distribution due to the action of the current upon itself. The second term

$$\left(\left| \frac{I_0}{I_1} \right|^2 + \left| \frac{I_0}{I_1} \right| \cos \theta \right) N_r,$$

accounts for the additional heating produced by the "eddy currents" due to the action of I_0 .

5. M_r and N_r are calculated from the data pertaining to the conductor in question and bear no relation to other conductors.

This equation (10) enables us to compute the ratio of alternating-current to direct-current resistance for any conductor carrying a specified current and for which a differential equation of the form given equation (1) can be written. In the case of solid conductors, the ratio only applies to the embedded portion. The following are the resistance ratios for some of the simpler arrangements of conductors.

1. The heat loss in an open-circuited bar with I_b amperes below it is:

$$\text{heat loss} = R I_b^2 N_r \quad (\text{equation 9a, } I_1 = 0)$$

2. The resistance ratio for the p th conductor of a one-coil-side-per-slot bar winding is:

$$\begin{aligned} K &= M_r + [(p-1)^2 + (p-1)] N_r \\ &= M_r + (p^2 - p) N_r \end{aligned}$$

3. The resistance ratio for a one-coil-side-per-slot winding having n layers is:

$$\begin{aligned} K &= 1/n \sum_1^n [M_r + (p^2 - p) N_r] \\ &= M_r + \frac{n^2 - 1}{3} N_r \end{aligned}$$

This is also the ratio for the lower coil side of any bar winding having n layers. The upper coil side has no effect on the resistance of the lower coil side.

4. The resistance ratio for the upper coil side of a two-coil-side-per-slot fractional pitch winding having n layers per coil side reduces to:

$$K = M_r + \left(\frac{4n^2 - 1}{3} + n^2 \cos \theta \right) N_r$$

θ is the phase angle between the currents in the upper and lower coil sides.

By combining this with the ratio just preceding, we obtain the resistance ratio for a coil, one side of which is above a coil side carrying a current which differs in phase by θ .

The hottest conductor is the one at the top of the coil side which has beneath it current of the same phase. The fact that, with solid bar windings, the heat developed is not uniformly distributed throughout the winding may be no inconsiderable argument against their use.

5. Our method of attack enables us to obtain a simple solution for the relation between the currents in a double squirrel-cage winding. Neglect the effect of the end rings. In this case the constant of integration, A , in equation (3) is determined by the fact that the resistance drop in the lowest element of the upper bar is the same as the impedance drop in the lower bar due to its resistance and to all of the leakage flux which does not link any portion of the upper bar. The vector equation for the current density in the upper bar may be written:

$$c = \frac{I_2 Z_2}{\rho} \cosh \alpha x + \frac{I_2 R_1}{\rho} \alpha d \sinh \alpha x$$

$I_2 Z_2$ is the vector impedance drop in the lower bar per centimeter;⁶ ρ and R_1 are respectively the resistivity and the true resistance per centimeter of the upper bar whose depth is d centimeters. α is calculated for the upper bar.

The vector current in the upper bar is:

$$I_1 = \frac{I_2 Z_2}{R_1} \frac{\sinh \alpha d}{\alpha d} + I_2 \cosh \alpha d - I_2$$

The process of calculating the heat loss in the upper bar by substitution in equation (9a) is much simplified if we let

$$P = \frac{m Z_2}{R_1} \cos (\theta_2 + \beta) + n \cos \delta$$

and

$$Q = \frac{m Z_2}{R_1} \sin (\theta_2 + \beta) + \sin \delta$$

where:

$$Z_2 = Z_2 / \theta_2; \quad \frac{\sinh \alpha d}{\alpha d} = m / \beta;$$

$$\cosh \alpha d = n / \delta$$

The expression for the loss in the upper bar is:

$$I_2^2 R_1 \{ [(P-1)^2 + Q^2] M_r + P N_r \}$$

This method of solution for the relation between the currents in a double squirrel cage should prove of considerable value in any analysis of the design of such windings.

