On the Development of certain Elliptic Functions as Continued Fractions. By L. J. ROGERS, M.A.

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1. In Crelle's *Journal*, Vol. xxxii., Heine has established certain theorems in connection with series of the form

$$1 + \frac{\sin \lambda \alpha}{\sin \lambda} \cdot \frac{\sin \lambda \beta}{\sin \lambda \gamma} x + \frac{\sin \lambda \alpha \sin \lambda (\alpha + 1) \sin \lambda \beta \sin \lambda (\beta + 1)}{\sin \lambda \sin 2\lambda \sin \lambda \gamma \sin \lambda (\gamma + 1)} x^2 + \dots,$$

analogous to Gauss's Hyper-geometric Series

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \dots,$$

and formed from the latter by substituting for every factor in the coefficients the sine of some constant angle λ multiplied by that factor.

Heine writes his series in the form

$$1 + \frac{1-q^\alpha}{1-q} \cdot \frac{1-q^\beta}{1-q'} x + \frac{1-q^\alpha}{1-q} \cdot \frac{1-q^{\alpha+1}}{1-q^3} \cdot \frac{1-q^\beta}{1-q'} \cdot \frac{1-q^{\beta+1}}{1-q'^3} x^2 + \dots,$$

to which the above can be reduced by an easy substitution. I have, however, thought it advisable to treat of the series in the trigonometrical form, in order to keep before the eye the analogy between Heine's series and that of Gauss.

If we call the above trigonometrical series

$$F \sin \lambda \{ \alpha, \beta, \gamma, x \},$$

Heine shows that the fraction

$$F \sin \lambda \{ \alpha, \beta+1, \gamma+1, x \} \div F \sin \lambda \{ \alpha, \beta, \gamma, x \}$$

can be reduced to a continued fraction of the form

$$\frac{e_0}{1-} \frac{e_1 x}{1-} \frac{e_2 x}{1-} \dots,$$

by a method precisely similar to that used by Gauss in establishing his analogous theorem for hyper-geometric series, explained in Todhunter's *Algebra*, § 801.

In fact we have

$$e_{2n-1} = \frac{\sin \lambda (\alpha + n - 1) \sin \lambda (\gamma + n - 1 - \beta)}{\sin \lambda (\gamma + 2n - 2) \sin \lambda (\gamma + 2n - 1)},$$

$$e_{2n} = \frac{\sin \lambda (\beta + n) \sin \lambda (\gamma + n - \alpha)}{\sin \lambda (\gamma + 2n - 1) \sin \lambda (\gamma + 2n)}.$$

It is my object in the following pages to show how certain elliptic functions can, by means of Heine's theorem, be brought into the form of a continued fraction.

2. It can be proved by induction or otherwise that the series

$$1 + \frac{\sin \lambda n}{\sin \lambda} x + \frac{\sin \lambda n}{\sin \lambda} \cdot \frac{\sin \lambda (n-1)}{\sin 2\lambda} x^2 + \dots \dots \dots (1),$$

formed by the process explained above from the binomial series, is decomposable into factors, provided n be a positive integer.

If $n = 2m$, it is equal to

$$(1 + 2x \cos \lambda + x^2)(1 + 2x \cos 3\lambda + x^2)(1 + 2x \cos 5\lambda + x^2) \dots \text{to } m \text{ factors.}$$

If $n = 2m + 1$, we get

$$(1 + x)(1 + 2x \cos 2\lambda + x^2)(1 + 2x \cos 4\lambda + x^2)(1 + 2x \cos 6\lambda + x^2) \dots$$

to $(m + 1)$ factors.

