WILEY

Isallobars of Moving Circular Depressions

Author(s): E. G. Bilham

Source: *Geografiska Annaler*, Vol. 3 (1921), pp. 336-357

Published by: Wiley on behalf of Swedish Society for Anthropology and Geography

Stable URL: <http://www.jstor.org/stable/519447>

Accessed: 06-05-2016 14:46 UTC

REFERENCES

Linked references are available on JSTOR for this article:

http://www.jstor.org/stable/519447?seq=1&cid=pdf-reference#references_tab_contents

You may need to log in to JSTOR to access the linked references.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at

<http://about.jstor.org/terms>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Swedish Society for Anthropology and Geography, Wiley are collaborating with JSTOR to digitize, preserve and extend access to *Geografiska Annaler*

ISALLOBARS OF MOVING CIRCULAR DEPRESSIONS.

BY *E. G. BILHAM*, LONDON.

INTRODUCTORY.

Since the appearance of Dr. Nils Ekholm's well-known paper ¹ in 1904 much attention has been devoted to the preparation and study of charts of pressure — differences, more especially in France and the United States. In the latter country the idea of using such charts as an aid to forecasting was not new. Charts of 8-hours and 24-hours differences of both pressure and temperature had in fact been prepared and used in U. S. Weather Bureau (Signal service) since 1872.² The method had also been previously discussed by Brounow ³ in 1878 and Sresnewsky ⁴ in 1895. It is however probably true to say that Ekholm's paper first stimulated interest in the subject among the general body of European meteorologists.

Since 1904 many changes have been made in the meteorological data which are interchanged among European countries. The introduction of "barometric tendency", or change of pressure in the three hours preceding the time of observation, in May 1911 has caused the subject to take on a new aspect, since forecasters now have at their disposal direct values of the pressure difference over a known period, from which charts can be drawn and studied side by side with the isobaric charts. An "isallobaric" chart drawn from such observations clearly possesses possibilities of usefulness quite distinct from those of a chart drawn by means of differences obtained by subtracting known values of the pressure at one epoch from the next. It is of great importance in forecasting to exhaust every possibility of extracting information from the barometric tendencies, because the changes in the last few hours are by far the most important so far as future

¹ "Wetterkarten der Luftdruckschwankungen." *Met. Zeitschrift* 39 1904 p. 345.

² U. S. Monthly Weather Review 34 May 1906 p. 205 (footnote).

³ "Von den Aenderungskarten und der Methode, etc." Anhang zu dem Bulletin für das Jahr 1879. Central Physical Observatory, St. Petersburg, Dec. 18th, 1878.

⁴ "Ueber starke Schwankungen des Luftdruckes im Jahre 1887." *Bull. Soc. Imp. Nat. Moscow* 1895, No. 3.

developments are concerned. Our main problem is therefore to determine the possibilities and limitations of the isallobaric method when applied to barometric tendencies. At the same time an attempt will be made to estimate the value of ordinary "pressure-change" charts drawn from calculated pressure differences. For this purpose it is proposed to adopt an experimental rather than an analytical method. That is to say we shall attempt to determine the form of the isallobars in certain cases where the barometric distribution and its changes are definitely laid down and see whether a forecaster, provided with the isallobaric chart could accurately deduce the barometric movements from it.

Pressure-difference charts have been used in the past as intermediaries in the prediction of the future pressure-distribution by the following method: — Values of the pressure being known at each station at times $t_1, t_2, t_3 \dots$ isallobaric charts are drawn representing the pressure-differences for the intervals $t_2 - t_1, t_3 - t_2 \dots$ etc. From a study of the movements of "highs" and "lows" on the series of isallobaric charts an estimate is made of the isallobaric distribution at an epoch say 12 hours or 24 hours in advance of the last synoptic chart. Superimposing the predicted pressure changes on the last pressure values, a forecast is made of the forms of the isobars at the future epoch under consideration.

This process has been criticised on various grounds but it has at any rate the advantage of expressing in a concrete form the exact effects of the continuity of the pressure changes in progress. Since changes in intensity of the isallobaric "highs" and "lows" may be taken into account it is clearly possible to take cognizance not only of the first but of the second difference of pressure with respect to time. On the other hand it has always appeared to the present author that it tends to develop the idea that isallobaric systems are entities more or less dissociated from the isobaric systems. This tendency would be combated to some extent if it were customary to draw the isobars and isallobars on the same chart, when their mutual relationships would be more strikingly displayed.

