On the proposal of the Chairman, a vote of thanks was passed to the Auditor.

The following communications were made:—

“Note on a Method of obtaining the q -formula for the Sine-amplitude in Elliptic Functions:” Mr. J. W. L. Glaisher, F.R.S.

“Note on a Numerical Theorem connected with the Cubical Division of Space:” the President.

“Notes on Curvature:” Mr. J. J. Walker, M.A.

“A Property of a Linkage:” Mr. A. B. Kempe, B.A.

The following presents were made to the Society:—

“Proceedings of the Royal Society,” Vol. xxix., No. 198.

“Educational Times,” December, 1879.

“Bulletin des Sciences Mathématiques et Astronomiques,” Août, 1879.

“American Journal of Mathematics, Pure and Applied,” Vol. ii., No. 3, September, 1879; Baltimore.

Note on a Method of obtaining the q -formula for the Sine-amplitude in Elliptic Functions. By J. W. L. GLAISHER, M.A., F.R.S.

[Read December 11th, 1879.]

The present Note contains a very short method of deducing the q -series for $\operatorname{sn} u$ from the q -products for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$.

1. The products are

$$\operatorname{sn} u = \frac{1}{k'} \cdot 2q^{\frac{1}{2}} \sin x \prod \frac{1 - 2q^{2n} \cos 2x + q^{4n}}{1 - 2q^{2n-1} \cos 2x + q^{4n-2}} \dots\dots\dots (i.),$$

$$\operatorname{cn} u = \frac{k'}{k} \cdot 2q^{\frac{1}{2}} \cos x \prod \frac{1 + 2q^{2n} \cos 2x + q^{4n}}{1 - 2q^{2n-1} \cos 2x + q^{4n-2}} \dots\dots\dots (ii.),$$

$$\operatorname{dn} u = k' \prod \frac{1 + 2q^{2n-1} \cos 2x + q^{4n-2}}{1 - 2q^{2n-1} \cos 2x + q^{4n-2}} \dots\dots\dots (iii.),$$

where u denotes $\frac{2Kx}{\pi}$, and the product symbol $\prod$ refers to the letter n , which is to have all positive values from $n = 1$ to $n = \infty$.

Taking the logarithm of (iii.), and differentiating, we have, as in the *Fundamenta Nova*, pp. 99, 100,

$$\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{\pi}{2k^2 K} \left\{ \frac{8q}{1-q^2} \sin 2x + \frac{8q^3}{1-q^2} \sin 6x + \&c. \right\};$$

and multiplying (i.) and (ii.), and dividing by (iii.),

$$\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{1}{k} \cdot 2q^{\frac{1}{2}} \sin 2x \Pi \frac{1-2q^{4n} \cos 4x + q^{8n}}{1-2q^{4n-2} \cos 4x + q^{8n-4}} \dots \dots (\text{iv.}),$$

so that $2q^{\frac{1}{2}} \sin 2x \Pi \frac{1-2q^{4n} \cos 4x + q^{8n}}{1-2q^{4n-2} \cos 4x + q^{8n-4}}$

$$= \frac{\pi}{2kK} \left\{ \frac{8q}{1-q^2} \sin 2x + \frac{8q^3}{1-q^6} \sin 6x + \&c. \right\}.$$

In this equation we observe that, if $\frac{1}{2}x$ be substituted for x , and $q^{\frac{1}{2}}$ for q , the left-hand member becomes equal to $k^{\frac{1}{2}} \operatorname{sn} u$, from (i.), so that

$$k^{\frac{1}{2}} \operatorname{sn} u = M \left\{ \frac{8q^{\frac{1}{2}}}{1-q} \sin x + \frac{8q^{\frac{3}{2}}}{1-q^3} \sin 3x + \&c. \right\},$$

where M is what $\frac{\pi}{2kK}$ becomes when q is replaced by $q^{\frac{1}{2}}$.

Now, by putting $x = 0$ in (iv.) and (i.),

$$\frac{2kK}{\pi} = 4q^{\frac{1}{2}} \Pi \left(\frac{1-q^{4n}}{1-q^{4n-2}} \right)^{\frac{1}{2}},$$

$$\frac{2k^{\frac{1}{2}}K}{\pi} = 2q^{\frac{1}{2}} \Pi \left(\frac{1-q^{2n}}{1-q^{2n-1}} \right)^{\frac{1}{2}};$$

the change of q into $q^{\frac{1}{2}}$, therefore, converts $\frac{2kK}{\pi}$ into $\frac{4k^{\frac{1}{2}}K}{\pi}$, so that

$$M = \frac{\pi}{4k^{\frac{1}{2}}K};$$

and therefore $\operatorname{sn} u = \frac{\pi}{2kK} \left\{ \frac{4q^{\frac{1}{2}}}{1-q} \sin x + \frac{4q^{\frac{3}{2}}}{1-q^3} \sin 3x + \&c. \right\}.$

2. By this method we arrive almost immediately at the q -series for $\operatorname{sn} u$; and it may be observed that this is the only one of the q -series of which Jacobi makes any important use in the subsequent portion of the *Fundamenta Nova*. This use is in connexion with the function $Z(u)$, which is obtained by squaring the series for $\operatorname{sn} u$ and integrating.

