40. The numbers E_n, I_n, J_n, H_n also serve to express the values of the following definite integrals :—

$$\int_0^\infty \frac{t^{2n}\,dt}{\cosh t} = E_n \left(\frac{\pi}{2}\right)^{2n+1},$$

$$\int_0^\infty t^{2n}\frac{\sinh \frac{1}{2}t}{\sinh \frac{3}{2}t}\,dt = \int_0^\infty \frac{t^{2n}\,dt}{2\cosh t+1} = \frac{I_n}{\sqrt{3}}\left(\frac{2\pi}{3}\right)^{2n+1},$$

$$\int_0^\infty t^{2n}\frac{\cosh \frac{1}{2}t}{\cosh \frac{3}{2}t}\,dt = \int_0^\infty \frac{t^{2n}\,dt}{2\cosh t-1} = \frac{J_n}{\sqrt{3}}\left(\frac{\pi}{3}\right)^{2n+1},$$

$$\int_0^\infty t^{2n}\frac{\sinh 2t}{\sinh 3t}\,dt = \int_0^\infty \frac{2t^{2n}\cosh t\,dt}{2\cosh 2t+1} = \frac{H_n}{\sqrt{3}}\left(\frac{\pi}{3}\right)^{2n+1}.*$$

41. In the papers in the *Quarterly Journal* and the *Messenger* which have been already referred to certain other quantities P_n, Q_n, R_n, S_n, T_n have been considered; the expansions in which they occur as coefficients were given in § 11 of the preceding paper (pp. 202, 203). These quantities are all integers, and their congruence properties are therefore similar to those of the Eulerian numbers.

On the Theory of Simultaneous Partial Differential Equations.

By J. E. Campbell. Read December 8th, 1898. Received, in revised form, May 24th, 1899.

The necessary and sufficient condition that any number of partial differential equations, of any orders whatever, in one dependent and n independent variables may be consistent is that by repeated differentiations of the equations and eliminations it should not be possible to deduce any relation between the independent variables.

Such a consistent system of differential equations is said to be integrable (Goursat, *Equations aux dérivées partielles du second ordre*, Tome II., p. 41). If p is the order of the lowest differential equation which can be deduced by mere algebra from the system, and if by successive differentiations and eliminations no equation algebraically

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* *Messenger*, Vol. xxvi., pp. 174, 175.

independent of the given equations of the system, and of order equal to or less than p, can be deduced, then the system is said to be completely integrable.

When an equation system is integrable, it is not to be expected that the most general solution of the system is a general solution of any one of the equations which make up the system; in fact, in the most ordinary case of an integrable system, the solution involves no arbitrary functions, but only a finite number of arbitrary constants. Thus, if we write down two partial differential equations $f_1 = 0$, $f_2 = 0$ at random, they will not be consistent; if $f_1 = 0$ and $f_2 = 0$ are consistent, it must be owing to a relation between the forms of f_1 and f_2; if we consider the form of one of these equations, say $f_1 = 0$, as known, then the form of the second f_2 considered as a function of the variables and the differential coefficients it contains must satisfy certain differential equations. Now, when f_2 satisfies these equations, $f_1 = 0$ and $f_2 = 0$ will be consistent; but the common solutions of this system will ordinarily involve only a finite number of constants.

It is here that we notice an essential difference between the theory of partial differential equations of the first order and those of the second and higher orders: given any partial differential equation of the first order $f_1 = 0$, then a second equation $f_2 = 0$, also of the first order, always exists such that $f_1 = 0$ and $f_2 = 0$ have common solutions involving $n-1$ arbitrary functions; on the other hand, if f_1 is of the second order, then it is not generally true that any other equation $f_2 = 0$ exists having in common with $f_1 = 0$ solutions involving any arbitrary function. If f_1 is of a special form, then $f_1 = 0$ may be an equation which belongs to a system having solutions involving an infinity of constants. Such systems have been called by Lie "systems of Darboux" (Goursat, *ibid.*, p. 41) or "systems in involution."

If we have any integrable system, by repeated differentiations and eliminations we can add new equations till, after a finite number of operations, we have a completely integrable system. If such a system is not in involution, it has the property that differential coefficients above a certain order can be expressed in terms of coefficients of lower order; the complete theory of such a system is given in Lie-Engel, *Transformations-Gruppen*, I., Kap. 10. The object of the present paper is to develop certain formulæ analogous to the Jacobian series of combinants, by aid of which it may be decided whether or no a system is integrable.

Before the results arrived at can be stated, certain preliminary explanations and definitions must be given. If f is a partial differential expression of order p, and, if we write for $\dfrac{\partial^{a_1+\ldots+a_n} z}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}}$ $z_{a_1 \ldots a_n}$, then the quantic in the set of auxiliary variables $\xi_1 \ldots \xi_n$,

$$\Sigma\, \xi_1^{a_1} \ldots \xi_n^{a_n} \frac{\partial f}{\partial z_{a_1 \ldots a_n}}.$$

(the summation being for all zero and positive integral values of $a_1 \ldots a_n$ such that $a_1 + \ldots + a_n = p$, and $\dfrac{\partial f}{\partial z_{a_1 \ldots a_n}}$ denoting the partial differential coefficient of f with respect to $z_{a_1 \ldots a_n}$), is said to correspond to the differential expression f.*

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* The following geometrical interpretation may be given to this quantic. Cauchy's existence theorem may be thus stated: "If a differential equation of order p contains the derivative $\dfrac{\partial^p z}{\partial x_1^p}$, then a definite number of solutions of the equation can be found which are of the form $z = F(x_1 \ldots x_n)$; F is a holomorphic function of $x_1 \ldots x_n$ which can be so chosen that the locus $z = F$ passes through the $n-1$-dimensional locus $\begin{cases} z = \phi(x_2 \ldots x_n) \\ x_1 = 0 \end{cases}$, where ϕ is arbitrarily assigned, and has contact of $(p-1)^{\text{st}}$ order at all points on $\begin{cases} z = \phi \\ x_1 = 0 \end{cases}$ with any arbitrarily assigned n-dimensional locus which passes through $\begin{cases} z = \phi \\ x_1 = 0 \end{cases}$." Apply now to the differential equation the point-transformation

$$\begin{aligned} z' &= z, \\ x_1' &= \psi(x_1 \ldots x_n), \\ x_2' &= x_2, \\ &\vdots \\ x_n' &= x_n; \end{aligned}$$

then the transformed equation will contain $\dfrac{\partial^p z'}{\partial x_1'^p}$ if, and only if,

$$\Sigma \left(\frac{\partial \psi}{\partial x_1}\right)^{a_1} \left(\frac{\partial \psi}{\partial x_2}\right)^{a_2} \ldots \left(\frac{\partial \psi}{\partial x_n}\right)^{a_n} \frac{\partial f}{\partial z_{a_1 \ldots a_n}} \neq 0;$$

so that Cauchy's theorem may be stated as follows:—"A definite number of solutions of the differential equation $f = 0$ can be found which are of the form $z = F(x_1 \ldots x_n)$; F is a holomorphic function of $x_1 \ldots x_n$ which can be so chosen that the locus $z = F$ passes through $\begin{cases} z = \phi(x_2 \ldots x_n) \\ \psi(x_1 \ldots x_n) = 0 \end{cases}$, and has contact of $(p-1)^{\text{st}}$ order at all points of $\begin{cases} z = \phi \\ \psi = 0 \end{cases}$ with any arbitrarily assigned n-dimensional

If we have any quantic $\Sigma a_{a_1 \dots a_n} \xi_1^{a_1} \dots \xi_n^{a_n}$, then $\Sigma a_{a_1 \dots a_n} \dfrac{d^{a_1+\dots+a_n}}{dx_1^{a_1} \dots dx_n^{a_n}}$ *

is said to be the operation which corresponds to the quantic. It follows that the operation which corresponds to the quantic which corresponds to f is

$$\Sigma \frac{\partial f}{\partial z_{\dots a_n}} \frac{d^{a_1+\dots+a_n}}{dx_1^{a_1} \dots dx_n^{a_n}}.$$

We may speak of this as the operation which corresponds to the differential expression f.

Let $w_1 \dots w_s$ be the s quantics which correspond to the s differential expressions $f_1 \dots f_s$ which are respectively of orders $p_1 \dots p_s$; let

$$\left.\begin{matrix} v_{11}, & v_{21}, & \dots, & v_s \\ v_{12}, & v_{22}, & \dots, & v_{s2} \\ \vdots & & & \\ v_{1r}, & v_{2r}, & \dots, & v_{sr} \end{matrix}\right\} \qquad (1)$$

be r sets of quantics such that for all values of κ from 1 up to r inclusive

$$v_{1\kappa} w_1 + v_{2\kappa} w_2 + \dots + v_{s\kappa} w_s \equiv 0\,; \qquad (2)$$

then, if $\lambda_1 \dots \lambda_r$ are any other arbitrary quantics, the identity

$$\sum_{\substack{\kappa=1 \\ h=1}}^{\substack{\kappa=r \\ h=s}} \lambda_\kappa v_{h\kappa} w_h \equiv 0$$

is merely an algebraic consequence of the identities (2), and is said to be reducible.