Finely laminated windings may be of three types: Those in which the laminations are joined at the ends of each half turn; those in which they are joined at the ends of each turn; and those in which the laminations are continuous throughout a single coil. The first is like a solid bar winding except that, as noted previously, the real resistivity of the copper should be multiplied by the ratio of the length of the half turn to the length of the core. The resistance ratio then applies to the whole winding and not to the embedded portion solely. The resistance ratios for the second type depend upon whether the end turn between the coil sides is untwisted or twisted. The resistance ratio for the third type depends upon whether the end turns are untwisted, twisted on one side only, or twisted on both sides. One considerable advantage of continuous laminations is that the heat developed is the same in all of the conductors.

6. Type two: End turn untwisted. The arrangement of the coil sides is shown in Fig. 5. The heavy line across the conductors indicates the same lamination. The current, I_2 , below the upper coil side may be in phase with the current above it or differ from it by 60 or 90 degrees. The differential equation of the

6. $I_2 Z_2$ is the voltage drop due to resistance and flux that does not link the upper bar.

7. See Table IV, Tables of Complex Hyperbolic and Circular Functions.

form (1) applying to this case for the p th layer from the bottom is:

$$2 \rho \frac{\partial c}{\partial x} - j \frac{4 \pi \omega}{s} \left\{ 2(p-1) I_1 + I_2 \right\} - j \frac{4 \pi \omega}{s} 2 \int_0^x w c \partial x = 0$$

Comparing this equation (1) shows that the vector constant I_0 equals $(p-1) I_1 + \frac{I_2}{2}$; $I_2 = n I_1 / \theta$,

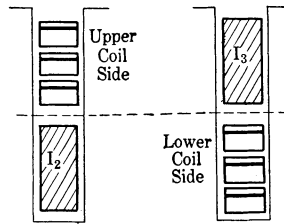


FIG. 5

n is the number of layers in the coil side and θ the phase angle between the currents in the upper and lower coil sides.

Making this substitution, equation (9) reduces to:

$$\text{Heat loss} = R I_1^2 \left\{ M_r + [p^2 - p + n(p-1/2) \cos \theta + n^2/4] N_r \right\}$$

The resistance ratio for the entire coil is one- n th of the summation of this expression from $p = 1$ to $p = n$. It reduces to:

$$K = M_r + \left(\frac{7n^2 - 4}{12} + \frac{n^2}{2} \cos \theta \right) N_r$$

7. Type three: End turns untwisted. In this case the heat developed is the same in each conductor. The differential equation now becomes:

$$2 n \rho \frac{\partial c}{\partial x} - j \frac{4 \pi \omega}{s} \sum_1^n \{ 2(p-1) I_1 + I_2 \} - j \frac{4 \pi \omega}{s} 2 n \int_0^x w c \partial x = 0$$

In this case $I_0 = \frac{n-1}{2} I_1 + I_2$. Making this substitution in equation (10) gives the resistance ratio for the whole coil.

$$K = M_r + \left(2 \frac{n^2 - 1}{4} + \frac{n^2}{2} \cos \theta \right) N_r$$

8. Type two: End turns twisted. When the end turns are twisted on one side only the top laminations in one coil side become the bottom laminations of the other coil side in corresponding layers. See Fig. 6. The lines across the layers trace the positions of one continuous lamination—type three. In type two, however, the laminations are joined at the beginning and end of each turn. The differential equation

which applies to the p th layer is:

$$2 \rho \frac{\partial c}{\partial x} - j \frac{4 \pi \omega}{s} I_2 + \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} \{ (p-1) I_1 (d-x) + (p-1) I_1 x \} - j \frac{4 \pi \omega}{s} \left\{ \int_0^x w c \partial x + \int_d^x w c \partial x \right\} = 0 \quad (11)$$

This readily reduces to:

$$\frac{\partial c}{\partial x} - j \frac{4 \pi \omega}{s \rho} \left(\frac{I_2}{2} - \frac{I_1}{2} \right) - j \frac{4 \pi \omega}{s \rho} \int_0^x w c \partial x = 0$$

Notice that the mutual effect of the layers upon each other is eliminated, by twisting the end connection (third term (11)). The resistance ratio is thus the same for each turn. It is independent of the current in the lower coil side, but it does depend upon the number of layers in the coil side.