In either case its logarithm is

$$\frac{\sin n\lambda}{\sin \lambda} x - \frac{1}{2} \frac{\sin 2n\lambda}{\sin 2\lambda} x^2 + \frac{1}{3} \frac{\sin 3n\lambda}{\sin 3\lambda} x^3 - \dots$$

If now, for brevity, we write the series (1)

$$1 + C_1 x + C_2 x^2 + \dots,$$

we can immediately deduce, by writing $x\sqrt{-1}$ for x , that

$$e^{(\sin n\lambda) xi / (\sin \lambda) + \frac{1}{2} (\sin 2n\lambda) x^2 / (\sin 2\lambda) - \dots} = 1 + C_1 xi - C_2 x^2 + \dots = e^{Li + M}, \text{ say.}$$

Hence, equating real and imaginary parts,

$$\tan L = \frac{C_1 x - C_3 x^3 + C_5 x^5 - \dots}{1 - C_2 x^2 + C_4 x^4 - \dots},$$

which can easily be shown to be equal to

$$C_1 x \frac{F \sin 2\lambda \left\{ -\frac{n-1}{2}, -\frac{n-2}{2}, \frac{3}{2}, -x^2 \right\}}{F \sin 2\lambda \left\{ -\frac{n-1}{2}, -\frac{n}{2}, \frac{1}{2}, -x^2 \right\}},$$

and this, by § 1, can be reduced to continued fraction form. We get after some reduction

$$\begin{aligned} \tan \left(\frac{\sin n\lambda}{\sin \lambda} x - \frac{\sin 3n\lambda}{3 \sin 3\lambda} x^3 + \frac{\sin 5n\lambda}{5 \sin 5\lambda} x^5 - \dots \right) \\ = \frac{x \sin n\lambda}{\sin \lambda -} \frac{x^2 (\sin^2 n\lambda - \sin^2 \lambda)}{\sin 3\lambda -} \frac{x^2 (\sin^2 n\lambda - \sin^2 2\lambda)}{\sin 5\lambda -} \dots (2). \end{aligned}$$

We have obtained this result by supposing n to be a positive integer; but, by the principle of permanence of algebraical forms, we may suppose it to be true for any value of n , consistent with convergency in the series, and definiteness in the continued fraction.

3. If $\lambda = 0$, we get the known form

$$\tan (n \tan^{-1} x) = \frac{nx}{1-} \frac{x^2 (n^2 - 1^2)}{3-} \frac{x^2 (n^2 - 2^2)}{5-} \frac{x^2 (n^2 - 3^2)}{7-} \dots$$

If $n = 0$, we get forms already investigated by Heine.

Let $\frac{x}{2i} e^{n\lambda i} = y$, and let n increase and x decrease indefinitely, while y remains constant.

$$\text{Then} \quad x \sin n\lambda = \frac{x}{2i} e^{n\lambda i} (1 - e^{-2n\lambda i}) = y,$$

$$x^3 \sin 3n\lambda = \frac{x^3}{2i} e^{3n\lambda i} (1 - e^{-6n\lambda i}) = -2^2 \cdot y^3, \text{ \&c. ;}$$

therefore

$$\tan \left(\frac{y}{\sin \lambda} + \frac{2^2 y^3}{3 \sin 3\lambda} + \frac{2^4 y^5}{5 \sin 5\lambda} + \dots \right) = \frac{y}{\sin \lambda -} \frac{y^3}{\sin 3\lambda -} \frac{y^5}{\sin 5\lambda -} \dots \dots \dots (1).$$

Writing $\frac{x}{2i}$ for y , and $\sqrt{q}$ for $e^{\lambda i}$, we get

$$\begin{aligned} \tan \left(\frac{x\sqrt{q}}{1-q} - \frac{1}{3} \cdot \frac{x^3\sqrt{q^3}}{1-q^3} + \frac{1}{5} \cdot \frac{x^5\sqrt{q^5}}{1-q^5} - \dots \right) \\ = \frac{x\sqrt{q}}{1-q} - \frac{x^3q^3}{1-q^3} + \frac{x^5q^5}{1-q^5} - \frac{x^7q^7}{1-q^7} + \dots \end{aligned}$$