An identity of the form (2) which is not a consequence of identities of the same form and of lower degree is said to be simple.

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locus which also passes through $\begin{cases} z = \phi \\ \psi = 0 \end{cases}$; the only limitations placed on the arbitrarily assigned holomorphic functions ϕ and ψ are that the direction cosines $\xi_1 \dots \xi_n$ of the normals to $\psi = 0$ must not satisfy the equation of the quantic which corresponds to f."

* The symbol $\dfrac{d}{dx_r}$ is used to denote total differentiation with respect to x_r, thus,

$$\frac{d}{dx_r} = \frac{\partial}{\partial x_r} + z_r \frac{\partial}{\partial z} + z_{1r} \frac{\partial}{\partial z_1} + z_{2r} \frac{\partial}{\partial z_2} + \dots.$$

It will be proved (§ 1) that, given the quantics $w_1 \dots w_n$, there are only a finite number of simple identities.

Let the r sets of quantics (1) generate simple identities, and let

$$\left.\begin{array}{l} \phi_{11}, \phi_{21}, \dots, \phi_{s1} \\ \phi_{12}, \phi_{22}, \dots, \phi_{s2} \\ \vdots \\ \phi_{1r}, \phi_{2r}, \dots, \phi_{sr} \end{array}\right\} \qquad (3)$$

be the set of operations which correspond to them; then

$$\phi_{1\kappa} f_1 + \phi_{2\kappa} f_2 + \dots + \phi_{s\kappa} f_s \qquad (4)$$

is said to be a combinant of the differential expressions $f_1 \dots f_s$.

From the definition here given of a combinant, and from the fact that there are only a finite number of simple identities, it at once follows that there are only a finite number of such combinants. If a combinant vanishes in consequence of the vanishing of $f_1 \dots f_s$ and the total differential coefficients of these expressions which do not contain differential coefficients of z of order higher than appear in the combinant, then the combinant is said to be satisfied.

The first theorem, then, to be stated is: "If all the combinants are satisfied, the differential equations $f_1 = 0 \dots f_s = 0$ will be integrable (§ 3).

If the combinants are not all satisfied, then we take those which are not satisfied as new equations, additional to $f_1 = 0 \dots f_s = 0$.

It may now happen that we have more equations than are algebraically sufficient to determine all the differential coefficients involved in them; in this case we see that the equation system is inconsistent; if not, we proceed as before with this increased system, and find its combinants. If these are satisfied, then the new equation system, and as a consequence the original one, is consistent; if not, we proceed further till we finally reach a satisfied system of combinants, or obtain more equations than are sufficient to determine the coefficients involved: in the former case the original system is consistent, in the latter it is not. That this question will be decided in every case by a finite number of operations is, I believe, true, but I have not yet succeeded in finding a general proof of its truth.

When all the differential equations are of the first order, combinants, as I have defined them, are easily seen to coincide with the Jacobian series of combinants and are all of the first order; the operations in this case then obviously form a closed series; so that the method of this paper will prove that Jacobi's conditions are necessary and sufficient, and may therefore be considered an extension of those conditions to the case of equations of higher order.

It may, perhaps, be worth while to point out that the question whether or no given equations are consistent is of interest quite apart from any light it may throw on the solution of equations; thus the question might be asked: "Is there any infinitesimal transformation $\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$, which leaves the equation

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} = 0$$

unaltered?" The answer to this would depend on the possibility of certain differential equations having common solutions, and it may be proved that these equations are inconsistent; so that

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} = 0$$

does not admit any infinitesimal transformation.

It will be proved (§ 2) that, if we have s quantics $w_1 \ldots w_s$, where $s \leqq n$, then, unless a special relation exists between their coefficients, the only simple identities are the obvious ones

$$w_h w_\kappa - w_\kappa w_h \equiv 0.$$

It follows then from the definition of a combinant that, if we have s differential equations $f_1 = 0 \ldots f_s = 0$, where $s \leqq n$, then, unless the quantics which correspond to them are of special form, the only combinants are of the form

$$\Sigma \frac{\partial f_h}{\partial z_{a_1 \ldots a_n}} \frac{d^{a_1 + \ldots + a_n} f_\kappa}{dx_1^{a_1} \ldots dx_n^{a_n}} - \Sigma \frac{\partial f_\kappa}{\partial z_{\beta_1 \ldots \beta_n}} \frac{d^{\beta_1 + \ldots + \beta_n} f_h}{dx_1^{\beta_1} \ldots dx_n^{\beta_n}},$$

the summation in first Σ being for all zero and positive integral values of $a_1 \ldots a_n$ such that $a_1 + \ldots + a_n = p_h$, and in the second for such values of $\beta_1 \ldots \beta_n$ as make $\beta_1 + \ldots + \beta_n = p_\kappa$.

As an example of the application of the methods discussed in this paper, it is proved (§ 4) that, if

$$F(x, y, z) = 0$$

is any minimum surface, then

$$F\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

are consistent, and their common solutions involve two arbitrary functions; and no other equation of the first order of the form

$$\phi\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0$$

has a solution satisfying $\nabla^2 u = 0$ and involving two arbitrary functions. It is shown how, given any minimum surface, a solution of $\nabla^2 u = 0$ can be made to depend on the solution of a partial differential equation of the second order in two independent variables. These results were suggested by Prof. Forsyth's paper in the *Messenger*, as was also the second example discussed. The second example (§ 5) proves that not only are the equations

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} = 0$$

and

$$\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_4}\right)^2 = 0$$

consistent (as the common solutions obtained in the *Messenger* show), but that they form with one other equation of the second order a completely integrable system whose common solutions involve four arbitrary functions of one argument.

1. If we take any number of *given* quantics $w_1 \dots w_s$ in any number of variables $x_1 \dots x_n$, and of any degrees, the question arises as to the form of s quantics $v_1 \dots v_s$ such that

$$v_1 w_1 + \dots + v_s w_s \equiv 0.$$

It must first be proved that there are only a finite number of simple identities of the above form—that is to say, there are only a limited number r of sets of quantics

$$\begin{array}{l} v_{11},\ v_{21},\ \dots,\ v_{s1}, \\ v_{12},\ v_{22},\ \dots,\ v_{s2}, \\ \vdots \\ v_{1r},\ v_{2r},\ \dots,\ v_{sr}, \end{array}$$

such that $$v_{1\kappa}w_1+v_{2\kappa}w_2+\ldots+v_{s\kappa}w_s=0 \quad (\kappa=1, 2, \ldots, r),$$
and that every other set $v_1 \ldots v_s$, such that
$$v_1w_1+\ldots+v_sw_s\equiv 0$$
is given by $$v_\rho=\lambda_1v_{\rho 1}+\lambda_2v_{\rho 2}+\ldots+\lambda_rv_{\rho s} \quad (\rho=1, 2, \ldots, s).$$
This theorem is almost an immediate consequence of Hilbert's very general theorem: "If S denotes *any* system of forms in n variables $x_1, x_2 \ldots x_n$, there can be so selected from S a finite number of forms $F_1, F_2 \ldots F_\mu$ that every form F of S can be expressed in the form
$$F\equiv A_1F_1+A_2F_2+\ldots+A_\mu F_\mu,$$
where $A_1, A_2 \ldots A_\mu$ are forms in the variables $x_1, x_2 \ldots x_n$" (Weber's *Algebra*, first edition, Vol. II., pp. 165–168). Now take for the system S that of forms which can be expressed in both the shapes
$$v_1w_1+v_2w_2+\ldots+v_{s-1}w_{s-1} \quad \text{and} \quad -v_sw_s.$$
The theorem which I wish to prove follows, except as regards sets of $v_1, v_2 \ldots v_s$ in which $v_s=0$. As to these, take for S the system of forms which can be expressed in both the shapes
$$v_1w_1+\ldots+v_{s-2}w_{s-2} \quad \text{and} \quad -v_{s-1}w_{s-1}.$$
It follows as to sets in which $v_s=0$, but $v_{s-1}\neq 0$. Continue in like manner. After at most $s-1$ repetitions, the theorem follows in its generality.*

An example on the calculation of simple identities having an interesting application to the differential equation
$$\frac{d^2V}{dx_1^2}+\frac{d^2V}{dx_2^2}+\frac{d^2V}{dx_3^2}+\frac{d^2V}{dx_4^2}=0$$
had perhaps best be given here.

Let $$w_1\equiv\left(\frac{y}{b}-\frac{z}{c}\right)^2\equiv X^2,$$
$$w_2\equiv\left(\frac{z}{c}-\frac{x}{a}\right)^2\equiv Y^2,$$
$$w_3\equiv\left(\frac{x}{a}-\frac{y}{b}\right)^2.$$

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* [I owe this reference and proof to the kindness of Prof. Elliott, who has also given me very much valued help in other parts of the paper. I desire to express to him and to both of the referees my thanks for the great trouble which they have taken in considering this paper.]