The isallobars due to the movement of a system of isobars capable of being represented mathematically are obviously amenable to mathematical calculation. Certain typical cases are considered in this paper. Before proceeding to their discussion we may however direct attention to a simple geometrical method by means of which any case may be studied. The method is as follows: — The forms of the isobars at two epochs t_A and t_B are drawn on the same chart, using ink or pencils of two different colours if thought necessary. The isallobars can then be drawn at once by joining the appropriate points of intersection of the isobars of the two systems. For example the " -1 mb " isallobar is formed by joining the points of intersection of 1,000 A with 999 B , 1,001 A with 1,000 B and so on. By applying this process to the various forms of isobaric distribution that occur in practice some very instructive results may be obtained.

E. H. Bowie¹ has made use of this method for the purpose of examining the forms of isallobars due to the simultaneous movements of a cyclone and anticyclone along the line joining their centres, or to be precise, along a parallel of latitude. The anticyclone is assumed to precede the cyclone in each case and attention is devoted exclusively to the area of barometric fall lying, roughly, between the centres of the two systems. Variations in the position of the area of maximum fall due to variations in depth, intensity and relative speed are well illustrated in the plates accompanying the paper.

In the present paper it is proposed to consider certain ideal cases by simple mathematical methods. We shall commence with the case of the cyclonic depression in which the isobars are circles whose radii increase in arithmetical progression. Such a system will be referred to as a "uniform depression". It will be assumed at first that the dimensions of the system are large compared with the distance through which it moves during the time interval under consideration. That is to say we shall impose no limits on the size of the isobars and therefore on the pressures represented by them. Later on, under the general heading "Small Depressions" we shall consider the effect of limiting the geographical extent of the systems. It will be convenient to refer to the change of pressure at a given point as the "barometric tendency" no matter what the time interval. Following the general practice the tendency will be taken as positive if the pressure at the end of the interval is higher than at the beginning, and *vice-versa*.

I. Large depressions.

1. Uniform circular depression moving unchanged.

Let p_0 be the pressure at the centre and γ the pressure gradient, supposed constant throughout the system. The pressure p at a point distant r from the centre is then given by

$$p = p_0 + \gamma \cdot r \dots \dots \dots (1)$$

Let A and B be the positions of the centre at times t_A and t_B (Fig. 1) and let $AB = 2a$. Since the isallobars depend only on the initial and final positions of the system they will be the same no matter what path has been taken between A and B , but to fix our ideas we will suppose that the centre has travelled with uniform velocity v along the straight line AB , which we will take as the axis of x .

Fig. 1.

¹ "On pressure-change charts", E. H. Bowie; U. S. Monthly Weather Review 44 March 1916, p. 132

Let P be a point on the isallobar representing the constant difference of pressure c' and let p_A and p_B represent the pressure at P at times t_A and t_B . Then from (1)

$$\begin{aligned} p_A &= p_0 + \gamma \cdot PA \\ p_B &= p_0 + \gamma \cdot PB \\ \text{whence } p_B - p_A &= \gamma(PB - PA) \\ \text{or } c' &= \gamma(PB - PA) \\ \text{i. e. } PB - PA &= c'/\gamma \dots \dots \dots (2) \end{aligned}$$

The isallobar is therefore defined by the property that the difference in the distances of any point on it from two fixed points, A and B is constant. This is the characteristic property of a hyperbola with foci at A and B . The isallobars are therefore a family of confocal hyperbolae.

Take as axis of y a line bisecting AB at right angles. Then from Fig 1

$$PA^2 = y^2 + (a + x)^2 \dots \dots \dots (3)$$

$$PB^2 = y^2 + (a - x)^2 \dots \dots \dots (4)$$

Eliminating PA and PB between (2), (3) and (4) and writing c for c'/γ we obtain the Cartesian equation

$$\frac{x^2}{c^2/4} - \frac{y^2}{a^2 - c^2/4} = 1 \dots \dots \dots (5)$$

$$\text{or } (4a^2 - c^2) \cdot x^2 - c^2 y^2 = (4a^2 - c^2) \cdot c^2/4 \dots \dots \dots (6)$$

Optical Analogy. The isallobars are analogous to the interference fringes produced by point sources of light situated at A and B . In their graphical representations both the isallobars a and interference fringes are the loci of constant differences in the *radii vectores* measured from two fixed points, and the solution of the two problems is the same, namely a set of hyperbolae with the fixed points as foci.