Again in § 2, (2), put $n\lambda = \frac{\pi}{2}$,

then we have

$$\tan \left(\frac{x}{\sin \lambda} + \frac{x^3}{3 \sin 3\lambda} + \frac{x^5}{5 \sin 5\lambda} + \dots \right) = \frac{x}{\sin \lambda} - \frac{x^3 \cos^2 \lambda}{\sin 3\lambda} + \frac{x^5 \cos^2 2\lambda}{\sin 5\lambda} - \dots \quad \dots\dots\dots(2).$$

Or, putting $x = 2y$,

$$\tan \left(\frac{2y}{\sin \lambda} + \frac{2^3 y^3}{3 \sin 3\lambda} + \dots \right) = \frac{2y}{\sin \lambda} - \frac{4y^3 \cos^2 \lambda}{\sin 3\lambda} + \dots$$

Now, the bracketed series on the left side is just twice that in (1), so that, if we call either side of that equation y , each side of (2) must be equal to $\frac{2y}{1-y^2}$ or $\frac{2}{1-y}$.

Hence
$$\frac{\sin \lambda}{2y} - \frac{2y \cos^2 \lambda}{\sin 3\lambda} + \frac{4y^3 \cos^2 2\lambda}{\sin 5\lambda} - \dots$$

$$= \frac{1}{2y} - \frac{y}{2} = \frac{\sin \lambda}{2y} - \frac{\frac{1}{2}y}{\sin 3\lambda} + \frac{y^3}{\sin 5\lambda} - \dots - \frac{\frac{1}{2}y}{\sin \lambda} + \frac{y^3}{\sin 3\lambda} - \dots$$

i.e.,
$$\frac{1}{\sin \lambda} - \frac{y^3}{\sin 3\lambda} + \frac{y^5}{\sin 5\lambda} - \dots + \frac{1}{\sin 3\lambda} - \frac{y^3}{\sin 5\lambda} + \dots$$

$$= \frac{4 \cos^2 \lambda}{\sin 3\lambda} - \frac{4y^3 \cos^2 2\lambda}{\sin 5\lambda} + \frac{4y^5 \cos^2 3\lambda}{\sin 7\lambda} - \dots$$

a curious result, as we have a linear relation connecting three infinite continued fractions. Transforming into q functions, we get, after some easy reductions,

$$\begin{aligned} \frac{q}{1-q} + \frac{zq^3}{1-q^3} + \frac{zq^4}{1-q^5} + \frac{zq^6}{1-q^7} + \dots + \frac{q^3}{1-q^3} + \frac{zq^4}{1-q^5} + \frac{zq^6}{1-q^7} + \dots \\ = \frac{q(q+1)^3}{1-q^3} + \frac{zq^3(1+q^2)^3}{1-q^5} + \frac{zq^5(1+q^2)^3}{1-q^7} + \dots \dots\dots(3), \end{aligned}$$

an identity which holds good provided $q < 1$, $z < 1$.

Either side, when expanded as far as q^6 , is found to be

$$q + 2q^2 - (z-1)q^3 - (2z-1)q^4 + (z^2-3z+2)q^5 + (3z^2-6z+1)q^6 + \dots$$

4. We now proceed to the investigation of certain elliptic functions, when converted into continued fractions.

In § 2, (2), let

$$e^{\lambda i} = q^{-1}, \quad x = 1, \quad n\lambda = \theta,$$

$$\begin{aligned} \text{then} \quad \tan 2i \left(\frac{\sqrt{q} \sin \theta}{1-q} - \frac{1}{3} \frac{\sqrt{q^3} \sin 3\theta}{1-q^3} + \dots \right) \\ = \frac{2i \sqrt{q} \sin \theta}{1-q-} \frac{q^3 \{4 \sin^2 \theta + q^{-1} (1-q)^2\}}{1-q^3-} \frac{q^4 \{4 \sin^2 \theta + q^{-2} (1-q^3)^2\}}{1-q^5-}. \end{aligned}$$