If, now, $$v_1 w_1 + v_2 w_2 + v_3 w_3 \equiv 0,$$

then $$v_1 X^2 + v_2 Y^2 + v_3 (X+Y)^2 \equiv 0,$$

or $$(v_1+v_3) X^2 + (v_2+v_3) Y^2 + 2v_3 XY \equiv 0;$$

therefore v_1+v_3 must be divisible by Y, so that

$$v_1+v_3 \equiv 2YP,$$

where P is some function of x, y, and z. Similarly,

$$v_2+v_3 = 2XQ;$$

consequently $$2YPX^2+2XQY^2+2XYv_3 \equiv 0;$$

and therefore $$2v_3+2PX+2QY \equiv 0,$$

that is, $$v_1 \equiv (2Y+X)P+YQ,$$

$$v_2 \equiv (2X+Y)Q+XP,$$

$$v_3 \equiv -XP-YQ.$$

The simple identities are then given by

$$\left.\begin{array}{lll} v_1 = \frac{y}{b} + \frac{z}{c} - \frac{2x}{a}, & v_2 = \frac{y}{b} - \frac{z}{c}, & v_3 = \frac{z}{c} - \frac{y}{b} \\ \text{and } v_1 = -\frac{z}{c} + \frac{x}{a}, & v_2 = \frac{x}{a} + \frac{z}{c} - \frac{2y}{b}, & v_3 = \frac{z}{c} - \frac{x}{a} \end{array}\right\}. \quad (10)'$$

2. It is now to be proved that when $s \leqq n$, unless the quantics $w_1 \dots w_s$ are such that their coefficients are connected by certain relations, there are no simple identities except those of the type

$$w_h w_\kappa - w_\kappa w_h \equiv 0.$$

Any term $x_1^{a_1} \dots x_n^{a_n}$ is said to be derived if $a_1 \geqq p_1$, or $a_2 \geqq p_2 \dots$ or $a_s \geqq p_s$, where s is a given integer $\not> n$ and $p_1 \dots p_s$ any s given integers; a term which is not derived is called arbitrary; Σa is the order of the above term.

It is easily seen that the number of arbitrary terms of order r is the coefficient of x^r in

$$(1-x^{p_1}) \dots (1-x^{p_s})(1-x)^{-n} = a_0 + a_1 x + \dots + a_r x^r + \dots;$$

$a_0, a_1, \dots$ are, of course, positive integers; if $s < n$, the series is infinite; if $s = n$, the series is finite. From the series

$$(1-x)^{-n} = (a_0 + a_1 x + \dots)(1-x^{p_1})^{-1} \dots (1-x^{p_s})^{-1},$$

it follows that, if Π_r denotes the coefficient of x^r in $(1-x)^{-n}$,

$$\Pi_r = a_r + \sum_{\kappa=1}^{\kappa=s} a_{r-p_\kappa} + \sum_{\substack{\kappa=1\\ t=1}}^{\substack{\kappa=s\\ t=s}} a_{r-p_\kappa-p_t} + \dots,$$

the summation on the right being continued so long as the suffixes are non-negative.

It should be noticed that $\Pi_r \nless a_r$, and that, if $s = n$, there are no arbitrary terms of order higher than $(p_1-1)(p_2-1) \dots (p_n-1)$.

If we have s quantics of order r in the n variables $x_1 \dots x_n$, we can form a matrix; thus the first row consists of the coefficients of the first r^{ic} in any assigned order, the second of the corresponding coefficients of the second in the same assigned order, and so on; *i.e.*, the coefficients of the same term in each r^{ic} form a column of the matrix.

From any quantic w of order p, we can form Π_{r-p} derived quantics $x_1^{a_1} \dots x_n^{a_n} w$, by taking all positive integral and zero values of a, such that $\Sigma a = r-p$. Let us therefore form the matrix of the $\sum_{\kappa=1}^{\kappa=s} \Pi_{r-p_\kappa}$ derived r^{ics} of $w_1 \dots w_s$.

Now it must be shown that in general not every $\Pi_r - a_r$-rowed determinant of this matrix will vanish. To prove this it will be sufficient to take

$$w_1 \equiv x_1^{p_1} \dots w_s \equiv x_s^{p_s}.$$

Then we can choose as $\Pi_r - a_r$ derived quantics the $\Pi_r - a_r$ derived terms, that is, the distinct terms which contain x_1^{p} or $x_2^{p} \dots$ or x_s^{p}. In this case, we see that in each row there is one, and only one, term which is not a zero coefficient; and no column can contain two non-zero coefficients, so that the matrix will contain one determinant of order $\Pi_r - a_r$ which does not vanish.

Unless, then, the quantics $w_1 \dots w_s$ are of "special form," not all

$H_r - a_r$-rowed determinants of the matrix will vanish; and we may assume without any real loss of generality that, in particular, the determinant of the derived terms will not vanish.

It will now be proved that all $H_r - a_r + 1$-rowed determinants of the matrix do vanish.

We may assume that, b_r denoting zero or some positive integer, not all $H_r - a_r + b_r$-rowed determinants vanish, but that all $H_r - a_r + b_r + 1$-rowed determinants do vanish.

It follows that we can express all derived terms, and a certain b_r arbitrary terms, in terms of $a_r - b_r$ remaining arbitrary terms and the derived quantics of $w_1 \dots w_s$.

Every r^{ic} can therefore be expressed in the form

$$w_1 v_1 + \dots + w_s v_s + P_r,$$

where $v_1 \dots v_s$ are respectively quantics of degree $r - p_1 \dots r - p_s$, and P_r is an r^{ic} which only contains the above $a_r - b_r$ arbitrary terms.

Treating v_1 by the same method, we see that it can be expressed in the form

$$w_1 v_{11} + \dots + w_s v_{s1} + P_{r-p_1},$$

where $v_{\kappa 1}$ is of degree $r - p_\kappa - p_1$, and P_{r-p_1} only contains $a_{r-p_1} - b_{r-p_1}$ arbitrary terms: proceeding thus, it is clear that every r^{ic} can be expressed in the form

$$P_r + w_1 P_{r-p_1} + \dots + w_s P_{r-p_s} + w_1^2 P_{r-2p_1} + w_1 w_2 P_{r-p_1-p_2} + \dots, \qquad (\text{I.})$$

where the term P_{r-p_κ}, for instance, represents a quantic of degree $r - p_\kappa$, which only contains $a_{r-p_\kappa} - b_{r-p_\kappa}$ arbitrary terms and no derived terms.

The number of arbitrary coefficients in the above form cannot then exceed

$$a_r - b_r + \sum_{\kappa=1}^{\kappa=s} (a_{r-p_\kappa} - b_{r-p_\kappa}) + \sum_{\substack{\kappa=1 \\ t=1}}^{\substack{\kappa=s \\ t=\kappa}} (a_{r-p_\kappa-p_t} - b_{r-p_\kappa-p_t}) + \dots .$$

It is said that the number of arbitrary coefficients cannot exceed the above limit, rather than that it is equal to it, because of possible identities of the form (I.).

Now the number of effective arbitrary constants in any r^{ic} is H_r, so that

$$H_r \leqq a_r - b_r + \sum_{\kappa=1}^{\kappa=s} (a_{r-p_\kappa} - b_{r-p_\kappa}) + \dots ;$$

but $$H_r = a_r + \sum_{\kappa=1}^{\kappa=s} a_{r-p_\kappa} + \dots ;$$

therefore $$b_r + \sum_{\kappa=1}^{\kappa=s} b_{r-p_\kappa} + \dots \leqq 0,$$

an inequality which (since b_κ is a positive integer or zero) can only hold when

$$b_\kappa = 0.$$

The conclusions that we draw are, firstly, that every $H_r - a_r + 1$-rowed determinant of the matrix of s non-special quantics does vanish, and that therefore the derived terms, and no others, can be expressed in terms of the arbitrary terms and the derived quantics; and, secondly, that every r^{ic} can be expressed in one definite way only in the form

$$P_r + \sum_{\kappa=1}^{\kappa=s} w_\kappa P_{r-p_\kappa} + \dots , \tag{II.}$$

where $P_r \dots$ denote quantics, of degree equal to their suffix, and only containing arbitrary terms; and consequently there can be no identity of this form. When an r^{ic} is so expressed, it is said to be in "standard form."