Deductions. A number of important deductions follow at once from the above equations.

(a) The zero isallobar is coincident with the axis of y .

(b) The maximum tendency is equal to $2a\gamma$. This is the value of the pressure change at A and all points on the x -axis to left of A . That portion of the x -axis which lies to left of A is therefore the isallobar $+2a\gamma$.

(c) The minimum tendency is equal to $-2a\gamma$. The isallobar $-2a\gamma$ is coincident with that portion of the x -axis which lies to right of B .

(d) The isallobar $+c'$ intersects the x -axis at the point $x = -c'/2\gamma$. If isallobars are drawn for equal increments in c' , say for each millibar, the "isallobaric gradient" between A and B will be uniform and equal to 2γ , that is to say, equal to twice the isobaric gradient.

The general discussion of the practical utility of these results may be postponed until other cases have been considered. We may notice here, however, that in a case where the data at hand are the pressure distribution at time t_B and the isallobars for the time interval $t_B - t_A$, the conclusions (a), (b) and (c) provide methods for the determination of a , that is to say, -- since $2a = v(t_B - t_A)$ --, of the velocity of the depression.

A theoretical chart of isallobars is shown in Fig. 2 which represents the case of a uniform depression for which $\gamma = 0.04$ mb/mi. moving from left to right at 25 m. p. h. The time interval is 5 hours, so that $2a$ is 125 miles. In this case the maximum and minimum tendencies happen to be whole numbers of millibars, so that it is possible to show the limiting isallobars as semi-infinite straight lines terminating at A and B respectively. The "characteristic of tendency" for different areas is indicated at the top of the diagram.

Fig. 2. Isallobars of a uniform depression moving unchanged. The centre moves from A to B in five hours travelling at 25 m. p. h; $\gamma = 0.04$ mb/mi. Isallobars are shown for each millibar change of pressure by the continuous lines. Pecked lines are isobars at end of the period.

2. Uniform depression becoming deeper or filling up without change in the gradient.

(I) Depression becoming deeper.

A fall of pressure at the centre unaccompanied by any change in the gradient is obviously equivalent to a uniform fall of pressure over the whole area. Thus if the central pressure changes from p_0 at t_A to $p_0 - n$ at t_B the final pressure

at P (Fig. 1) will be given by $p_B = p_0 - n + \gamma.PB$ and equation (2) becomes

$$PB - PA = (c' + n)/\gamma \dots \dots \dots (7)$$

The Cartesian equation is obtained by putting $c + n/\gamma$ for c in equations (5) or (6).

It is clear that the solution in this case is a set of hyperbolae similar in all respects to those previously considered except that a hyperbola which previously represented a tendency c' mb. now represents the tendency $c' - n$ mb. The chief results may be summarized as follows: —

- (a) The isallobar $-n$ is coincident with the y -axis.
- (b) The maximum tendency is equal to $2a\gamma - n$ and is represented by an isallobar coincident with the x -axis to left of A . The maximum tendency is clearly negative if n is greater than $2a.\gamma$.
- (c) The minimum tendency is equal to $-2a.\gamma - n$ and is represented by an isallobar coincident with the x -axis to right of B .
- (d) The isallobaric gradient between A and B is uniform and equal to twice the pressure gradient.

(II) *Depression filling up.*

The case of a depression filling up without change of pressure gradient is equivalent to a uniform rise of pressure over the whole area. The results for this case differ from those of the previous section only in the reversal of the sign of n , and are as follows: —

- (a) The isallobar $+n$ mb. is coincident with the y -axis.
- (b) The maximum tendency is equal to $2a\gamma + n$ and is represented by an isallobar coincident with the x -axis to left of A .
- (c) The minimum tendency is equal to $-2a\gamma + n$ and is represented by an isallobar coincident with the x -axis to right of B . The minimum tendency is clearly positive if $n > 2a.\gamma$.
- (d) The isallobaric gradient between A and B is uniform and equal to twice the pressure gradient.

It is clear that a representation of the theoretical isallobars of a uniform depression changing its depth but not its intensity can be obtained by re-numbering the isallobars shown in Fig. 2. Thus, if at time t_B the pressure at the centre is 2 mb. lower than t_A , other things remaining the same, the isallobars of Fig. 2 should be marked reading from left to right, 3, 2, 1, 0, -1 , -2 , -3 , -4 , -5 , -6 , -7 mb. For a rise of 2 mb. at the centre the numbering would be 7, 6, 5, 4, 3, 2, 1, 0, -1 , -2 , -3 mb.