$$K = M_r + \frac{n^2 - 1}{4} N_r$$

9. Type three: End connections twisted on one side only. The differential equation of the form (1) is:

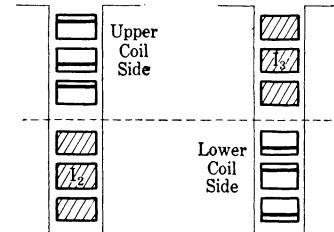


FIG. 6

$$2 n \rho \frac{\partial c}{\partial x} + \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} I_2 \{ (d-x) + x + (d-x) + \dots \} + \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} I_1 \left\{ \begin{array}{l} (d-x) + 2x + 3(d-x) \\ + x + 2(d-x) + 3x \\ + \dots \end{array} \right\} - j \frac{4 \pi \omega}{s} \{ n \int_0^x w c \partial x + n \int_d^x w c \partial x \}$$

The third term of this equation is always zero, but the second term may or may not be. It is thus apparent that there are two cases to be considered, viz., when the number of layers is even and when it is odd. If n is even the second term is zero and the equation reduces to:

$$\frac{\partial \epsilon}{\partial x} - j \frac{4 \pi \omega}{s \rho} \left(-\frac{I_1}{2} \right) - j \frac{4 \pi \omega}{s \rho} \int_0^x w \epsilon \partial x = 0$$

Notice that this equation is independent of the number of layers and the current in the lower coil side. This arrangement of an even number of continuously laminated layers whose end turns are twisted on one side only gives the smallest resistance ratio of any of the cases considered.

$$K = M_r - 1/4 N_r$$

From the forms of M_r and N_r this is readily shown to be the M_r for a conductor one-half as deep. This same condition of current distribution is obtained by having an even number of laminated conductors side by side in the slots if the end connections are twisted on one side only. The resistance ratio is then independent of the number of layers in the coil. If n is odd, the differential equation reduces to:

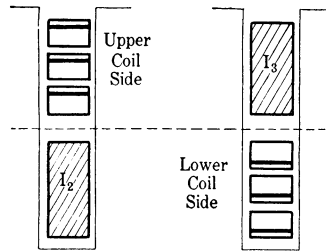


FIG. 7

$$\frac{\partial \epsilon}{\partial x} - j \frac{4 \pi \omega}{s \rho} \left\{ \frac{I_2}{2n} - \frac{I_1}{2} \right\} - j \frac{4 \pi \omega}{s \rho} \int_0^x w \epsilon \partial x = 0$$

The resistance ratio is now:

$$K = M_r$$

This ratio is the same as in the case of a *single* solid conductor of the *same* depth, whereas if there are an even number of layers the ratio is the same as for a single solid conductor of one-half the depth of the laminated one.

10. Type three: End connections twisted on both sides (See Fig. 7). The differential equation of the form (1) is:

$$2n\rho \frac{\partial \epsilon}{\partial x} + \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} I_2 (d-x)n + \frac{\partial}{\partial x} j \frac{4 \pi \omega}{s} I_1 \left\{ \begin{array}{l} (d-x) + 2x + 3(d-x) \\ + \dots \dots \dots \\ + x + 2(d-x) + 3x \\ + \dots \dots \dots \end{array} \right\} - j \frac{4 \pi \omega}{s} \left\{ n \int_0^x w \epsilon \partial x + n \int_0^x w \epsilon \partial x \right\} = 0$$

This reduces to:

$$\frac{\partial \epsilon}{\partial x} - j \frac{4 \pi \omega}{s \rho} \left(\frac{I_2}{2} - \frac{I_1}{2} \right) - j \frac{4 \pi \omega}{s \rho} \int_0^x w \epsilon \partial x = 0$$

The resistance ratio is:

$$K = M_r + \frac{n^2 - 1}{4} N_r$$

It is independent of the current in the lower coil side. Notice that this is the same ratio as was obtained for case 8.

