Note on Secondary Tucker-Circles. By JOHN GRIFFITHS, M.A.

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The principal theorems discussed in this paper are particular cases of the following proposition, viz., If DEF denote a triangle of given species having its vertices D, E, F respectively on the sides BC, CA, AB of a given triangle ABC, or on these sides produced, then DEF will belong to one or other of a pair of systems of similar inscribed triangles, and each of these systems will have a common centre of similitude. In fact, if D, E, F be the given angles of the inscribed triangle DEF, there will be a primary system of similar inscribed triangles whose common centre of similitude is the point given by the isogonal coordinates

$$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F},$$

and also a secondary system whose centre of similitude is the point represented by

$$x = \frac{\sin(D-A)}{\sin D}, \quad y = \frac{\sin(E-B)}{\sin E}, \quad z = \frac{\sin(F-C)}{\sin F}.$$

These two centres of similitude are the inverse of each other with respect to the circumcircle ABC. For example, if $D = A$, $E = B$, $F = C$, then the centre of similitude of the primary system of inscribed triangles will be the point $(2\cos A, 2\cos B, 2\cos C)$, *i.e.*, the centre of the circumcircle ABC, while that of the secondary system will be an infinitely distant point (0, 0, 0).

Moreover, the circle circumscribing any triangle DEF of the first system will have double contact with the inscribed conic

$$\Sigma \sqrt{x \sin D \sin(D+A)} = 0,$$

and the corresponding conic for the circumcircle of a triangle of the second system will be represented by

$$\Sigma \sqrt{x \sin D \sin(D-A)} = 0.$$

The particular examples considered in the note are the primary and secondary systems of inscribed triangles corresponding to the values (1) $\angle D = B$, $\angle E = C$, $\angle F = A$; (2) $\angle D = C$, $\angle E = A$, $\angle F = B$.

For the primary systems of in-triangles under consideration, the centres of similitude are the Brocard points, and the corresponding circumscribed circles are well-known as Tucker-circles.

The secondary systems, whose centres of similitude are the inverse with respect to the circumcircle ABC of the Brocard points, have not hitherto, so far as I know, been noticed.

In each of these we have a series of triangles directly similar to each other, but—unlike the Tucker-triangles—inversely similar to the triangle of reference ABC.

Section I.

The results arrived at will be more readily understood by an explanation regarding what I have called isogonal coordinates, which can be employed to investigate the properties of systems of circles connected with the triangle (see my "Notes on the Recent Geometry of the Triangle").

Briefly, if α, β, γ be the trilinear coordinates of a point G, in the plane of a given triangle of reference ABC, the isogonal coordinates of G are given by

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{a\alpha + b\beta + c\gamma}{a\beta\gamma + b\gamma\alpha + c\alpha\beta},$$

where a, b, c denote the sides BC, CA, AB.

It is thus easily seen that these coordinates x, y, z satisfy the relation

$$ax + by + cz = ayz + bzx + cxy, \quad \text{or} \quad \Sigma x \sin A = \Sigma yz \sin A,$$

which is unaltered by writing therein x^{-1}, y^{-1}, z^{-1} for x, y, z.

Again, if we take a point G inside the circumcircle ABC, and draw perpendiculars GD, GE, GF from it to the sides BC, CA, AB, it may be proved that the isogonal coordinates of G are

$$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F},$$

where D, E, F are the angles of the pedal triangle DEF. These coordinates satisfy the relation

$$\Sigma x \sin A = \Sigma yz \sin A.$$

For example, let G coincide with O, the centre of the circumcircle ABC, then $D = A$, $E = B$, $F = C$, so that

$$x = 2\cos A, \quad y = 2\cos B, \quad z = 2\cos C$$

are the isogonal coordinates of the centre of the circumcircle ABC.

If G be taken outside the circumcircle ABC, the isogonal coordinates of G, in terms of the angles D, E, F of its pedal triangle DEF, are

$$x = \frac{\sin(D-A)}{\sin D}, \quad y = \frac{\sin(E-B)}{\sin E}, \quad \text{and} \quad z = \frac{\sin(F-C)}{\sin F}.$$

Supposing then we have two points, G and g, whose isogonal coordinates are respectively

$$x = \frac{\sin(D+A)}{\sin D}, \quad x' = \frac{\sin(D-A)}{\sin D}, \text{ \&c.,}$$

it follows that

$$x+x' = 2\cos A, \quad y+y' = 2\cos B, \quad z+z' = 2\cos C.$$

G and g will then, in fact, be a pair of points inverse to each other with respect to the circumcircle ABC, so that

$$OG \,.\, Og = R^2,$$

where O denotes the centre and R the radius of the circle ABC.