In the *Fundamenta Nova* the q -products arise as limiting cases of formulæ in the transformation theory; these formulæ afford products, not only for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, but also for

$$\left(\frac{1-\operatorname{sn} u}{1+\operatorname{sn} u} \right)^{\frac{1}{2}} \text{ and } \left(\frac{1-k \operatorname{sn} u}{1+k \operatorname{sn} u} \right)^{\frac{1}{2}};$$

and it is from the latter of these expressions, which, differentiated logarithmically, gives $k \operatorname{sn}(K-u)$, that Jacobi deduces the series for $\operatorname{sn} u$. But if the q -products are derived otherwise than from the transformation theory (as, *e.g.*, from the Abel double products), so that it is convenient to start with the products for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, the deduction of the series for $\operatorname{sn} u$ requires an imaginary transformation

of either the product or the series; i.e., we may either form the q -product for $\frac{\text{cn}(\frac{1}{2}u + \frac{1}{2}iK') \text{dn}(\frac{1}{2}u + \frac{1}{2}iK')}{\text{sn}(\frac{1}{2}u + \frac{1}{2}iK')}$, which, when differentiated logarithmically, gives $-k \text{sn } u$;* or, we may form the q -product for $\frac{\text{cn} \frac{1}{2}u \text{dn} \frac{1}{2}u}{\text{sn} \frac{1}{2}u}$, which, when differentiated logarithmically, gives $-\frac{1}{\text{sn } u}$; and then transform the series by the substitution of $u + iK'$ for u . This latter method is followed by Durège (*Theorie der elliptischen Functionen*, 1861, and second edition, 1868).

In the process given above, the substitution of $u + iK'$ for u , either in the products or the series, is avoided, and the series for $\text{sn } u$ is obtained very readily from the q -formulæ; but the method is only applicable in the case of the $\text{sn } u$, and does not give the series for $\text{cn } u$, $\text{dn } u$, $\frac{1}{\text{sn } u}$, &c.

January 8th, 1880.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Prof. W. Snow Burnside, M.A., Trinity College, Dublin, was elected a Member.

Prof. Cayley read a paper, "A Theorem in Spherical Trigonometry," and made some remarks upon a posthumous paper of Prof. Clifford's, "On the Theory of Distances."

The Chairman supplemented his communication of December 11th by a few further remarks.

Mr. J. Hammond, M.A., gave an Expression for the Complementary Function in Fractional Differentiation; upon which a discussion arose, in which Prof. Cayley, the Chairman, Mr. Roberts, Mr. Glaisher, and Mr. A. Freeman took part.

The following presents were received:—

"Zur Theorie der Elliptischen Modulfunctionen," von F. Klein (aus den Sitzungsberichten der Akad. der Wissenschaften zu München, vom 6 Dezember, 1879).

"The Melbourne University Calendar for the Academic years 1878—1880," Melbourne: from the Chancellor of the University.

"Factor Table for the Fourth Million, containing the least factor of every number not divisible by 2, 3, or 5, between 3,000,000 and 4,000,000," by J. Glaisher, F.R.S., 1879: from the author.

"Crelle's Journal," 88 Band, 3^{te} Heft, Berlin, 1879; 4^{te} Heft, Berlin, 1880.

* This method is employed in "A Chapter in Elliptic Functions" (*Quarterly Journal*, T. xvii., pp. 50—65, 1880).

"Beiblätter zu den Annalen der Physik und Chemie," Band iii., Stück 12, Leipzig, 1879; Band iv., Stück 1, Leipzig, 1880.

"Educational Times," January, 1880.

"Teorica delle Forze Newtoniane e sue applicazioni all' elettrostatica e al magnetismo," del Prof. Enrico Betti; Pisa, 1879.

"Journal of Institute of Actuaries," No. cxvii., Vol. xxii., Part i., October, 1879.

"Proceedings of the Royal Society," Vol. xxix., No. 199.

"Monatsbericht" (Sept. and Oct.), Nov., 1879.

"Atti della R. Accademia dei Lincei—Transunti," Vol. iv., Fasc. 1^o, Dic. 1879; Roma, 1880.

"Bulletin de la Société Mathématique de France," Tome vii., No. 6; Paris, 1879.

"Archives Néerlandaises des Sciences Exactes et Naturelles," publiées par la Société Hollandaise des Sciences à Harlem, Tome xiv. (3^{me}, 4^{me}, et 5^{me} Livr.); La Haye, 1879.

"The Physical Society of London—Proceedings," Vol. iii., Part 3, Nov. 1879—Feb. 1880.

"Tidsskrift for Mathematik," udgivet af H. G. Zeuthen, Fjerde Række, Tredie Aargang, Første . . . Sjette Hefte; Kjøbenhavn, 1879.

A Theorem in Spherical Trigonometry. By Prof. CAYLEY.

[Read January 8th, 1880.]

In a spherical triangle, where a, b, c are the sides, and A, B, C the opposite angles, we have

$$-\tan \frac{1}{2}c \tan \frac{1}{2}a \tan \frac{1}{2}b \sin (A-B) = \tan \frac{1}{2}b \sin A - \tan \frac{1}{2}a \sin B,$$

$$\tan \frac{1}{2}c \{1 - \tan \frac{1}{2}a \tan \frac{1}{2}b \cos (A-B)\} = \tan \frac{1}{2}b \cos A + \tan \frac{1}{2}a \cos B;$$

which are both included in the form

$$\tan \frac{1}{2}a (\cos B - i \sin B) = \frac{\tan \frac{1}{2}c - \tan \frac{1}{2}b (\cos A + i \sin A)}{1 + \tan \frac{1}{2}c \tan \frac{1}{2}b (\cos A + i \sin A)}.$$

For the first of the two identities: from

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C},$$