Enough illustrations of the method and the simplicity of its application have been given. There follows a numerical calculation of the resistance ratios for a given arrangement of conductors of various types. The winding data are: Three-phase with four slots per pole per phase; coil pitch of 10 slots; two turns per coil; conductors 1.5 cm. deep; length of embedded portion and of end turns the same; frequency 60 cycles, the ratio of width of copper to width of slot, 0.6; resistivity, 2100 c. g. s. units.

For solid conductors:

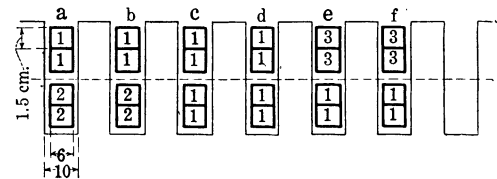


FIG. 8

$$\alpha d = 1.5 \sqrt{\frac{8 \pi^2 \times 60 \times 0.6}{2100}} / 45^\circ = 1.74 / 45^\circ$$

From curve

$$M_r = 1.20$$

$$N_r = 0.73$$

For laminated conductor:

$$\alpha d = 1.5 \sqrt{\frac{8 \pi^2 \times 60 \times 0.6}{2 \times 2100}} / 45^\circ = 1.23 / 45^\circ$$

From curve

$$M_r = 1.05$$

$$N_r = 0.20$$

The arrangement of the conductors of one phase before one pole is shown in Fig. 8. The resistance ratios for phase one are given in the following table for the various cases considered.

TABLE I	
SOLID CONDUCTORS	
Lower coil sides (slots e, d, e & f).....	1.93*
Upper " " (" a and b).....	6.31*
" " " (" e and d).....	7.77*
Entire winding including end turns.....	2.75†

INFINITELY LAMINATED CONDUCTORS, TYPE 1.	
Lower coil sides (slots c, d, e & f).....	1.25
Upper " " (" a and b).....	2.45
" " " (" c and d).....	2.85
Entire winding.....	1.95
INFINITELY LAMINATED CONDUCTORS, TYPE 2.	
Untwisted (slots c and d).....	1.85
" " (" a, b, e and f).....	1.65
" " (entire winding).....	1.75
Twisted (entire winding).....	1.20
INFINITELY LAMINATED CONDUCTORS, TYPE 3.	
Untwisted (slots c and d).....	1.80
" " (" a, b, e and f).....	1.60
" " (entire winding).....	1.70
Twisted in one end connection (entire winding).....	1.005
Twisted in both end connections (entire winding)....	1.20

*Embedded portion.

†The ratio of the heat developed in the top conductor of slots, b or c to that developed in the bottom conductor is as $\frac{9.96}{1.20}$ or as 8.3 to 1.

This may be a very important consideration, even more than the resistance ratio for the entire winding.

Leakage Reactance Volts. As previously stated, the entire leakage impedance due to resistance and slot leakage flux lying wholly within the conductors themselves may be computed by the proper combination of equations (7a) and (8a). This method offers little or no advantage when calculating the resistance. It has been shown that in the case of solid bar winding I_0 equals I_b , (equation 8a). Thus it is probable that the expressions for reactance are similar to those for resistance except that M_x and N_x would replace M_r and N_r . Such proves to be the case. In the case of laminated conductors, however, there is an added term in the expressions for reactance. Consider the general case of a three-phase fractional pitch winding, a typical arrangement of which is shown in Fig. 8. There will usually be slots in which both coil sides are in the same phase. There will also be slots in which the top coil side is in phase one, for example, and the lower coil side in phase two, together with an equal number of slots in which the lower coil side is in phase one and the upper coil side in phase three. In the general polyphase case of p phases there will be slots occurring in pairs in which the currents differ in phase by plus and minus π/p radians. Let us designate these symmetrical pairs as fractional pitch slots and those in which the upper and lower coil sides are in the same phase as full pitch slots. Accordingly there are two fractional pitch and two full pitch slots per pole per phase in the winding illustrated in Fig. 8. Let n be the number of layers per coil side and R the true resistance of the conductors considered

1. Solid conductors: Full pitch slots. Apply equations (7a) and (8a), but divide by the phase current, I_1 , in order to obtain the impedance directly. With solid conductors the coefficient of $\alpha^2 d^2$ in equation (8a) is zero. Let p be the number of any conductor measured from the bottom of the slot. There are $2n$ conductors in both layers in the slot.