Conversely, if G, g be two inverse points with respect to the circumcircle ABC, then the pedal triangles of these points are similar. This result is an important one in the geometry of the triangle.

The following theorem, for example, is deduced at once from it. If the cotangents of the angles of the pedal triangle DEF, with respect to ABC, of a point P, be connected by a linear relation

$$\Sigma\lambda \cot D = \text{const.},$$

then the locus of P will, in general, be a pair of circles inverse to each other with respect to the circumcircle ABC. As a particular case, it follows that, if P be any point either on the Brocard circle

$$\Sigma x \operatorname{cosec} A = 2\cot\omega$$

or on its inverse line $\quad \Sigma x \operatorname{cosec} A = 0,$

the pedal triangle of P with respect to ABC has the same Brocard angle as ABC.

Again, the pedal triangle DEF of a point G can be turned in its own plane round G as a fixed centre of similitude, with its vertices

D, E, F moving on the sides BC, CA, AB of the triangle of reference. The angles D, E, F will, in fact, remain constant, and the circle DEF will have double contact with an inscribed conic which has $G(x, y, z)$ for a focus, the chord of contact being parallel to the transverse axis of the conic. The equation of this curve may be written in various forms, one of which is

$$\Sigma \sqrt{ax(bz+cy-a)\,\alpha} = 0.$$

For example, let G coincide with the negative Brocard point $\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$; here

$$x = \frac{\sin(C+A)}{\sin C}, \quad y = \frac{\sin(A+B)}{\sin A}, \quad z = \frac{\sin(B+C)}{\sin B},$$

so that, for the pedal triangle of the point, we have

$$\angle D = C, \quad \angle E = A, \quad \angle F = B,$$

and, for the equation of the inscribed conic in question,

$$\Sigma \sqrt{b\gamma\alpha} = 0.$$

This curve is known as the Brocard ellipse, and the circle DEF is a Tucker-circle.

SECTION 2. *Secondary Tucker-Circles.*

As a particular case of the above general theorem, viz., that the pedal triangles of a pair of inverse points with respect to the circum-circle ABC are similar, I consider the systems of circles corresponding to the Brocard points and their inverse points as here defined.

1. Taking the negative Brocard point $\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$, we have seen that the angles of its pedal triangle are $D = C$, $E = A$, $F = B$, and, if the triangle be turned round this point in the manner explained in Section 1, the Tucker-circle DEF will have double contact with the Brocard ellipse

$$\Sigma \sqrt{b\gamma\alpha} = 0.$$

Again, the pedal triangle def of the inverse point g will be similar to the corresponding Tucker triangle DEF, or $\angle d = C$, $\angle e = A$, $\angle f = B$, and if the triangle def be turned round g as above, the circle *def* will have double contact with the conic

$$\Sigma \sqrt{a \sin C \sin (C-A)} = 0.$$

The coordinates of g are, in fact,

$$x = 2\cos A - \frac{b}{c} = \frac{\sin (C-A)}{\sin C}, \quad y = 2\cos B - \frac{c}{a} = \frac{\sin (A-B)}{\sin A},$$

and
$$z = 2\cos C - \frac{a}{b} = \frac{\sin (B-C)}{\sin B},$$

so that the equation $\Sigma \sqrt{ax\,(bz+cy-a)\,a} = 0$

becomes $\Sigma \sqrt{a \sin C \sin (C-A)} = 0.$

I propose to call the system of circles corresponding to the point inverse to a Brocard point a secondary Tucker system.