Or, calling the bracketed series S ,

$$\begin{aligned} i \frac{e^{4S} - 1}{e^{4S} + 1} &= \text{the same fraction as before} \\ &= \frac{2i \sqrt{q} \sin \theta}{1-q+} \frac{q (1-2q \cos 2\theta + q^3)}{1-q^3+} \frac{q^3 (1-2q^3 \cos 2\theta + q^4)}{1-q^5+...} \end{aligned}$$

$$\text{Now,} \quad 4S = \log \sqrt{\frac{1+k \operatorname{sn} \frac{2K\theta}{\pi}}{1-k \operatorname{sn} \frac{2K\theta}{\pi}}} = \log \frac{1+k \operatorname{sn} \frac{2K\theta}{\pi}}{\operatorname{dn} \frac{2K\theta}{\pi}} \quad \left(\begin{array}{l} \text{Jacobi,} \\ \text{Fund. Nov.,} \\ \text{p. 156} \end{array} \right)$$

$$\text{therefore} \quad \frac{e^{4S} - 1}{e^{4S} + 1} = \frac{1+k \operatorname{sn} \frac{2K\theta}{\pi} - \operatorname{dn} \frac{2K\theta}{\pi}}{1+k \operatorname{sn} \frac{2K\theta}{\pi} + \operatorname{dn} \frac{2K\theta}{\pi}},$$

$$\text{which} \quad = \frac{k \operatorname{sn} \frac{2K\theta}{\pi}}{1+\operatorname{dn} \frac{2K\theta}{\pi}} = \frac{k \operatorname{sn} \frac{K\theta}{\pi} \operatorname{cn} \frac{K\theta}{\pi}}{\operatorname{dn} \frac{K\theta}{\pi}}.$$

We shall always write $u \doteq \frac{2K\theta}{\pi}$ for shortness.

$$\begin{aligned} \text{Hence} \quad \frac{k \operatorname{sn} u}{1+\operatorname{dn} u} &= \frac{2\sqrt{q} \sin \theta}{1-q+} \frac{q (1-2q \cos 2\theta + q^3)}{1-q^3+...} \\ &= \frac{2\sqrt{q} \sin \theta}{f^n}, \text{ say } \dots\dots\dots (1). \end{aligned}$$

Now,
$$\frac{k \operatorname{sn} u}{1 + \operatorname{dn} u} = \frac{1 - \operatorname{dn} u}{k \operatorname{sn} u} = \frac{2\sqrt{q} \sin \theta}{F};$$

and, since
$$\frac{1 + \operatorname{dn} u}{k \operatorname{sn} u} = \frac{F}{2\sqrt{q} \sin \theta},$$

we see that $\frac{2}{k \operatorname{sn} u}$ is equal to the sum of two continued fractions.

This reduces to the equation

$$\begin{aligned} \frac{4\sqrt{q} \sin \theta}{k \operatorname{sn} u} &= 1 - q + \frac{q(1 - 2q \cos 2\theta + q^2)}{1 - q^2 +} \frac{q^2(1 - 2q^2 \cos 2\theta + q^4)}{1 - q^6 + \dots} \\ &+ \frac{4q \sin^2 \theta}{1 - q +} \frac{q(1 - 2q \cos 2\theta + q^2)}{1 - q^2 + \dots} \dots \dots \dots (2), \end{aligned}$$

while
$$\begin{aligned} \frac{4\sqrt{q} \sin \theta}{k} \cdot \frac{\operatorname{dn} u}{\operatorname{sn} u} &= 1 - q + \frac{q(1 - 2q \cos 2\theta + q^2)}{1 - q^2 +} \dots \\ &- \frac{4q \sin^2 \theta}{1 - q +} \frac{q(1 - 2q \cos 2\theta + q^2)}{1 - q^2 + \dots} \dots \dots (3). \end{aligned}$$

Adding $\frac{\pi}{2}$ to θ , and K to u , we get

$$\begin{aligned} \frac{4k'\sqrt{q} \cos \theta}{k \operatorname{cn} u} &= 1 - q + \frac{q(1 + 2q \cos 2\theta + q^2)}{1 - q^2 +} \frac{q^2(1 + 2q^2 \cos 2\theta + q^4)}{1 - q^6 +} \\ &- \frac{4q \cos^2 \theta}{1 - q +} \frac{q(1 + 2q \cos 2\theta + q^2)}{1 - q^2 + \dots} \dots \dots \dots (4). \end{aligned}$$