It is now required to investigate the form of s quantics $v_1 \dots v_s$, such that

$$v_1 w_1 + \dots + v_s w_s \equiv 0. \tag{III.}$$

Remembering that $v_1 \dots v_s$ can each be thrown into standard form, and that there can be no identical relation between $w_1 \dots w_s$ and arbitrary terms of the form (II.), we conclude that the coefficients of each arbitrary in the above identity must be zero. The problem is therefore really reduced to finding the forms of s rational integral functions of $w_1 \dots w_s$, such that

$$v_1 w_1 + \dots + v_s w_s \equiv 0.$$

Now any rational integral function of $w_1 \dots w_s$ may be written in the form

$$v_1 \equiv P_1 + w_2 P_{12} + \dots + w_s P_{1s} + \sum_{\substack{\kappa=1 \\ t=1}}^{\substack{\kappa=s \\ t=s}} w_\kappa w_t P_{1\kappa t} + \dots + w_2 w_3 \dots w_s P_{12\dots s},$$

where $P_{1\kappa t}$, for instance, denotes a rational integral function of w_1, w_κ, and w_t only

Expressing $v_2 \dots v_s$ also in similar forms, we deduce from the equation (III.)

$$P_m = P_h = P_t = \dots = 0,$$
$$P_{hm} + P_{mh} = 0, \quad P_{hmt} + P_{mht} + P_{tmh} = 0,$$
$$P_{qmht} + P_{mqht} + P_{tmhq} + P_{hmqt} = 0.$$

It is clear that in the P functions all the suffixes except the first may be interchanged without altering the form of the functions.

In case of equal suffixes the equations deduced differ slightly; thus, if $q = m$, he equations last written would be replaced by

$$P_{mmht} + P_{tmhm} + P_{hmmt} = 0;$$

and, if $q = m = t$, by $\quad P_{mmhm} + P_{hmmm} = 0;$

if, finally, $h = m$, by $\quad P_{mmmm} = 0.$

It is not difficult to see that consequently $v_1 \dots v_s$ may be written in the form

$$\left.\begin{array}{l} v_1 = Q_{12} w_2 + Q_{13} w_3 + \dots + Q_{1s} w_s \\ v_2 = Q_{21} w_1 + Q_{23} w_3 + \dots + Q_{2s} w_s \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ v_s = Q_{s1} w_1 + Q_{2s} w_2 + \dots + Q_{s,s-1} w_{s-1} \end{array}\right\}, \qquad \text{(IV.)}$$

where $$Q_{hm} + Q_{mh} \equiv 0,$$

but except for this restriction the Q's are any functions whatever of $w_1 \dots w_s$.

It follows that the only simple identities are of the form

$$w_h w_\kappa - w_\kappa w_h \equiv 0.$$

In case all the quantics are linear forms in $x_1 \dots x_n$, it is obvious that the system is non-special; in fact, we lose no essential generality in taking $w_1 \equiv x_1 \dots w_s \equiv x_s$, in which it has been shown that not all $H_r - a_r$-rowed determinants of the matrix disappear.

We can now write down the combinants of the differential expressions $f_1 \dots f_s$ for the case here considered, viz., when $n \not< s$, and the quantics $w_1 \dots w_s$ which correspond to $f_1 \dots f_s$ are non-special. Since the only quantics which now generate simple identities are $w_1 \dots w_s$, we see that in (1) $v_{mh} = w_\kappa$, where κ is some integer $\not> s$ and $\neq h$, and $v_{m\kappa} = -w_h$, and all other quantics in the row which contains v_{mh} and $v_{m\kappa}$ are zero.

It follows that the operations ϕ_{mh} and $\phi_{m\kappa}$ in (3) which correspond to these are

$$\Sigma \frac{\partial f_\kappa}{\partial z_{\beta_1 \dots \beta_n}} \frac{d^{\beta_1+\dots+\beta_n}}{dx_1^{\beta_1} \dots dx_n^{\beta_n}},$$

the summation being for all non-negative integral values of $\beta_1 \dots \beta_n$, such that

$$\beta_1+\dots+\beta_n = p_\kappa,$$

and
$$-\Sigma \frac{\partial f_h}{\partial z_{\alpha_1 \dots \alpha_n}} \frac{d^{\alpha_1+\dots+\alpha_n}}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}},$$

the summation being for such values of $\alpha_1 \dots \alpha_n$ that

$$\alpha_1+\dots+\alpha_n = p_h;$$

and therefore we get the typical form of combinant for non-special cases to be

$$\Sigma \frac{\partial f_h}{\partial z_{\alpha_1 \dots \alpha_n}} \frac{d^{\alpha_1+\dots+\alpha_n} f_\kappa}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}} - \Sigma \frac{\partial f_\kappa}{\partial z_{\beta_1 \dots \beta_n}} \frac{d^{\beta_1+\dots+\beta_n} f_h}{dx_1^{\beta_1} \dots dx_n^{\beta_n}}.$$

We can easily verify the fundamental property of this combinant, that all partial derivatives of order $p_\kappa+p_h$ disappear from it, for the derivatives $z_{\alpha_1+\beta_1 \dots \alpha_n+\beta_n}$ appear under each summation with the coefficient

$$\frac{\partial f_h}{\partial z_{\alpha_1 \dots \alpha_n}} \frac{\partial f_\kappa}{\partial z_{\beta_1 \dots \beta_n}},$$

and consequently the terms cancel.

3. From the s differential equations $f_1 = 0 \dots f_s = 0$ respectively of orders $p_1 \dots p_s$, we obtain, to determine the differential coefficients of the r^{th} order, the system of equations

$$\frac{d^{\alpha_1+\dots+\alpha_n} f_\kappa}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}} = 0,$$

where all zero and positive integral values of $\alpha_1 \dots \alpha_n$ are to be taken, such that

$$\Sigma \alpha = r - p_\kappa,$$

and κ is to have any value from 1 up to s inclusive.

After r attains a certain value there will be more equations of this system than there are differential coefficients of the r^{th} order; so that we can eliminate the coefficients of the r^{th} order, and obtain a reduced system of equations not containing any coefficients of order higher than $r-1$. The system of the r^{th} order may then be divided into two parts: the first will not contain more equations than are sufficient to determine the coefficients of the r^{th} order in terms of coefficients of lower order (and it may not contain so many)—it will be convenient to speak of these equations as the effective ones of the r^{th} order; the second part will consist of reduced equations not containing coefficients of the r^{th} order. The system of the r^{th} order will then contain effective and reduced equations; the reduced equations of the r^{th} order may of course be effective in determining coefficients of the $(r-1)^{\text{st}}$ orders. It is now necessary to examine the forms of these reduced equations.

The highest differential coefficients which occur in $\dfrac{d^{a_1+a_2+\dots+a_n} f_\kappa}{dx_1^{a_1}\dots dx_n^{a_n}}$ occur in the part

$$\Sigma z_{a_1+l_1,\, a_2+l_2\dots a_n+l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2\dots l_n}},$$

where the summation is to be taken for all positive integral and zero values of $l_1, l_2 \dots l_n$, such that

$$l_1+l_2+\dots+l_n = p_\kappa,$$

p_κ being the order of the highest derivative in f_κ. They occur linearly.

For
$$\sum^{\kappa} \sum^{a_1\dots a_n} \lambda_{\kappa\, a_1 a_2\dots a_n} \frac{d^{a_1+a_2+\dots+a_n} f_\kappa}{dx_1^{a_1}\dots dx^{a_n}} = 0, \tag{11}$$

where $\lambda_{\kappa a_1\dots a_n}$ is some function of $x_1 x_2 \dots x_n z$, and differential coefficients of order not exceeding p_κ, and where the summations cover all non-negative integral values of $a_1 \dots a_n$, for which $\Sigma a = r - p_\kappa$, and all integral values of κ from 1 to s inclusive, to be free from differential coefficients of order exceeding $r-1$, and so to be an equation of the reduced system, it is then necessary and sufficient that

$$\sum^{\kappa} \sum^{a_1 a_2\dots a_n} \lambda_{\kappa a_1 a_2\dots a_n} \sum^{l_1 l_2\dots l_n} z_{a_1+l_1,\, a_2+l_2\dots a_n+l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2\dots l_n}} \tag{12}$$

vanish identically, where the summations for $l_1 l_2 \dots l_n$, for $a_1 a_2 \dots a_n$, and for κ are as explained above. And this sum will vanish identically if, and only if, the sum

$$\sum^{\kappa} \sum^{a_1 a_2\dots a_n} \lambda_{\kappa a_1 a_2\dots a_n} \sum^{l_1 l_2\dots l_n} \xi_1^{a_1+l_1} \xi_2^{a_2+l_2} \dots \xi^{a_n+l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2\dots l_n}},$$

where $\xi_1, \xi_2 \dots \xi_n$ are any distinct quantities or symbols, vanishes identically; *i.e.*, if

$$\sum^{\kappa} \sum^{a_1 \dots a_n} \lambda_{\kappa a_1 \dots a_n} \xi_1^{a_1} \xi_2^{a_2} \dots \xi_n^{a} \sum^{l_1 l_2 \dots l_n} \xi_1^{l_1} \xi_2^{l_2} \dots \xi_n^{l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2 \dots l_n}} \quad (13)$$

vanishes identically.

Now here, for any κ, $\sum^{a_1 \dots a_n} \lambda_{\kappa a_1 a_2 \dots a_n} \xi_1^{a_1} \xi_2^{a_2} \dots \xi_n^{a_n}$ is what we have earlier defined as the quantic which corresponds to the operation

$$\sum^{a_1 \dots a_n} \lambda_{\kappa a_1 \dots a_n} \frac{d^{a_1+a_2+\dots+a_n}}{dx_1^{a_1} \dots dx_n^{a_n}},$$

and

$$\sum^{l_1 l_2 \dots l_n} \xi_1^{l_1} \xi_2^{l_2} \dots \xi_n^{l_n} \frac{\partial f_\kappa}{\partial z_{l_1 l_2 \dots l_n}}$$

is what we have defined as the quantic which corresponds to f_κ.