With regard to the “characteristic of tendency”, we have noticed above that either positive or negative tendencies may be caused to disappear by a suitable

choice of the change of pressure at the centre. For the general case in which both positive and negative tendencies appear, the distribution of the characteristic is as follows:

To right of ordinate through B.

"Falling continuously" or "falling at first, now steady".

Between ordinate through B and zero isallobar.

"Falling at first, now steady or rising" (tendency negative).

Between zero isallobar and ordinate through A.

"Steady or falling at first, now rising" (tendency positive).

To left of ordinate through B.

"Rising continuously.

The depression is assumed always to be moving from left to right.

3. Uniform depression varying in depth and intensity.

Up to the present we have confined our attention to cases in which the pressure gradient γ remains constant. It usually happens in practice however that a depression which is increasing in depth also increases in intensity and *vice versa*. We shall therefore now proceed to consider the effect of increasing γ and decreasing p_0 simultaneously.

(I) Depression deepening and increasing in intensity.

We will suppose that in travelling from A to B the gradient increases from γ_A to γ_B while the pressure at the centre decreases from p_0 to p'_0 where $p_0 - p'_0 = n$ mb. Retaining the previous notation and referring to Fig. 1 we have

$$\begin{aligned} p_A &= p_0 + \gamma_A \cdot PA \\ p_B &= p'_0 + \gamma_B \cdot PB \\ p_B - p_A &= p'_0 - p_0 + \gamma_B \cdot PB - \gamma_A \cdot PA \end{aligned}$$

For an isallobar $p_B - p_A$ is constant and equal to c' , say

$$\begin{aligned} c' &= -n + \gamma_B \cdot PB - \gamma_A \cdot PA \\ \text{or} \quad \gamma_B \cdot PB - \gamma_A \cdot PA &= c' + n \dots \dots \dots (8) \end{aligned}$$

This equation is of the general form

$$a \cdot r_1 \pm b \cdot r_2 = \text{const.}$$

which is the bi-polar equation of a "Cartesian oval" referred to its foci. The Cartesian equation obtained by the substitution of the values of PA and PB is a bi-quadratic of somewhat cumbrous form and need not be given here. It can be shown that the corresponding solution for the infinitesimal case (i. e. when the isallobars represent values of dp/dt) is a family of limaçons.

The main features of the curves represented by (8) may be exhibited very simply. Consider for example the isallobar $c' = 0$. Since pressure has diminished at the centre and the gradient has increased there will be a certain isobar whose radius remains unaltered. Let R be this radius, then R may be found from the equations.

$$\begin{aligned} p &= p_0 + R \cdot \gamma_A \\ \text{and} \quad p &= p'_0 + R \cdot \gamma_B \\ \text{i. e.} \quad R &= (p_0 - p'_0)/(\gamma_B - \gamma_A) = n/(\gamma_B - \gamma_A) \dots \dots \dots (9) \end{aligned}$$

For values of r greater than R pressure will have increased between t_A and t_B . In any ordinary case R will be very large compared with other linear quantities since it is not customary to observe a decrease in the area enclosed by any isobar forming part of a cyclone which is increasing in depth and intensity. For present purposes it will suffice if we suppose $R > a$. Then this isobar will intersect the y -axis at two points which are obviously points on the zero isallobar. The ordinates of these points are $\pm \sqrt{R^2 - a^2}$. We have to note therefore that the zero-isallobar intersects the y -axis at two points whose ordinates are $\pm \sqrt{R^2 - a^2}$.

Fig. 3.

Next, let Q be a point on the line AB which satisfies the necessary conditions. That is to say

$$\gamma_B \cdot QB - \gamma_A \cdot QA = n$$

which may be written

$$\gamma_B \cdot QB = \gamma_A (QA + n/\gamma_A)$$

Take a point A' on the x -axis to left of A (Fig. 3) such that $AA' = n/\gamma_A$.