As I have just explained, if the Brocard point be $\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$, the Tucker inscribed triangle DEF and the corresponding triangle *def* are similar; also the Tucker-circle DEF has double contact with the conic

$$\Sigma \sqrt{a \sin B \sin C} = 0,$$

and the secondary circle *def* with the conic

$$\Sigma \sqrt{a \sin C \sin (C-A)} = 0.$$

It may be here observed that one form of the equation of a circle *def* of this secondary system is

$$k_1 (1+k_2) x_1 x + k_2 (1+k_3) y_1 y + k_3 (1+k_1) z_1 z + 1 = 0,$$

where $x_1 = \frac{\sin (C-A)}{\sin C}, \quad y_1 = \frac{\sin (A-B)}{\sin A}, \quad z_1 = \frac{\sin (B-C)}{\sin A},$

and k_1, k_2, k_3 are connected by the relations

$$k_1 + k_2 + k_3 + 2 = 0, \quad a^2 (1+k_2) + b^2 (1+k_3) + c^2 (1+k_1) = 0.$$

2. In a similar manner, if we take G to coincide with the positive Brocard point $\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$, the inverse point g will be given by the coordinates

$$x = 2\cos A - \frac{c}{b}, \quad y = 2\cos B - \frac{a}{c}, \quad z = 2\cos C - \frac{b}{a},$$

or $$x = \frac{\sin(B-A)}{\sin B}, \quad y = \frac{\sin(C-B)}{\sin C}, \quad z = \frac{\sin(A-C)}{\sin A};$$

also in the triangles DEF, def the angles will be

$$D = d = B, \quad E = e = C, \quad \text{and} \quad F = f = A.$$

Lastly, the Tucker-circle DEF will have double contact with the Brocard ellipse

$$\Sigma \sqrt{a \sin B \sin C} = 0,$$

and the secondary circle def with the conic

$$\Sigma \sqrt{a \sin B \sin (B-A)} = 0.$$

As an example of a secondary circle, illustrating the difference between it and an ordinary Tucker-circle, I notice the following, viz.:— Let the tangent at A to the circumcircle ABC meet BC produced in d; through d draw de parallel to AB and meeting AC produced in e; then the circle Aed is a secondary Tucker-circle. Here the point f coincides with A, and the angles of the triangle Aed or fed are $d = C$, $e = A$, $f = B$. Also, if the other points where this circle meets the lines BC, CA, AB be denoted by d', e', f', then e' coincides with A, and the angles of the triangle $d'e'f'$ are $d' = A-C$, $e' = 180° - (A-B)$, $f' = C-B$, supposing $A > C > B$. If the circle were an ordinary Tucker-circle, we could have $d = C$, $e = A$, $f = B$, as before, but for the triangle $d'e'f'$ the angles would be $d' = B$, $e' = C$, $f' = A$.

In the case of the secondary circle in question, the equation of the double-contact inscribed conic is

$$\Sigma \sqrt{a \sin C \sin (C-A)} = 0,$$

whereas for an ordinary Tucker-circle it would be

$$\Sigma \sqrt{a \sin C \sin (C+A)} = 0.$$

It may be remarked that two of the foci of the curves

$$\Sigma \sqrt{a \sin B \sin (B-A)} = 0$$

and $$\Sigma \sqrt{a \sin C \sin (C-A)} = 0,$$

viz., the points $$\left(\frac{\sin(B-A)}{\sin B}, \ \frac{\sin(C-B)}{\sin C}, \ \frac{\sin(A-C)}{\sin A}\right)$$

and $$\left(\frac{\sin(C-A)}{\sin C}, \ \frac{\sin(A-B)}{\sin A}, \ \frac{\sin(B-C)}{\sin B}\right),$$

lie on the circle $\Sigma (b^2c^2 - a^4)\, bcx + 3a^2b^2c^2 - \Sigma a^6 = 0$, which touches the Brocard circle at the centre of the circumcircle (ABC). The first-mentioned circle is the inverse with respect to (ABC) of the line joining the Brocard points, since the expression $\Sigma\,(b^2c^2-a^4)\,bcx$ is transformed into $a^6+b^6+c^6-3a^2b^2c^2-\Sigma\,(b^2c^2-a^4)\,bcx$ by writing therein $2\cos A-x$, $2\cos B-y$, $2\cos C-z$ for x, y, z.

Section 3.

I here notice briefly some additional properties connected with the above pair of secondary systems of similar in-triangles, and the double-contact conics of the circumcircles.