We have, then, established that, if $v_1, v_2 \dots v_s$ be the quantics which correspond to the s operations

$$\sum^{a_1 a_2 \dots a_n} \lambda_{\kappa a_1 \dots a_n} \frac{d^{a_1+a_2+\dots+a_n}}{dx_1^{a_1} dx_2^{a_2} \dots dx_n^{a_n}} \quad (\kappa = 1, 2 \dots s),$$

and if $w_1, w_2 \dots w_s$ be the quantics which correspond to $f_1, f_2 \dots f_s$ respectively, what is necessary and sufficient that (11) may be an equation of the reduced system is that

$$v_1 w_1 + v_2 w_2 + \dots + v_s w_s = 0.$$

Here $w_1, w_2 \dots w_s$ are definitely given quantics. We apply then the conclusion of § 1, and are enabled to state that every v_κ must be of the form

$$\lambda_1 v_{\kappa 1} + \lambda_2 v_{\kappa 2} + \dots + \lambda_m v_{\kappa m} \quad (\kappa = 1, 2 \dots s),$$

where

$$\begin{array}{cccc} v_{11}, & v_{21} & \dots & v_{s1}, \\ v_{12}, & v_{22} & \dots & v_{s2}, \\ \dots & \dots & & \dots \\ v_{1m}, & v_{2m} & \dots & v_{sm} \end{array}$$

are the m sets of quantics which occur in the simple identities of $w_1, w_2 \dots w_s$; and where $\lambda_1, \lambda_2 \dots \lambda_m$ are of orders $r-p_1, r-p_2 \dots r-p_m$.

We proceed to apply this conclusion to the supposed reduced equation (11).

If we write it

$$\phi_1 f_1 + \phi_2 f_2 + \dots + \phi_s f_s = 0,$$

then ϕ_κ ($\kappa = 1, 2 \dots s$) is the operation to which corresponds the quantic v_κ, *i.e.*, is the result of replacing $\xi_1, \xi_2 \dots \xi_n$ by $\frac{d}{dx_1}, \frac{d}{dx_2} \dots \frac{d}{dx_n}$ in that quantic. Now let $\phi_{\kappa\nu}$ ($\kappa = 1, 2 \dots s$; $\nu = 1, 2 \dots m$) be the operations to which correspond in like manner the quantics $v_{\kappa\nu}$, and let $\mu_1, \mu_2 \dots \mu_m$ be the operations to which correspond the quantics $\lambda_1 \dots \lambda_m$. What we have learned is that ϕ_κ is the result of omitting from

$$\mu_1 \phi_{\kappa 1} + \mu_2 \phi_{\kappa 2} + \dots + \mu_m \phi_{\kappa m}$$

all operations of differentiation of lower order than the highest which occur, *i.e.*, than order $r - p_\kappa$.*

Consequently (11) differs from

$$\mu_1 \sum_{\kappa=1}^{\kappa=s} \phi_{\kappa 1} f_\kappa + \mu_2 \sum_{\kappa=1}^{\kappa=s} \phi_{\kappa 2} f_\kappa + \dots + \mu_m \sum_{\kappa=1}^{\kappa=s} \phi_{\kappa m} f_\kappa = 0$$

only by terms which have for factors derivatives

$$\frac{d^{a_1 + \dots + a_n}}{dx_1^{a_1} \dots dx_n^{a_n}} \quad (\kappa = 1, 2 \dots s),$$

for which $a_1 + a_2 + \dots + a_n < r - p_\kappa$, *i.e.*, by terms whose vanishing is a result of the vanishing of total derivatives of $f_1, f_2 \dots f_s$ which are not of high enough order to involve partial derivatives of z with regard to $x_1, x_2 \dots x_n$ of order exceeding $r-1$; that is, all reduced equations of the r^{th} order [since (11) was the general form of such equations] can be obtained by differentiation of the combinants, and by the addition of total derivatives of $f_1, f_2 \dots f_s$ which are not of high enough order to involve partial derivatives of z with regard to $x_1, x_2 \dots x_n$ of order exceeding $r-1$.

Now suppose that all the combinants are satisfied; then there will be no reduced equations of the r^{th} order; and, proceeding similarly with the equations of the $(r-1)^{\text{st}}$ and lower orders, we see that there are none except the effective ones. The number of effective equations is never greater than is sufficient to determine the coefficients,

* In the actual v_κ, $\lambda_1 v_{\kappa 1}$, $\lambda_2 v_{\kappa 2}$, &c. are mere algebraical products; whereas the operator $\mu_1 \phi_{\kappa 1}$ is the sum of such an algebraic product and other parts resulting from operations of μ_1 on the coefficients of symbols of differentiation in $\phi_{\kappa 1}$.

so that in this case the system must be integrable; and by subtracting the number of effective equations of any order from the number of differential coefficients of that order we measure the generality of the common solution possible.

4. It is required to find the form of the most general differential equation

$$F\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0,$$

such that

$$F = 0$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

may have common solutions involving two arbitrary functions.

We have wherewith to determine the derivatives of the second order

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

$$\frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 0;$$

F must therefore be of such form that it is not possible to deduce any equation of the second order independent, algebraically, of these four; it follows that the first combinant of F and $\nabla^2 u$ must vanish identically by aid of $F = 0$ and these four equations. It will lighten the labour of determining F if we use the following notation

$$\frac{\partial u}{\partial x} = \lambda_1, \quad \frac{\partial u}{\partial y} = \lambda_2, \quad \frac{\partial u}{\partial z} = \lambda_3, \quad \frac{\partial^2 u}{\partial y \partial z} = \lambda_{23},$$

with similar expressions for $\dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial x \partial z}, \ldots,$

$$\frac{\partial F}{\partial \lambda_\kappa} \equiv F_\kappa, \quad \frac{\partial^2 F}{\partial \lambda_\kappa \partial \lambda_h} \equiv F_{\kappa h}.$$

Since there are only two equations $F = 0$ and $\nabla^2 u = 0$, and the quantics which correspond to these are respectively a line, and $\xi_1^2 + \xi_2^2 + \xi_3^2$, and the latter does not break into factors, we see that the only combinant is

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) F - \left(F_1 \frac{d}{dx} + F_2 \frac{d}{dy} + F_3 \frac{d}{dz}\right) \nabla^2 u.$$

Now $$\frac{dF}{dx_\kappa} = \lambda_{1\kappa} F_1 + \lambda_{2\kappa} F_2 + \lambda_{3\kappa} F_3,$$

so that, remembering that derivatives higher than the second disappear identically from the combinant, we obtain without much labour that the combinant is

$$F_{11}(\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2) + \ldots + 2F_{23}\{\lambda_{12}\lambda_{13} + \lambda_{23}(\lambda_{22} + \lambda_{33})\};$$

then, from the fact that

$$\frac{dF}{dx_1} = \frac{dF}{dx_2} = \frac{dF}{dx_3} = 0,$$

we have
$$\left.\begin{aligned} \lambda_{11} F_1 + \lambda_{12} F_2 + \lambda_{13} F_3 &= 0 \\ \lambda_{12} F_1 + \lambda_{22} F_2 + \lambda_{23} F_3 &= 0 \\ \lambda_{13} F_1 + \lambda_{23} F_2 + \lambda_{33} F_3 &= 0 \end{aligned}\right\}. \tag{14}$$

If now we write $\lambda_{11} = a,\quad \lambda_{22} = b,\quad \lambda_{33} = c,$

$\lambda_{12} = h,\quad \lambda_{23} = f,\quad \lambda_{31} = g,$

and employ the notation usual in the theory of conics, we have (since $a+b+c = 0$) as combinant

$$-(B+C)F_{11} - (C+A)F_{22} - (A+B)F_{33} + 2GHF_{23} + 2FHF_{31} + 2FGF_{12}.$$

From (14) we deduce

$$\frac{F_1}{GH} = \frac{F_2}{FH} = \frac{F_3}{FG},$$

since the discriminant

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$$

and
$$B + C = \frac{FH}{G} + \frac{GH}{F}$$

for the same reason, so that, finally, the combinant takes the simple form

$$(F_2^2 + F_3^2)F_{11} + (F_3^2 + F_1^2)F_{22} + (F_1^2 + F_2^2)F_{33}$$
$$= 2F_2F_3F_{23} + 2F_3F_1F_{13} + 2F_1F_2F_{12},$$

that is, when $F = 0$ is looked on as a surface in space whose coordinates are $\lambda_1, \lambda_2, \lambda_3$, it has the sum of its principal curvatures everywhere zero, that is, is a minimum surface.

We may verify this result, and at the same time see how to obtain particular classes of solutions of the equation

$$\nabla^2 u = 0$$

in the following method.