Changing $k, k', q, \operatorname{cn} u$ respectively into

$$\frac{ik}{k'}, \quad \frac{1}{k'}, \quad -q, \quad \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

we get

$$\begin{aligned} \frac{4\sqrt{q} \cos \theta}{k} \frac{\operatorname{dn} u}{\operatorname{cn} u} &= 1 + q - \frac{q(1 - 2q \cos 2\theta + q^2)}{1 + q^2 +} \frac{q^2(1 + 2q^2 \cos 2\theta + q^4)}{1 + q^6 - \dots} \\ &+ \frac{4q \cos^2 \theta}{1 + q -} \frac{q(1 + 2q \cos 2\theta + q^2)}{1 + q^2 + \dots} \dots \dots \dots (5). \end{aligned}$$

Adding $\frac{\pi}{2}$ to θ , and K to u , we get

$$\begin{aligned} \frac{4\sqrt{q} \sin \theta}{k \operatorname{sn} u} &= 1 + q - \frac{q(1 + 2q \cos 2\theta + q^2)}{1 + q^2 +} \frac{q^2(1 - 2q^2 \cos 2\theta + q^4)}{1 + q^6 -} \\ &+ \frac{4q \sin^2 \theta}{1 + q -} \frac{q(1 - 2q \cos 2\theta + q^2)}{1 + q^2 + \dots} \dots \dots \dots (6). \end{aligned}$$

We have thus by a series of transformations brought the left-hand side of the equation (2) into its original form, while the right-hand side is different. This is due to certain identities which exist between certain continued fractions, and which will be established in the following section.

5. The left-hand side of § 4, (1) is easily shown by elementary principles to be equal to

$$k \operatorname{sn} \frac{u}{2} \operatorname{cn} \frac{u}{2} / \operatorname{dn} \frac{u}{2},$$

so that, by writing $2u$ for u , 2θ for θ , we get

$$\frac{k \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{2\sqrt{q} \sin 2\theta}{1-q+} \frac{q(1-2q \cos 4\theta + q^2)}{1-q^3+} \frac{q^2(1-2q^2 \cos 4\theta + q^4)}{1-q^5+...} \dots\dots\dots(1).$$

By changing q into $-q$, we change $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ respectively into

$$\frac{k' \operatorname{sn} u}{\operatorname{dn} u}, \quad \frac{\operatorname{cn} u}{\operatorname{dn} u}, \quad \frac{1}{\operatorname{dn} u},$$

and (1) becomes

$$\frac{k \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{2\sqrt{q} \sin 2\theta}{1+q-} \frac{q(1+2q \cos 4\theta + q^2)}{1+q^3+} \frac{q^2(1-2q^2 \cos 4\theta + q^4)}{1+q^5-...} \dots\dots\dots(2).$$

The right sides of (1) and (2) equated give an identity which may be written

$$2q = \frac{q(1+2q \cos 2\theta + q^2)}{1+q^3+} \frac{q^2(1-2q^2 \cos 2\theta + q^4)}{1+q^5-...} + \frac{q(1-2q \cos 2\theta + q^2)}{1-q^3+} \frac{q^2(1-2q^2 \cos 2\theta + q^4)}{1-q^5+...} \dots\dots\dots(3),$$

and by which we may establish the identity of equations (2) and (6) in the preceding section.