The vector expression for the impedance is:

$$Z = \frac{R}{2n} \left\{ \sum_i^{2n} \left(M + \frac{p-1}{2} N \right) + \sum_i^{2n} (p-1) (1/2 + p-1) N \right\}$$

The first term is the summation of the resistance drops (equation 7a) divided by the current; the second term is the summation of the reactive drops per ampere produced in each of the $(p-1)$ conductors below the p th conductor (equation 8a) by the flux within the latter.

This reduces to:

$$Z = R \left\{ M + \frac{4n^2 - 1}{3} N \right\}$$

The resistance is the real portion of this expression and the reactance the imaginary portion. Thus:

$$r = R \left\{ M_r + \frac{4n^2 - 1}{3} N_r \right\}$$

$$x = R \left\{ M_x + \frac{4n^2 - 1}{3} N_x \right\}$$

2. Solid conductors: Fractional pitch slots (taken in pairs as described). The expression for the impedance reduces to

$$Z = \frac{R}{2n} \left\{ \sum_i^n \left[M + \left(\frac{p-1}{2} + n/\theta \right) N \right] + \sum_i^n \left[M + \frac{p-1}{2} N \right] + \sum_i^n (p-1) \left[\frac{1}{2} + (p-1) + n/\theta \right] N + n \sum_i^n \left[\frac{1-\theta}{2} + (p-1)/-\theta + n \right] N + \sum_i^n (p-1) \left[\frac{1}{2} + (p-1) \right] N \right\}$$

The first two terms are respectively the summations of the resistance drops per ampere (equation 7a) in the upper and lower coil sides; the third term is the summation of the reactive drops per ampere produced in each of the $(p-1)$ conductors below the p th conductor of the upper coil side (equation 8a) by the flux within the latter due to its own current and all of that below it in the slot; the fourth term is the reactive drop (equation 8a) in the lower coil side due to the flux within the coil side above it which carries

a current having a relative phase angle of $\angle - \theta$; the last term is the summation of the reactive drops per ampere produced in each of the $(p - 1)$ conductors below the p th conductor of the lower coil side (equation 8a) by the flux within the latter.

This reduces to:

$$Z = R \left\{ M + \left(\frac{5n^2 - 2}{6} + \frac{n^2}{2} \cos \theta \right) N \right\}$$

The resistance is:

$$r = R \left\{ M_r + \left(\frac{5n^2 - 2}{6} + \frac{n^2}{2} \cos \theta \right) N_r \right\}$$

The reactance is:

$$x = R \left\{ M_x + \left(\frac{5n^2 - 2}{6} + \frac{n^2}{2} \cos \theta \right) N_x \right\}$$

This expression is general for both fractional and full pitch slots. For the latter θ equals zero.

3. Finely and continuously laminated conductors (type three) with untwisted end connections. Consider the general case of symmetrical pairs of fractional pitch slots in which the currents in the coil sides lying in the same slot differ in phase by plus and minus θ . R is the true resistance of one coil. In this case

$$I_0 = \frac{n-1}{2} I_1 + \frac{I_2}{2};$$

where I_2 is the current in the lower coil side.

$$Z = \frac{R}{2n}$$

$$\begin{aligned} & \left\{ 2 \sum_1^n \left[M + \left(\frac{n-1}{2 \times 2} + \frac{n/\theta}{2 \times 2} \right) N \right] \right. \\ & + \sum_1^n (p-1) \left[\left(\frac{1}{2} + \frac{n-1}{2} + \frac{n/\theta}{2} \right) N \right. \\ & \quad \left. + \left(p-1 + n/\theta - \frac{n-1}{2} - \frac{n/\theta}{2} \right) \alpha^2 d^2 \right] \\ & + n \sum_1^n \left[\left(\frac{\angle - \theta}{2} + \frac{n-1}{2} \angle - \theta + \frac{n}{2} \right) N \right. \\ & \quad \left. + \left((p-1) \angle - \theta + n - \frac{n-1}{2} \angle - \theta - \frac{n}{2} \right) \alpha^2 d^2 \right] \\ & \left. + \sum_1^n (p-1) \left[\left(-\frac{1}{2} + \frac{n-1}{2} + \frac{n/\theta}{2} \right) N \right. \right. \\ & \quad \left. \left. + \left(p-1 - \frac{n-1}{2} - \frac{n/\theta}{2} \right) \alpha^2 d^2 \right] \right\} \end{aligned}$$