Let $$z = f(x, y)$$

be any solution whatever of the equation

$$(1+p^2)\,t+(1+q^2)\,r-2pqs = 0,$$

that is, any minimum surface. It is well known that

$$u = ax+by+f(a, b)\,z+\phi(a, b),$$

where we consider x, y, and z as independent variables, and $\phi(a, b)$ is any arbitrary function of a and b, and a and b are given by

$$\frac{\partial u}{\partial a} = 0, \quad \frac{\partial u}{\partial b} = 0,$$

is the general integral of

$$\frac{du}{dz} = f\left(\frac{du}{dx}, \frac{du}{dy}\right).$$

We wish to find the form of ϕ in order that this may be an integral of

$$\nabla^2 u = 0.$$

Let us write $$\frac{\partial^2 f}{\partial a^2} = A', \quad \frac{\partial^2 f}{\partial a \partial b} = H', \quad \frac{\partial^2 f}{\partial b^2} = B',$$

$$\frac{\partial^2 \phi}{\partial a^2} = A, \quad \frac{\partial^2 \phi}{\partial a \partial b} = H, \quad \frac{\partial^2 \phi}{\partial b^2} = B,$$

$$\frac{\partial f}{\partial a} = P', \quad \frac{\partial f}{\partial b} = Q', \quad \frac{\partial \phi}{\partial a} = P, \quad \frac{\partial \phi}{\partial b} = Q,$$

$\frac{\partial u}{\partial a} = 0$ is then $$x+P'z+P = 0,$$

$\frac{\partial u}{\partial b} = 0$ is $$y+Q'z+Q = 0.$$

Differentiating these two equations with respect to x, y, z, we get

$$1+(zA'+A)\frac{\partial a}{\partial x}+(zH'+H)\frac{\partial b}{\partial x}=0,$$

$$(zA'+A)\frac{\partial a}{\partial y}+(zH'+H)\frac{\partial b}{\partial y}=0,$$

$$P'+(zA'+A)\frac{\partial a}{\partial z}+(zH'+H)\frac{\partial b}{\partial z}=0,$$

$$1+(zB'+B)\frac{\partial b}{\partial y}+(zH'+H)\frac{\partial a}{\partial y}=0,$$

$$(zB'+B)\frac{\partial b}{\partial x}+(zH'+H)\frac{\partial a}{\partial x}=0,$$

$$Q'+(zB'+B)\frac{\partial b}{\partial z}+(zH'+H)\frac{\partial a}{\partial z}=0.$$

Now $$\frac{\partial u}{\partial x}=a,\quad \frac{\partial u}{\partial y}=b,\quad \frac{\partial u}{\partial z}=f(a,\,b),$$

so that $\nabla^2 u=0$, if, and only if,

$$\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}+P'\frac{\partial a}{\partial z}+Q'\frac{\partial b}{\partial z}=0.$$

Solving the first six equations, we obtain

$$\frac{\partial a}{\partial x}=-\frac{zB'+B}{D},$$

$$\frac{\partial b}{\partial y}=-\frac{zA'+A}{D},$$

$$\frac{\partial a}{\partial z}=-\frac{P'(zB'+B)-Q'(zH'+H)}{D},$$

$$\frac{\partial b}{\partial z}=-\frac{Q'(zA'+A)-P'(zH'+H)}{D},$$

where $D=z^2(A'B'-H'^2)+z(AB'+BA'-2HH')+AB-H^2$,

and we at once deduce

$$z\{A'(1+Q'^2)+B'(1+P'^2)-2H'P'Q'\}$$
$$+A(1+Q'^2)+B(1+P'^2)-2HP'Q'=0;$$

but the coefficient of z vanishes, from the definition of f, and we see that ϕ must satisfy the equation

$$\frac{\partial^2\phi}{\partial a^2}\left\{1+\left(\frac{\partial f}{\partial b}\right)^2\right\}+\frac{\partial^2\phi}{\partial b^2}\left\{1+\left(\frac{\partial f}{\partial a}\right)^2\right\}-2\frac{\partial^2\phi}{\partial a\,\partial b}\,\frac{\partial f}{\partial a}\,\frac{\partial f}{\partial b}=0.$$

Knowing now the form of f, and choosing ϕ so as to satisfy the above equation, we see that

$$u = ax+by+f(a,\,b)\,z+\phi\,(a,\,b)$$

will be a solution of $\nabla^2 u = 0$, provided that we choose a and b so as to satisfy

$$\frac{\partial u}{\partial a}=0,\quad \frac{\partial u}{\partial b}=0.$$

5. In the *Messenger of Mathematics* (November, 1897, p. 100), Prof. Forsyth proves, amongst other theorems, that, if p_1, p_2, p_3, p_4 denote four arbitrary functions of u subject to the single condition

$$p_1^2+p_2^2+p_3^2+p_4^2=0,$$

and if u be determined as a function of x_1, x_2, x_3, x_4 by the equation

$$au = x_1p_1+x_2p_2+x_3p_3+x_4p_4,$$

where a is a constant, then, if v denote any arbitrary function of u, it satisfies the equation

$$\frac{\partial^2 v}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_2^2}+\frac{\partial^2 v}{\partial x_3^2}+\frac{\partial^2 v}{\partial x_4^2}=0,$$

and also the equation

$$\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial v}{\partial x_2}\right)^2+\left(\frac{\partial v}{\partial x_3}\right)^2+\left(\frac{\partial v}{\partial x_4}\right)^2=0.$$

Now, it is very easily verified that u not only satisfies the above two equations, but also the equation

$$\frac{\partial^2 u}{\partial x_1^2}\left(\frac{\partial u}{\partial x_2}\right)^2+\frac{\partial^2 u}{\partial x_2^2}\left(\frac{\partial u}{\partial x_1}\right)^2-2\frac{\partial^2 u}{\partial x_1\partial x_2}\left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right)=0$$

and five others of the same type. The question is thus suggested whether these six are mere consequences of the first two; it will be found that, though consistent with them, as of course they must be, they are not necessary consequences. The system

$$\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2+\left(\frac{\partial u}{\partial x_3}\right)^2+\left(\frac{\partial u}{\partial x_4}\right)^2=0,$$

$\nabla^2 u = 0$, and the combinant of these two will, however, be proved to form a complete system whose common solutions involve four arbitrary functions of one argument.

Let us write

$$\frac{\partial u}{\partial x_1} = \lambda_1 \dots \frac{\partial^2 u}{\partial x_1^2} = \lambda_1^2 a_{11}, \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = \lambda_1 \lambda_2 a_{12};$$

the equations which we have to consider are

$$f_1 \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 0,$$

$$f_2 \equiv a_{11}\lambda_1^2 + a_{22}\lambda_2^2 + a_{33}\lambda_3^2 + a_{44}\lambda_4^2 = 0.$$

Forming the combinant (here there is obviously only one)

$$(f_1, f_2) \equiv \left(\frac{d^2}{dx_1} + \dots + \frac{d^2}{dx_4^2}\right)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)$$

$$-2\left(\lambda_1 \frac{d}{dx_1} + \dots + \lambda_4 \frac{d}{dx_4}\right)(a_{11}\lambda_1^2 + \dots + a_{44}\lambda_4^2),$$

we get

$$\lambda_1^2(\lambda_1^2 a_{11}^2 + \lambda_2^2 a_{12}^2 + \lambda_3^2 a_{13}^2 + \lambda_4^2 a_{14}^2)$$

$$+\lambda_2^2(\lambda_1^2 a_{21}^2 + \lambda_2^2 a_{22}^2 + \lambda_3^2 a_{23}^2 + \lambda_4^2 a_{24}^2)$$

$$+\lambda_3^2(\lambda_1^2 a_{31}^2 + \lambda_2^2 a_{32}^2 + \lambda_3^2 a_{33}^2 + \lambda_4^2 a_{34}^2)$$

$$+\lambda_4^2(\lambda_1^2 a_{41}^2 + \lambda_2^2 a_{42}^2 + \lambda_3^2 a_{43}^2 + \lambda_4^2 a_{44}) = 0.$$

Differentiating $f_1 = 0$ with respect to $x_1 \dots x_4$, we get

$$\left.\begin{aligned} a_{11}\lambda_1^2 + a_{12}\lambda_2^2 + a_{13}\lambda_3^2 + a_{14}\lambda_4^2 &= 0 \\ a_{21}\lambda_1^2 + a_{22}\lambda_2^2 + a_{23}\lambda_3^2 + a_{24}\lambda_4^2 &= 0 \\ a_{31}\lambda_1^2 + a_{32}\lambda_2^2 + a_{33}\lambda_3^2 + a_{34}\lambda_4^2 &= 0 \\ a_{41}\lambda_1^2 + a_{42}\lambda_2^2 + a_{43}\lambda_3^2 + a_{44}\lambda_4^2 &= 0 \end{aligned}\right\}. \qquad (15)$$

If now we write

$$a_{11} + a_{22} - 2a_{12} = b_{12},$$

$$a_{11} + a_{33} - 2a_{13} = b_{13},$$

$$a_{22} + a_{33} - 2a_{23} = b_{23},$$

$$a_{11} + a_{44} - 2a_{14} = b_{14},$$

$$a_{22} + a_{44} - 2a_{24} = b_{24},$$

$$a_{33} + a_{44} - 2a_{34} = b_{34}.$$

(Notice Prof. Forsyth's solutions require all the b's to vanish.)