It is known, moreover, that by transforming k into $\frac{1-k}{1+k'}$, we change u , K , K' , q , $\operatorname{sn} u$ into

$$(1+k')u, \quad \frac{1}{2}(1+k')K, \quad (1+k')K', \quad q^2, \quad \frac{(1+k') \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

Conversely, (4) becomes

$$\sqrt{k} \operatorname{sn} u = \frac{2\sqrt{q} \sin \theta}{1-\sqrt{q}+ \frac{\sqrt{q}(1-2\sqrt{q} \cos 2\theta + q)}{1-\sqrt{q^3+} \frac{q(1-2q \cos 2\theta + q^2)}{1-\sqrt{q^5+} \dots}} \dots\dots\dots(4).$$

Adding K to u , we get a form for $\frac{\text{cn } u}{\text{dn } u}$, from which, by retransformation, we get

$$\sqrt{\frac{1+k'}{1-k'}} \frac{\text{dn}^2 u - k'}{\text{dn}^2 u + k'} = \frac{2\sqrt{q} \cos 2\theta}{1-q+} \frac{q(1+2q \cos 4\theta + q^2)}{1-q^2+...} \dots\dots(5).$$

6. Again, in § 2, (2), let $e^{\lambda} = q^{-1}$, $x = q^{\frac{1}{2}}$, $n\lambda = \theta$, so that

$$\begin{aligned} & \tan 2i \left(\frac{q \sin \theta}{1-q} - \frac{1}{3} \cdot \frac{q^3 \sin 3\theta}{1-q^3} + \dots \right) \\ &= \frac{2iq \sin \theta}{1-q+} \frac{q^3(1-2q \cos 2\theta + q^2)}{1-q^2+} \frac{q^5(1-2q^2 \cos 2\theta + q^4)}{1-q^4+...} \\ &= Hi, \text{ say} \dots\dots\dots(1). \end{aligned}$$

Then $\tan 4i \left(\frac{q \sin \theta}{1-q} - \dots \right) = \frac{2Hi}{1+H^2}$

which by Jacobi, *Fund. Nov.*, p. 156, becomes

$$\frac{\frac{1+\text{sn } u}{1-\text{sn } u} - \frac{1+\sin \theta}{1-\sin \theta}}{\frac{1+\text{sn } u}{1-\text{sn } u} + \frac{1+\sin \theta}{1-\sin \theta}} = \frac{2H}{1+H^2}$$

that is, $\frac{\text{sn } u - \sin \theta}{1 - \text{sn } u \sin \theta} = \frac{2H}{1+H^2}$

This ultimately reduces to

$$\begin{aligned} \frac{4q \cos^2 \theta \sin \theta}{\text{sn } u - \sin \theta} &= 1-q+4q \sin^2 \theta + \frac{q^3(1-2q \cos 2\theta + q^2)}{1-q^2+...} \\ &+ \frac{4q^2 \sin^2 \theta}{1-q+} \frac{q^3(1-2q \cos 2\theta + q^2)}{1-q^2+...} \dots\dots\dots(2). \end{aligned}$$

From this formula may be deduced a series of like formulæ for

$$\frac{\text{cn } u}{\text{dn } u}, \quad \text{cn } u, \quad \frac{\text{sn } u}{\text{dn } u},$$

by the same methods as were used in the last section.

For instance,

$$\begin{aligned} \frac{4q \sin^2 \theta \cos \theta}{\cos \theta - \text{cn } u} &= 1+q-4q \cos^2 \theta + \frac{q^3(1-2q \cos 2\theta + q^2)}{1+q^2-...} \\ &+ \frac{4q^2 \cos^2 \theta}{1+q+} \frac{q^3(1-2q \cos 2\theta + q^2)}{1+q^2-...} \dots\dots\dots(3). \end{aligned}$$

Again, with the same notation, we have

$$\frac{\frac{1+\operatorname{sn} u}{\operatorname{cn} u} - \frac{1+\sin \theta}{\cos \theta}}{\frac{1+\operatorname{sn} u}{\operatorname{cn} u} + \frac{1+\sin \theta}{\cos \theta}} = H.$$