The first term is the summation of the resistance drops per ampere (equation 7a) in the upper and lower coil sides. Due to the fact that laminations are continuous the current distribution is the same in each conductor

of the coil. Thus the resistance drops are also the same for each conductor. The second, third and fourth terms respectively correspond to the third, fourth and fifth terms in the preceding case.

This reduces to:

$$Z = R \left\{ M + \left(\frac{2n^2 - 1}{4} + \frac{n^2}{2} \cos \theta \right) N + \frac{4n^2 - 1}{12} \alpha^2 d^2 \right\}$$

The resistance and reactance are respectively the real and imaginary portions of this expression. Thus:

$$r = R \left\{ M_r + \left(\frac{2n^2 - 1}{4} + \frac{n^2}{2} \cos \theta \right) N_r \right\}$$

$$x = R \left\{ M_x + \left(\frac{2n^2 - 1}{4} + \frac{n^2}{2} \cos \theta \right) N_x + \frac{4n^2 - 1}{12} \left| \alpha \right|^2 d^2 \right\}$$

$\left| \alpha \right|^2$ is the square of the numerical value of α , viz.,

$$\frac{8 \pi^2 w f}{\rho s}.$$

4. Finely and continuously laminated conductors (type three) with end connections twisted on both sides. Consider the general case of symmetrical pairs of fractional pitch slots in which the currents in the coil sides lying in the same slot differ in phase by θ . Due to the twist in the end connections the current density is the same at points equally distant from the bottom of half of the conductors, and from the top of the other half. The expression for the flux within the conductor has already been given for the first condition. When current density is measured from the top of the conductor, the expression for the flux within it is:

$$\varphi = \frac{4 \pi}{s} \int_a^0 \partial x \int_a^x w \epsilon \partial x$$

This readily reduces to:

$$\varphi = \frac{1}{j \omega} \frac{\rho}{w d} \left\{ (I_1 + I_0) \alpha^2 d^2 - \left(\frac{I_1}{2} + I_0 \right) N \right\}$$

The total flux within the conductor including that produced by the current, I_b , below it is:

$$\varphi_0 = \frac{1}{j \omega} \frac{\rho}{w d} \left\{ (I_1 + I_0 + I_b) \alpha^2 d^2 - \left(\frac{I_1}{2} + I_0 \right) N \right\}$$

The voltage produced in every conductor below the one in question by this flux is:

$$E = R \left\{ (I_1 + I_0 + I_b) \alpha^2 d^2 - \left(\frac{I_1}{2} + I_0 \right) N \right\}$$

where R is the true resistance of a half turn.

For conductors in which the current density is given for values of x measured from the top, rather than from the bottom, the resistance drop $\rho \epsilon$ is that in the bottom element. The flux within this conductor then produces an *additional* voltage in it. In this case

$$I_0 = \frac{I_2}{2} - \frac{I_1}{2}$$

$$Z = \frac{R}{2n}$$

$$\begin{aligned} & \left\{ 2 \sum_{i=1}^n \left[M + \left(\frac{n/\theta}{2 \times 2} - \frac{1}{2 \times 2} \right) N \right] \right. \\ & + \sum_{i=1}^n (p-1) \left[\left(\frac{1}{2} + \frac{n/\theta}{2} - \frac{1}{2} \right) N \right. \\ & \quad \left. + \left((p-1) + n/\theta - \frac{n}{2} / \theta + \frac{1}{2} \right) \alpha^2 d^2 \right] \\ & + n \sum_{i=1}^n \left[\left(\frac{\angle - \theta}{2} + \frac{n}{2} - \frac{\angle - \theta}{2} \right) N \right. \\ & \quad \left. + \left((p-1) / \angle - \theta + n - \frac{n}{2} + \frac{\angle - \theta}{2} \right) \alpha^2 d^2 \right] \\ & \left. + \sum_{i=1}^n p \left[\left(1 + \frac{n}{2} / \theta - \frac{1}{2} + (p-1) \right) \alpha^2 d^2 \right. \right. \\ & \quad \left. \left. - \left(\frac{1}{2} + \frac{n}{2} / \theta - \frac{1}{2} \right) N \right] \right\} \end{aligned}$$

In this expression R is the true resistance of the coil. The terms are written in the same order as in the previous case.