The above four equations take the simpler form (by aid of $f_1=0$ and $f_2=0$)

$$\left.\begin{aligned} b_{12}\lambda_2^2+b_{13}\lambda_3^2+b_{14}\lambda_4^2 &= 0\\ b_{21}\lambda_1^2+b_{23}\lambda_3^2+b_{24}\lambda_4^2 &= 0\\ b_{31}\lambda_1^2+b_{32}\lambda_2^2+b_{34}\lambda_4^2 &= 0\\ b_{14}\lambda_1^2+b_{24}\lambda_2^2+b_{34}\lambda_3^2 &= 0 \end{aligned}\right\}, \tag{16}$$

which equations may also be written in the form

$$\left.\begin{array}{l} \lambda_2^2\lambda_3^2b_{23}+\lambda_3^2\lambda_1^2b_{13}+\lambda_1^2\lambda_2^2b_{12}=0\\ \lambda_2^2\lambda_3^2b_{23}=\lambda_1^2\lambda_4^2b_{14},\quad \lambda_1^2\lambda_3^2b_{13}=\lambda_2^2\lambda_4^2b_{24},\quad \lambda_1^2\lambda_2^2b_{12}=\lambda_3^2\lambda_4^2b_{34} \end{array}\right\}. \tag{17}$$

Expressing all such terms as $2a_{12}$ in the equivalent form $a_{11}+a_{22}-b_{12}$, we see [by aid of $f_1=0$, $f_2=0$ and (16)] that the combinant which is

$$\begin{aligned} 2\Sigma\lambda_1^4a_{11}^2+\Sigma\lambda_1^2\lambda_2^2(a_{11}+a_{22}-b_{12})^2 &= \Sigma\lambda_1^4a_{11}^2+\Sigma a_{11}^2\lambda_1^2(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)+2\Sigma\lambda_1^2\lambda_2^2a_{11}a_{22}\\ &\quad -2\Sigma a_{11}\lambda_1^2(\lambda_2^2b_{12}+\lambda_3^2b_{13}+\lambda_4^2b_{14})+\Sigma\lambda_1^2\lambda_2^2b_{12}^2 \end{aligned}$$

may be written $$\Sigma\lambda_1^2\lambda_2^2b_{12}^2. \tag{18}$$

The combinant can also be thrown into the form

$$\begin{aligned} (\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_4^2)(b_{12}+b_{34}-b_{13}-b_{24})^2+(\lambda_3^2\lambda_1^2+\lambda_2^2\lambda_4^2)(b_{12}+b_{34}-b_{23}-b_{14})^2\\ +(\lambda_1^2\lambda_2^2+\lambda_3^2\lambda_4^2)(b_{23}+b_{14}-b_{13}-b_{24})^2. \quad (18)' \end{aligned}$$

To prove the identity of these two expressions (18) and (18)′ write

$$x=b_{23},\quad y=b_{31},\quad z=b_{12},$$

and let us write for sake of brevity

$$a=\lambda_2^2\lambda_3+\lambda_1^2\lambda_4^2,\quad b=\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_4^2,\quad c=\lambda_1^2\lambda_2^2+\lambda_3^2\lambda_4^2;$$

then the expression (18)′ is equal to

$$\Sigma a\left(\frac{cz}{\lambda_3^2\lambda_4^2}-\frac{by}{\lambda_2^2\lambda_4^2}\right)^2,$$

by aid of (17); but $$\left(\frac{x}{\lambda_1^2}+\frac{y}{\lambda_2^2}+\frac{z}{\lambda_3^2}\right)^2=0,$$

by the first of equations (17); therefore (18)′ may be written in the form

$$\frac{(bc+ca+ab)}{\lambda_4^4\lambda_1^4\lambda_2^4\lambda_3^4}\,(ax^2\lambda_2^4\lambda_3^4+by^2\lambda_1^4\lambda_3^4+cz^2\lambda_1^4\lambda_2^4), \tag{18''}$$

and this, by (17),

$$\equiv \frac{(bc+ca+ab)}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2}\,(\lambda_2^2\lambda_3^2b_{23}^2+\lambda_1^2\lambda_4^2b_{14}^2+\lambda_3^2\lambda_1^2b_{13}^2+\lambda_2^2\lambda_4^2b_{24}^2+\lambda_1^2\lambda_2^2b_{12}^2+\lambda_3^2\lambda_4^2b_{34}^2),$$

that is, the equations obtained by equating (18) and (18)′ to zero are equivalent.

We must now prove that the system $f_1=0$, $f_2=0$, and $(f_1f_2)=0$ is complete; and first we shall prove that the combinant $[f_1(f_1f_2)]$ is satisfied.

Notice that

$$\frac{1}{\lambda_\kappa}\,\frac{d}{dx_\kappa}\,a_{12}=a_{12\kappa}-a_{12}\,(a_{1\kappa}+a_{2\kappa}), \tag{19}$$

where $$a_{12\kappa}\equiv\frac{\partial^3 u}{\partial x_1\partial x_2\partial x_\kappa}\div\lambda_1\lambda_2\lambda_\kappa;$$

therefore $$\left(\lambda_1\frac{d}{dx_1}+\lambda_2\frac{d}{dx_2}+\lambda_3\frac{d}{dx_3}+\lambda_4\frac{d}{dx_4}\right)a_{h\kappa}$$

$$=\lambda_1^2a_{h\kappa1}+\lambda_2^2a_{h\kappa2}+\lambda_3^2a_{h\kappa3}+\lambda_4^2a_{h\kappa4},$$

the other terms disappearing by (15).

Forming the combinant of $a_{h\kappa}$ and f_1, we get

$$\frac{1}{\lambda_h\lambda_\kappa}\,\frac{d^2}{dx_hdx_\kappa}(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)-2\left(\lambda_1\frac{d}{dx_1}+\ldots+\lambda_4\frac{d}{dx_4}\right)a_{h\kappa}$$

$$=2\,(\lambda_1^2a_{1\kappa}a_{1h}+\lambda_2^2a_{2\kappa}a_{2h}+\lambda_3^2a_{3\kappa}a_{3h}+\lambda_4^2a_{4\kappa}a_{4h}).$$

If, then, we form the combinant of $a_{12}+a_{34}-a_{13}-a_{24}$ with f_1, we get

$$\sum_{\kappa=1}^{\kappa=4}\lambda_\kappa\,(a_{\kappa1}a_{\kappa2}+a_{\kappa3}a_{\kappa4}-a_{\kappa1}a_{\kappa3}-a_{\kappa2}a_{\kappa4}).$$

Expressing every term $2a_{12}$ in this in its equivalent form $a_{11}+a_{22}-b_{12}$ as before, and using (17) to reduce this expression into terms in b_{12}, b_{13}, b_{23} only, we see that it vanishes identically; but

$$a_{12}+a_{34}-a_{13}-a_{24}\equiv b_{12}+b_{34}-b_{13}-b_{24};$$

therefore the combinant of the latter with f_1 is satisfied.

It follows that the combinant of $(b_{12}+b_{34}-b_{13}-b_{24})^2$ with f_1 is also satisfied.

Notice now that $\sum_{\kappa=1}^{\kappa=4} \lambda_\kappa \frac{d}{dx_\kappa}$ annihilates any function of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ only, since, in operating on such a function, the coefficient of $\frac{\partial}{\partial \lambda_\kappa}$ is

$$\lambda_1^2 a_{1\kappa}+\lambda_2^2 a_{2\kappa}+\lambda_3^2 a_{3\kappa}+\lambda_4^2 a_{4\kappa},$$

which is zero, by (15). Using these results, we see that the combinant of f_1 with $(f_1 f_2)$ is satisfied.

We must now prove that the combinant $[f_2(f_1 f_2)]$ is satisfied; when this is done, we can say that $f_1 = 0$, $f_2 = 0$ and $(f_1 f_2) = 0$ form a complete system.

Instead of directly proving this, it will be sufficient to prove that the combinant of $(f_1 f_2)$ with ϕ_2 is satisfied, where $\phi_2 = 0$ is any expression of the form

$$\mu_2 f_2 + \sum_{\kappa=1}^{\kappa=4} \mu_{1\kappa} \frac{\partial f_1}{\partial x_\kappa} + \mu f_1 = 0,$$

$\mu_{11} \ldots \mu_{14}$ being any functions which do not contain derivatives of the second or higher orders, μ a function not containing derivatives of the first or higher orders, and μ_2 a function which does not vanish identically nor contain derivatives higher than the first order.