1. If $d = B$, $e = C$, $f = A$, the sides de, ef, fd of the in-triangle def are parallel respectively to lines AP, BP, CP, which meet in a point P on the circumcircle ABC. See Fig. 1.

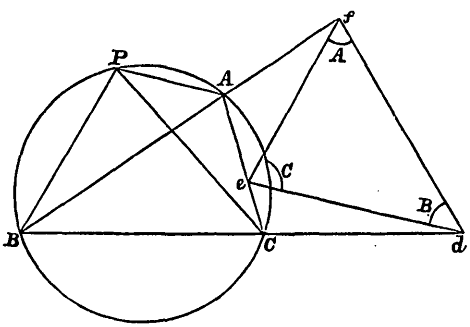

Fig. 1.

2. In like manner, when $d = C$, $e = A$, $f = B$, the sides de, ef, fd are parallel to lines BP, CP, AP, which also meet in P on the circumcircle. See Fig. 2.

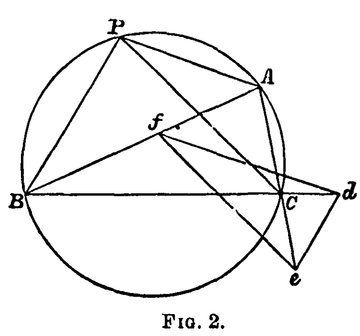

Fig. 2.

3. If a_1, b_1, e_1; a_2, b_2, e_2 denote the semi-axes and eccentricities of the above double-contact conics, and ω be the Brocard angle of the triangle ABC, then

$$a_1^2 = a_2^2 = R^2 \frac{\sin^2\omega}{1-4\sin^2\omega},$$

where $R =$ radius of circle ABC,

$$-b_1^2 = b_2^2 = 4R^2 \frac{\sin^4\omega}{(1-4\sin^2\omega)^2} \frac{\sin(B-C)\sin(C-A)\sin(A-B)}{\sin A \sin B \sin C},$$

$$e_1^2 - 1 = 1 - e_2^2 = \frac{4\sin^2\omega}{1-4\sin^2\omega} \frac{\sin(B-C)\sin(C-A)\sin(A-B)}{\sin A \sin B \sin C},$$

or $$a_1 = a_2, \quad b_1^2 + b_2^2 = 0, \quad e_1^2 + e_2^2 = 2.$$

It thus follows that one of the conics is an ellipse and the other a hyperbola.

4. If a triangle $A'B'C'$ be inscribed in the circumcircle ABC, and circumscribed to the Brocard ellipse, it is known that $A'B'C'$ has the same Brocard angle ω and the same Brocard points as ABC. Hence, by means of the formulæ for the secondary double-contact conics, given above, I have deduced the following theorem, viz. :—

One focus of each of the secondary double-contact conics corresponding to a triangle $A'B'C'$—as defined above—remains fixed if the triangle of reference ABC be fixed, and the other two foci lie respectively on one of two fixed circles which have a common radius $2R\sin\omega$, and whose centres coincide with the fixed Brocard points of ABC.

5. If $A = \frac{\pi}{7}$, $B = \frac{2\pi}{7}$, and $C = \frac{4\pi}{7}$, one of the secondary double-contact conics is a circle, and the other an equilateral hyperbola. In this case one of the secondary Tucker systems consists of a system of concentric circles.

6. The inverse of the Brocard points of a triangle ABC, with respect to the circle ABC, are the Brocard points of a triangle similar and similarly placed to ABC, the centre of similitude being the centre of the circle ABC. If R', R denote the radii of the circumcircles of these two triangles, then

$$R' = R \div \sqrt{1-4\sin^2\omega}.$$

[The following is added in answer to questions suggested by one of the referees, viz.: What is the centre of similitude of a given triangle ABC, and an in-triangle inversely similar to it; and what is the locus of this point as the latter triangle moves?

So far as I have studied the problem, I have arrived at the following results with regard to centre of similitude (S, say) of ABC, and an inversely similar escribed triangle def, where $d=B$, $e=C$, $f=A$. See Fig. 1.