This reduces to:

$$Z = R \left\{ M + \frac{n^2 - 1}{4} N + \left(\frac{7n^2 - 1}{12} + \frac{n^2}{2} \cos \theta \right) \alpha^2 d^2 \right\}$$

The method of calculating the impedance should now be sufficiently clear. The final equations for the impedance of finely and continuously laminated conductors whose end turns are twisted on one side only are given without showing their detailed construction.

There are two cases to consider,—one with an even number of layers per coil side, and the other with an odd number of layers.

For n , even

$$Z = R \left\{ M - \frac{N}{4} + \left(\frac{10n^2 - 1}{12} + \frac{n^2}{2} \cos \theta \right) \alpha^2 d^2 \right\}$$

For n , odd

$$Z = R \left\{ M + \left(\frac{10n^2 - 1}{12} + \frac{n^2}{2} \cos \theta \right) \alpha^2 d^2 \right\}$$

The formulas are given in such detail that it must be evident how the effects of unbalanced currents may be calculated. If there are marked harmonics in the currents the heating loss for each harmonic may be calculated as if the others were absent. The resulting loss is the sum of the component losses. The resistance ratios increase with the frequency so that higher harmonics of any considerable magnitude may prove troublesome. For example, if the currents should contain 20 per cent fifth and seventh harmonics, the resistance ratio for the entire winding—solid conductors—would increase from 2.75 as given in Table I to 2.95. This neglects any skin effect in the end turns which would probably be considerable for these harmonics. It also neglects the fact that with higher harmonics there would be a marked magnetic skin effect in the laminations surrounding the conductor which might raise the saturation to such a point that the fundamental assumptions would no longer hold.

The increase in the ratio for the embedded portion only is much more marked. The ratio for the embedded portion of the entire winding as calculated from Table I is

$$\frac{4 \times 1.93 + 2 \times 6.31 + 2 \times 7.77}{8} = 4.49$$

The ratio for the entire embedded portion with harmonics present becomes 6.73. The ratio of the heats developed in top and bottom conductors of slots c or d

becomes $\frac{15.8}{1.39}$ or 11.4 when these harmonics are present instead of the value of 8.3 as given in the table.

By making the proper assumptions, this method of analysis allows us to account for the hysteresis and eddy current losses in the armature teeth and core due to the leakage flux, the effect of which we are discussing. Assume that, due to these iron losses, each tube of flux lags behind the net current that is producing it by the same angle, η . If this be the case the reactive drop will lead the resistance drop by $(\pi/2 - \eta)$ radians instead of by $\pi/2$ radians as we have assumed.

$$\text{Thus } \alpha^2 = \frac{8 \pi^2 f w}{\rho s} \angle -\frac{\pi}{2} - \eta$$

$$\text{and } \alpha = \sqrt{\frac{8 \pi^2 f w}{\rho s}} \angle \frac{\pi}{4} - \frac{\eta}{2}$$

New values of the complex quantities M and N may be calculated for this value of α and substituted in the expressions for effective resistance and reactance already obtained. Whether or not this method will produce accurate results can only be determined by much experimental research.