$$\text{Then} \quad \gamma_B \cdot QB = \gamma_A \cdot QA'$$

That is to say the point Q divides the line $A'B$ in the ratio of $\gamma_B : \gamma_A \cdot \gamma_B$ being supposed greater than γ_A there will be two positions of Q , one of which divides $A'B$ internally and the other externally in the required ratio. Consequently the zero isallobar will intersect the x -axis in two points, one lying between A' and B and the other to right of B . If x is the abscissa of the point of intersection between A' and B we have

$$\begin{aligned}
 QB &= a - x; \quad QA' = a + x + n/\gamma_A \\
 \therefore \gamma_B(a - x) &= \gamma_A(a + x) + n \\
 \text{whence } x &= (\gamma_B - \gamma_A)a - n / (\gamma_A + \gamma_B) = (a - R)(\gamma_B - \gamma_A) / (\gamma_B + \gamma_A) \dots (10)
 \end{aligned}$$

which is essentially negative since $\gamma_B > \gamma_A$ and $R > a$. That is to say, the point where the zero isallobar intersects the path is nearer to the initial than to the final position of the centre of the depression. This result is similar to that obtained in the case where γ was kept constant.

Similarly we may find the points of intersection of the other isallobars with the x -axis. It can be shown that the points of intersection along AB are given by

$$x = \{a(\gamma_B - \gamma_A) - (n + c')\} / (\gamma_B + \gamma_A) \dots (11)$$

and those to right of B by

$$x = \{a(\gamma_B + \gamma_A) + (n + c')\} / (\gamma_B - \gamma_A) \dots (12)$$

Special interest attaches to the isallobar for which $c' = -n$. From (8) we have in this case

$$\gamma_B \cdot PB - \gamma_A \cdot PA = 0$$

By a well-known proposition the locus of P in this case is a circle on Q_1Q_2 as diameter where Q_1 and Q_2 are the points which divide the line of AB internally and externally in the ratio $\gamma_B : \gamma_A$. Its centre is at the point $x = a \cdot (\rho^2 + 1) / (\rho^2 - 1)$ and its radius is $2a \cdot \rho / (\rho^2 - 1)$, where $\rho = \gamma_B / \gamma_A$.

The more important results may be summarized as follows:

(a) There is a minimum value of the tendency given by $c'_m = -2a\gamma_A - n$. The isallobar for this value reduces to a single point, namely the point B .

(b) There is no upper limit to the positive tendencies.

(c) Along the path AB the isallobaric gradient is uniform and equal to $\gamma_A + \gamma_B$.

(d) To right of B and to left of A the isallobaric gradient is also uniform and is equal to $\gamma_B - \gamma_A$.

(e) The isallobar representing a fall of pressure equal to that at the centre of the depression is a circle centred at a point in advance of B .

Fig. 4 illustrates the case of a uniform cyclone in which the gradient increases in the ratio 4 : 3 between A and B while the pressure at the centre falls by 2 mb. As before the velocity of the depression is 25 m. p. h. and the time interval 5 hours. The initial gradient is .04 mb/mi

(II) *Depression filling up and decreasing in intensity.*

The case of a depression filling up, — that is to say pressure is rising at the centre while the gradient is decreasing — is obviously the exact converse of that just considered. The

mean pressure gradient during the movement. Now in the case of a depression increasing in intensity the mean pressure gradient will be less than the final gradient and *vice versa*.

There is therefore a relationship between the ratio of the isallobaric gradient to the final pressure gradient and the change of intensity in the cyclone which may be expressed by the following rule: —

In a uniform depression the isallobaric gradient is less than, equal to, or greater than twice the final pressure gradient according as the depression is increasing, remaining constant, or decreasing in intensity.

4. Summary of Results for Uniform Depression.

We may now proceed to summarize the more important of the general conclusions arrived at in the preceding sections. It is supposed that a forecaster has before him a chart of isobars at time t_B and of isallobars for the interval $t_B - t_A$, preferably drawn on the same chart. His object is to obtain from his data as much information as possible relating to the changes in position, depth and intensity of the uniform circular depression which we suppose to be the sole feature of the pressure distribution. For this purpose it is convenient to summarize the results obtained in the form of a number of rules.

(a) The path of the depression is an axis of symmetry of the isallobars. Along the path the isallobaric gradient is uniform and a maximum.

(b) The position of the centre of the depression at the beginning of the interval can be identified on the isallobaric chart as the limiting point on the positive side of the line of maximum isallobaric gradient.

(c) If the isallobars are confocal hyperbolae, no change of intensity has occurred.

(d) If the zero isallobar is a straight line bisecting the path of the centre at right angles, no change of intensity or depth has occurred.

(e) If the isallobars are hyperbolae but positive and negative values are unsymmetrical, a change of pressure at the centre has occurred. In such a case, if there is a zero isallobar, it will be concave towards that position of the centre at which pressure is higher and will intersect the path line nearer to that position than to the other.