This reduces to

$$\frac{\frac{1+\operatorname{sn} u}{\operatorname{cn} u} - 1}{\frac{1+\operatorname{sn} u}{\operatorname{cn} u} + 1} = \cot \frac{\theta}{2} - \frac{\cot^2 \frac{\theta}{2} - 1}{\cot \frac{\theta}{2} + H},$$

$$\text{or} \quad \tan \frac{\theta}{2} \frac{\operatorname{sn} \frac{u}{2} \operatorname{dn} \frac{u}{2}}{\operatorname{cn} \frac{u}{2}} = 1 - \frac{1 - \tan^2 \frac{\theta}{2}}{1 + H \tan \frac{\theta}{2}} \dots\dots\dots(1).$$

Hence

$$\tan \theta \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} = 1 - \frac{1 - \tan^2 \theta}{1 +} \frac{4q \sin^2 \theta}{1 - q +} \frac{q^2 (1 - 2q \cos 4\theta + q^2)}{1 - q^3 + \dots} \dots(2).$$

By adding K to u , we do not get any new identity; but, by changing q into $-q$, we have

$$\tan \theta \frac{k' \operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} = 1 - \frac{1 - \tan^2 \theta}{1 -} \frac{4q \sin^2 \theta}{1 + q +} \frac{q^2 (1 + 2q \cos 4\theta + q^2)}{1 + q^3 + \dots}$$

7. From equations § 5, (1), (5), we get

$$\begin{aligned} \frac{2kK}{\pi} &= \frac{4\sqrt{q}}{1-q+} \frac{q(1-q)^2}{1-q^3+} \frac{q^2(1-q^2)^2}{1-q^5+\dots}, \\ \frac{k}{1+k'} &= \frac{2\sqrt{q}}{1-q+} \frac{q(1+q)^2}{1-q^3+} \frac{q^2(1+q^2)^2}{1-q^5+\dots}, \\ \frac{2K}{\pi} + 1 &= \frac{4q}{1-q+} \frac{q^2(1-q)^2}{1-q^3+} \frac{q^4(1-q^2)^2}{1-q^5+\dots}; \end{aligned}$$

but they can be as well derived from Heine's results, after replacing the left-hand sides by their known equivalents as q -series.

Moreover, we may derive many identities by comparing the different fractional expression obtained for $\operatorname{sn} u$ in §§ 4 and 5, and deduce values for K , &c. as continued fractions.

8. Again, since

$$\log \operatorname{tn} u = \log \tan \theta - \log \sqrt{k'} - \frac{4q^2}{1-q^2} \cos 2\theta - \frac{4q^6}{1-q^6} \frac{\cos 6\theta}{3} - \dots,$$

we get, by § 2, (2),

$$\frac{\tan \theta - \sqrt{k'} \tan u}{\tan \theta + \sqrt{k'} \tan u} = \frac{2q^2 \cos 2\theta}{1-q^2+} \frac{q^4(1-2q^2 \cos 4\theta + q^4)}{1-q^6+} \frac{q^8(1-2q^4 \cos 4\theta + q^8)}{1-q^8+...},$$

by the same method as before.

This may be written

$$\sqrt{k'} \tan u + \tan \theta = \frac{2 \tan \theta}{1+} \frac{2q^2 \cos 2\theta}{1-q^2+} \frac{q^4(1-2q^2 \cos 4\theta + q^4)}{1-q^6+...} \dots (1).$$

In the same way, we get from the equation

$$\log \operatorname{dn} u = \log \sqrt{k'} + \frac{4q \cos 2\theta}{1-q^2} + \frac{4q^3 \cos 6\theta}{3(1-q^6)} + \dots,$$

that

$$\frac{\operatorname{dn} u - \sqrt{k'}}{\operatorname{dn} u + \sqrt{k'}} = \frac{2q \cos 2\theta}{1-q^2+} \frac{q^2(1+2q^2 \cos 4\theta + q^4)}{1-q^6+} \frac{q^4(1+2q^4 \cos 4\theta + q^8)}{1-q^{10}+...}$$

$$\text{or} \quad \operatorname{dn} u + \sqrt{k'} = \frac{2\sqrt{k'}}{1-} \frac{2q \cos 2\theta}{1-q^2+} \frac{q^2(1+2q^2 \cos 4\theta + q^4)}{1-q^6+...} \dots (2).$$