1. The point S is given by the trilinear equations

$$(1+k_2)\,a\alpha = (1+k_3)\,b\beta = (1+k_1)\,c\gamma,$$

where k_1, k_2, k_3 are connected by the relations

$$k_1+k_2+k_3+1=0 \quad \text{and} \quad \Sigma a^2k_1 = 0.$$

2. The locus of S is the circumscribed conic represented by

$$\Sigma\beta\gamma \sin C \cos B = 0, \quad \text{or} \quad \Sigma yz \sin C \cos B = 0.$$

This curve passes through the point

$$x = \frac{\sin C}{\sin(C-A)}, \quad y = \frac{\sin A}{\sin(A-B)}, \quad z = \frac{\sin B}{\sin(B-C)},$$

i.e., through the isogonal conjugate with respect to the triangle ABC of the following point, viz., the inverse with regard to the circumcircle ABC of the negative Brocard point of ABC. The point in question is a focus of one of the double-contact conics discussed in the note.

3. The point P, on the circumcircle ABC, in which the lines AP, BP, CP, parallel to sides de, ef, fd, meet, is given by

$$k_1 a\alpha' = k_2 b\beta' = k_3 c\gamma',$$

where k_1, k_2, k_3 are subject to the same relations as before, viz.,

$$1+\Sigma k_1 = 0 \quad \text{and} \quad \Sigma a^2 k_1 = 0.$$

It thus appears that there is a correspondence between P and the centre of similitude S, which is expressed by the equations

$$\left(\frac{1}{a\alpha'}+\frac{1}{c\gamma'}\right)a\alpha = \left(\frac{1}{b\beta'}+\frac{1}{a\alpha'}\right)b\beta = \left(\frac{1}{c\gamma'}+\frac{1}{b\beta'}\right)c\gamma.$$

4. Again, the referee suggests the following question:—Let ABC be a given triangle, $A'B'C'$ an inversely similar escribed triangle.

Then, as $A'B'C'$ moves, its centre of similitude (U, say) is the inverse of a Brocard point of ABC. What relation has U to $A'B'C'$?

If U be taken to be the inverse, with respect to the circumcircle ABC, of the positive Brocard point of ABC, I have found that the isogonal coordinates of U, with reference to $A'B'C'$, are

$$x' = \frac{\sin C'}{\sin(C'-A')}, \quad y' = \frac{\sin A'}{\sin(A'-B')}, \quad z' = \frac{\sin B'}{\sin(B'-C')}.$$

It follows, then, that U is the isogonal conjugate, with reference to $A'B'C'$, of the following point, viz., the inverse, with respect to the circumcircle $A'B'C'$, of the negative Brocard point of $A'B'C'$.

The position of U with reference to $A'BC'$ may be determined by considering the angles subtended at it by the sides $B'C'$, $C'A'$, $A'B'$. These are C', $\pi-A'$, B'; C', A', $\pi-B'$; $\pi-C'$, A', B', according as A', B' or C' is the least angle.

These results have been deduced by means of the following theorem, viz.: If the angles subtended by the sides BC, CA, AB of a triangle ABC, at a point G situated outside ABC, be $\pi-\theta$, ϕ, ψ; θ, $\pi-\phi$, ψ; or θ, ϕ, $\pi-\psi$, the isogonal coordinates of G will be

$$x = \frac{\sin\theta}{\sin(\theta-A)}, \quad y = \frac{\sin\phi}{\sin(\phi-B)}, \quad z = \frac{\sin\psi}{\sin(\psi-C)}.$$

Similarly, if $\pi-\theta$, $\pi-\phi$, $\pi-\psi$ denote the angles in question for an internal point G, then the isogonal coordinates of G with reference to ABC will be

$$x = \frac{\sin\theta}{\sin(\theta+A)}, \quad y = \frac{\sin\phi}{\sin(\phi+B)}, \text{ and } z = \frac{\sin\psi}{\sin(\psi+C)}.$$

In all cases the angles θ, ϕ, ψ are the angles of the pedal triangle, with respect to ABC, of the following point, viz., the isogonal conjugate of G with reference to ABC.

For example, the angles subtended at the centre of the circumcircle of an acute-angled triangle ABC by the sides BC, CA, AB are $2A$, $2B$, $2C$, so that $\theta = \pi-2A$, $\phi = \pi-2B$, $\psi = \pi-2C$ are the angles of the pedal triangle of the orthocentre of ABC. The centre of the circumcircle and the orthocentre are, of course, isogonal conjugates.]