If, then, we prove that

$$(f_1 f_2) = 0 \quad \text{and} \quad \lambda_2^2\lambda_3^2 b_{23}+\lambda_3^2\lambda_1^2 b_{13}+\lambda_1^2\lambda_2^2 b_{12} = 0,$$

or any two equations algebraically equivalent with these, are complete in themselves, that is, if all their combinants are satisfied, we may conclude that $f_1 = 0, f_2 = 0$, and $(f_1 f_2) = 0$ form a complete system.

Now, from $(f_1 f_2) = 0$, in its form (18)″, and

$$\lambda_2^2\lambda_3^2 b_{23}+\lambda_3\lambda_1^2 b_{13}+\lambda_1^2\lambda_2^2 b_{12} = 0,$$

we deduce by algebraical solution

$$\frac{b_{23}}{\lambda_1^2(\lambda_1\lambda_2 \pm \lambda_3\lambda_4)^2} = \frac{b_{31}}{\lambda_2^2(\lambda_1^2+\lambda_3^2)^2} = \frac{b_{12}}{\lambda_3^2(\lambda_2\lambda_3 \mp \lambda_1\lambda_4)^2}.$$

Writing $\qquad x = b_{23}, \quad y = b_{13}, \quad z = b_{12},$

we have now to test two equations of the forms

$$x = p^2 z,$$

$$y = q^2 z,$$

p and q being homogeneous functions of λ_1, λ_2, λ_3, λ_4 of zero degree, and connected by the relation

$$p+q+1 \equiv 0.$$

We shall now prove that any such pair of equations is complete.

The quantics which correspond respectively to x, y, and z are w_1, w_2, w_3, where

$$w_1 \equiv \left(\frac{\xi_2}{\lambda_2}-\frac{\xi_3}{\lambda_3}\right)^2, \quad w_2 \equiv \left(\frac{\xi_3}{\lambda_3}-\frac{\xi_1}{\lambda_1}\right)^2, \quad w_3 = \left(\frac{\xi_1}{\lambda_1}-\frac{\xi_2}{\lambda_2}\right)^2,$$

and we proved (p. 243) that there were two simple identities in this case (10)

$$v_1 \equiv \frac{\xi_3}{\lambda_3}+\frac{\xi_2}{\lambda_2}-\frac{2\xi_1}{\lambda_1}, \quad v_2 \equiv \frac{\xi_2}{\lambda_2}-\frac{\xi_3}{\lambda_3}, \qquad v_3 \equiv \frac{\xi_3}{\lambda_3}-\frac{\xi_2}{\lambda_2},$$

and $v_1 \equiv \dfrac{\xi_1}{\lambda_1}-\dfrac{\xi_3}{\lambda_3}$, $\qquad v_2 \equiv \dfrac{\xi_1}{\lambda_1}+\dfrac{\xi_3}{\lambda_3}-\dfrac{2\xi_2}{\lambda_2} \qquad v_3 \equiv \dfrac{\xi_3}{\lambda_3}-\dfrac{\xi_1}{\lambda_1}.$

There are therefore two combinants,

$$\left(\frac{1}{\lambda_2}\frac{d}{dx_2}-\frac{1}{\lambda_3}\frac{d}{dx_3}\right)(b_{13}+b_{23}-b_{12})+2\left(\frac{1}{\lambda_3}\frac{d}{dx_3}-\frac{1}{\lambda_1}\frac{d}{dx_1}\right)b_{23} \qquad (20)$$

and

$$\left(\frac{1}{\lambda_3}\frac{d}{dx_3}-\frac{1}{\lambda_1}\frac{d}{dx_1}\right)(b_{13}+b_{23}-b_{12})+2\left(\frac{1}{\lambda_2}\frac{d}{dx_2}-\frac{1}{\lambda_3}\frac{d}{dx_3}\right)b_{13}. \qquad (21)$$

Using (19), we see that

$$\left(\frac{1}{\lambda_2}\frac{d}{dx_2}-\frac{1}{\lambda_3}\frac{d}{dx_3}\right)a_{h\kappa} = a_{h\kappa 2}-a_{h\kappa 3}-a_{h\kappa}(a_{h2}+a_{\kappa 2}-a_{h3}-a_{\kappa 3}).$$

Expressing all terms b which are to be operated upon in terms of $a_{h\kappa}\ldots$, we verify that the combinants (as we expected) do not contain derivatives above the second order, and are, in fact, the first

$$(a_{13}-a_{12})(b_{13}+b_{32}-b_{12})-2(a_{23}-a_{12})\,b_{23}, \qquad (22)$$

and the second

$$-(a_{23}-a_{12})(b_{13}+b_{23}-b_{12})+2(a_{13}-a_{12})\,b_{13}. \qquad (23)$$

If then we write $\quad x'$ for $\dfrac{1}{\lambda_2}\dfrac{d}{dx_2}-\dfrac{1}{\lambda_3}\dfrac{d}{dx_3}$,

$\qquad y'$ for $\dfrac{1}{\lambda_3}\dfrac{d}{dx_3}-\dfrac{1}{\lambda_1}\dfrac{d}{dx_1}$, we may express these results by the formulæ

$$x'(y+x-z)+2y'x \equiv (a_{13}-a_{12})(y+x-z)-2(a_{23}-a_{12})x, \quad (24)$$

$$y'(x+y-z)+2x'y \equiv -(a_{23}-a_{12})(y+x-z)+2(a_{13}-a_{12})y. \quad (25)$$

Now $\{(1-q)y'-qx'\}(x-p^2z)+\{(1-p)x'-py'\}(y-q^2z)$

is easily seen to be the only combinant of $x-p^2z$ and $y-q^2z$, and, expanding it, we get

$$(1-q)y'x-(1-q)p^2y'z-qx'x+qp^2x'z+(1-p)x'y$$
$$-(1-p)q^2x'z-py'y+pq^2y'z+z\{-(1-q)y'+qx'\}p^2$$
$$+z\{py'-(1-p)x'\}q^2.$$

The last two terms taken together are

$$z\{q(x'+y')p^2+p(x'+y')q^2-y'p^2-x'q^2\}$$
$$=z\{2qp(x'+y')(p+q)-2py'p-2qx'q\},$$

which becomes (since $p+q=-1$, a constant and therefore annihilated by $x'+y'$)

$$-2z(py'p+qx'q);$$

the other terms reduce (using the fact $p+q=1$) to

$$-p\{y'(x+y-z)+2x'y\}-q\{x'(x+y-z)+2y'x\}.$$

Using (24) and (25) and remembering that

$$x=p^2z, \quad y=q^2z,$$

we see that these terms all disappear.

We have now only to prove that $py'p+qx'q$ or $(py'-qx')p$ vanishes. First, we shall prove that

$$(py'-qx')\frac{\lambda_1}{\lambda_3}=0;$$

this is to prove that

$$\left(\frac{1}{\lambda_3}\frac{d}{dx_3}+\frac{p}{\lambda_1}\frac{d}{dx_1}+\frac{q}{\lambda_2}\frac{d}{dx_2}\right)\frac{\lambda_1}{\lambda_3}=0;$$

that is to prove that

$$-a_{13}+a_{33}-p(a_{11}-a_{13})-q(a_{12}-a_{23})=0.$$

The expression on the left may be written

$$-p(a_{11}-2a_{13}+a_{33})-q(a_{12}+a_{33}-a_{23}-a_{13})=-py+\frac{q}{2}(z-x-y),$$

which vanishes when we put

$$x = p^2 z, \quad y = q^2 z.$$

Similarly, we may prove that

$$(py' - qx') \frac{\lambda_2}{\lambda_3} = 0,$$

and therefore $py' - qx'$ annihilates $\frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{\lambda_3^2}$, and therefore $\frac{\lambda_4}{\lambda_3}$; that is, $py' - qx'$ annihilates any homogeneous function of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ which is of zero degree, so that

$$(py' - qx')\, p = 0.$$

We have seen that

$$\{(1-q)\, y' - qx'\}\, (x - p^2 z) + \{(1-p)\, x' - py'\}\, (y - q^2 z)$$

is the combinant of $x - p^2 z$ and $y - q^2 z$, and we have now proved that it is satisfied.

The system $f_1 = 0$, $f_2 = 0$, and $(f_1 f_2) = 0$ is now proved to be complete, and, as we have two equations of the second order and one of the first in four independent variables, the formula

$$(1-x^2)^2 (1-x)^1 (1-x)^{-4} = (1+x)^2 (1-x)^{-1} = 1 + 3x + 4x^2 + 4x^3 + \ldots$$

shows us that four derivatives of any order above the first are arbitrary; we could take these to be

$$\frac{\partial^r u}{\partial x_1^r}, \quad \frac{\partial^r u}{\partial x_2 \partial x_1^{r-1}}, \quad \frac{\partial^r u}{\partial x_3 \partial x_1^{r-1}}, \quad \text{and} \quad \frac{\partial^r u}{\partial x_4 \partial x_1^{r-1}},$$

so that the most general common solution could be taken to be

$$u = a_0 + a_2 x_2 + a_3 x_3 + a_4 x_4 + \ldots,$$

a series in powers of x_2, x_3, x_4, the coefficients being functions of x_1, the first four arbitrary and the remaining ones given in terms of these.