From the above, we get

$$\begin{aligned} \frac{2K}{\pi} \sqrt{k'} + 1 &= \frac{2}{1+} \frac{2q^2}{1-q^2+} \frac{q^4(1-q^2)^2}{1-q^6+...}, \\ \frac{1-\sqrt{k'}}{1+\sqrt{k'}} &= \frac{2q}{1-q^2+} \frac{q^2(1+q^2)^2}{1-q^6+} \frac{q^4(1+q^4)^2}{1-q^{10}+...}, \end{aligned}$$

as may be proved by Heine's theorem.

9. The question of convergency, as regards the continued fractions above employed, is of easy solution.

It is known that a continued fraction

$$\frac{1}{1-} \frac{e_1}{1-} \frac{e_2}{1-} \frac{e_3}{1-} \dots$$

is convergent, provided the limit of e_n is numerically less than unity. (See Todhunter's *Algebra*, Art. 783.)

In all the fractions met with above in the later sections, the value of e_n is of the form

$$\frac{q^m(1-2q^n \cos 2\theta + q^{2n})}{(1-q^r)(1-q^s)},$$

where m, n, r, s all tend to infinite limit. Hence e_n tends to zero

limit, since $q < 1$. Similarly, in § 3 (3), all the continued fractions are convergent if $zq < 1$. We may therefore suppose that $z = 1$.

10. There are a very large number of algebraic identities which may be derived from the several equations established in the above section.

We have already obtained one in § 5.

If we make $2\theta = \pi$ in the identity § 5 (3), we get

$$2q = \frac{q(1+q^3)}{1+q^3+} \frac{q^2(1+q^4)}{1+q^4-} \dots + \frac{q(1+q^3)}{1-q^3+} \frac{q^2(1+q^4)}{1-q^4+} \dots$$

But, by § 3 (3), we can split up each of these fractions into two others, so that we have ultimately

$$\begin{aligned} 2q = & \frac{q}{1-q+} \frac{q^2}{1-q^3+} \dots + \frac{q^2}{1-q^3+} \frac{q^4}{1-q^6+} \dots \\ & + \frac{q}{1+q+} \frac{q^2}{1+q^3+} \dots - \frac{q^2}{1+q^3+} \frac{q^4}{1+q^6+} \dots \end{aligned}$$

Again, from § 5, if we multiply the fraction in (4) with that obtained by adding K to u , we shall get a product equal to $\frac{k \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}$. Comparing this with § 5 (2), we get the following identity after some substitutions and reductions—

$$\begin{aligned} & 1 - q^2 + \frac{q^2(1 - 2q^2 \cos 2\theta + q^4)}{1 - q^6 +} \frac{q^4(1 - 2q^4 \cos 2\theta + q^8)}{1 - q^{10} +} \dots \\ = & \left\{ 1 - q + \frac{q(1 - 2q \cos \theta + q^2)}{1 - q^3 +} \frac{q^2(1 - 2q^2 \cos \theta + q^4)}{1 - q^6 +} \dots \right\} \\ \times & \left\{ 1 - q + \frac{q(1 + 2q \cos \theta + q^2)}{1 - q^3 +} \frac{q^2(1 + 2q^2 \cos \theta + q^4)}{1 - q^6 +} \dots \right\}. \end{aligned}$$

If $\cos \theta = 0$, we have

$$\left\{ 1 - q + \frac{q(1+q^3)}{1-q^3+} \frac{q^2(1+q^4)}{1-q^4-} \dots \right\}^2 = 1 - q^2 + \frac{q^2(1+q^3)^2}{1-q^6+} \frac{q^4(1+q^4)^2}{1-q^{10}+} \dots$$

We may get many other identities, *e.g.*, by comparing § 8 (2) with § 5 (5). We do not arrive at very interesting results, as they are for the most part very complicated and involve more than three infinite continued fractions.