SUMMARY OF FORMULAS

SOLID CONDUCTORS

$$\text{Ratio} = \frac{\text{Alternating-current resistance}}{\text{Direct-current resistance}}$$

p th conductor from bottom one-coil-side-per-slot bar winding.

$$M_r + p(p-1)N_r$$

One-coil-side-per-slot with n layers, or lower coil side with n layers.

$$M_r + \frac{n^2 - 1}{3} N_r$$

Upper coil side, n layers, two-coil-side-per slot, fractional pitch.

$$M_r + \left(\frac{4n^2 - 1}{3} + n^2 \cos \theta \right) N_r$$

FINELY LAMINATED CONDUCTORS (laminations soldered at beginning and end of each turn) fractional pitch¹

$$\text{Ratio} = \frac{\text{A-C. resistance.}}{\text{D-C. resistance}}$$

End turn untwisted, p th conductor from bottom of upper coil side.

$$M_r + [p^2 - p + n(p - 1/2) \cos \theta + n^2/4] N_r$$

End turn untwisted, each coil side.

$$M_r + \left(\frac{7n^2 - 4}{12} + \frac{n^2}{2} \cos \theta \right) N_r$$

End turn twisted, each coil side.

1. Two coil sides per slot, n layers per coil side.

$$M_r + \frac{n^2 - 1}{4} N_r$$

FINELY LAMINATED CONDUCTORS, soldered at the beginning and end of each coil. Ratio of impedance to direct current resistance is given for a pair of coil sides below one of which is current lagging by θ and above the other current leading by θ . n layers per coil side.²

End turns untwisted.

$$M + \left(\frac{2n^2 - 1}{4} + \frac{n^2}{2} \cos \theta \right) N + \frac{4n^2 - 1}{12} \alpha^2 d^2$$

End turns twisted both sides.

$$M + \frac{n^2 - 1}{4} N + \left(\frac{7n^2 - 1}{12} + \frac{n^2}{2} \cos \theta \right) \alpha^2 d^2$$

End turns twisted one side, n even.

$$M - \frac{N}{4} + \left(\frac{10n^2 - 1}{12} + \frac{n^2}{2} \cos \theta \right) \alpha^2 d^2$$

End turns twisted one side, n odd.

$$M + \left(\frac{10n^2 - 1}{12} + \frac{n^2}{2} \cos \theta \right) \alpha^2 d^2$$

2. In calculating the impedance only leakage flux that lies within the conductors is considered. There are well known methods for calculating the reactance due to other leakage flux.

Carrier Current Telephony and Telegraphy

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(Continued from page 315 of JOURNAL for April, 1921.)

Repeaters. From the discussion of the transmission characteristics of lines which will be given later in the paper, the very great practical importance of amplifying apparatus at intermediate points on a line employing carrier frequencies will be evident. For this purpose fortunately we have available, first, the vacuum tube, and, second, a large variety of methods of applying this tube which have been developed to a high state of efficiency in connection with the voice frequency telephone repeater. While, as just indicated, the same general considerations apply to repeaters for carrier current circuits as to repeaters on circuits operated at voice frequencies, the conditions peculiar to carrier current operation require that the repeaters for this service differ quite considerably from standard voice frequency repeaters.

In the first place, on a multiplex carrier current circuit a single repeater installation must handle the energy associated with a number of independent conversations. This could be accomplished by making the installation include a number of repeaters in parallel with suitably associated filter combinations, but it is at once obvious that it is much preferable to install but one repeater channel capable of amplifying all the carrier transmission. The requirements for

the repeater set are made still more severe by the fact that modulation in the repeater tubes, which tends to increase with the load, introduces disturbing factors in carrier operation which are not serious in ordinary repeater operation. The reason for this is that the combination frequencies resulting from the interaction of the currents in two channels may lie in the frequency

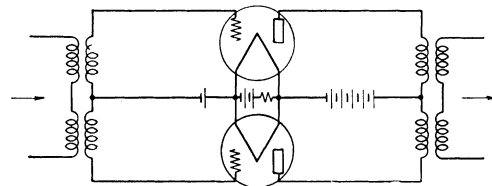


FIG. 21

range of a third, in which case they are transmitted through the selective circuits of that channel and appear as an interfering noise or tone at the subscriber's station. To obtain sufficient energy carrying capacity, and to overcome to some degree this difficulty of intermodulation, we use a number of tubes in parallel in the so-called "push-pull" arrangement. The principle of the push-pull amplifier is shown in Fig. 21.

In this arrangement the input voltage is applied in such a way as to increase the grid voltage of one tube