(f) If the isallobars form a family of closed curves, a change of intensity has occurred, the gradient being higher for that position of the centre which is enclosed by the isallobars.

In addition to these relationships which are of a general (qualitative) character, a considerable number of numerical relationships have been found in the course

of the work. Although these are of some theoretical interest their practical value would probably be small, chiefly because with any reasonable number of stations, however accurate their reports, it would not usually be possible to draw the isallobars with sufficient accuracy. It must also be remembered of course that the large depressions seen on our charts are usually but very rough approximations to the ideal form assumed here, and it would be idle to suppose that any but the most characteristic features of the isallobaric systems would be recognizable in practice.

5. Generalized method for circular depressions.

We have considered the case of a uniform circular depression in some detail because the assumption of a uniform gradient probably gives the nearest approach to the truth when very large systems are considered. On account of their theoretical interest, it is desirable however to extend the enquiry to certain other forms of circular depressions. Before proceeding further it will be convenient to generalize the method of procedure.

The distribution of pressure in a circular depression of any form may be represented by:

$$p - p_0 = A \cdot f(r) \dots \dots \dots (13)$$

where both the central pressure p_0 and the coefficient A may vary with the time. If the centre of the depression moves with uniform velocity v from A ($x = -a$) to B ($x = +a$) in time $t_B - t_A$ we obtain the bifocal equation of the isallobars referred to A and B as follows: — Referring to Fig. 1 and putting $PA = r_1$, $PB = r_2$, we have for the pressure at P

$$\begin{aligned} p_A &= p_0 + A \cdot f(r_1) \text{ at time } t_A \\ p_B &= p'_0 + A' \cdot f(r_2) \text{ at time } t_B \end{aligned}$$

Subtracting and remembering that for an isallobar

$$\begin{aligned} p_B - p_A &= c', \text{ (a constant)} \\ c' &= p'_0 - p_0 + A' \cdot f(r_2) - A \cdot f(r_1) \dots \dots \dots (14) \end{aligned}$$

The Cartesian equation referred to the axes we have already employed may be obtained by putting

$$r_1 = \sqrt{y^2 + (a + x)^2}; \quad r_2 = \sqrt{y^2 + (a - x)^2}$$

in (14).

For the simple case in which the depression moves unchanged, $p_0' - p_0$ is zero and $A' = A$; that is to say (14) reduces

$$\text{to} \quad c' = A \{f(r_2) - f(r_1)\} \dots \dots \dots (15)$$

If $p_0' - p_0$ is not zero, we have already seen that its effect is merely equivalent to a uniform rise or fall of pressure. The isallobars have the same form as those given by (15) but the numbering is different, the isallobar of $c' + p_0' - p_0$ being identical with that of c' in (15).

6. Normal Cyclone moving unchanged.

The normal cyclone is defined by Shaw as one in which the vorticity is uniform. If V is the air speed at a distance r from the centre, $V/r = K$ or $V = K \cdot r$, where K is a constant. We can replace V by $K' \cdot dp/dr$ (where K' is another constant) since the air speed is proportional to the pressure gradient.

We have, therefore, for a normal cyclone

$$dp = A' \cdot r \cdot dr, \text{ where } A' = K/K'$$

Integrating, we obtain

$$p - p_0 = A \cdot r^2 \text{ where } A = \frac{1}{2} A' \dots \dots \dots (16)$$

r^2 therefore replaces $f(r)$ in the general equation and the bifocal equation of the isallobars is

$$c' = A \{r_2^2 - r_1^2\}$$

or, substituting the value of r_2 and r_1

$$c' = -A \cdot 4ax$$

i. e.

$$x = -c'/4A \cdot a \dots \dots \dots (17)$$

(17) represents a straight line perpendicular to the path of the depression, the abscissa being constant and equal to $-c'/4A \cdot a$. If c' is varied by successive increments of 1 mb, the constant abscissa increases by $1/4A \cdot a$, so that the isallobaric gradient is everywhere constant and equal to $4A \cdot a$. Since a is proportional to the speed of the depression we see that *the isallobaric gradient depends upon the speed of the depression as well as upon its intensity*. This result is of importance since we found in dealing with a uniform depression that the isallobaric gradient along the path depended only on the pressure gradient.

We observe that there is no theoretical limit to the value of c' in either direction. A point of some practical importance is that the isallobaric chart contains no feature whereby the path of the centre could be identified. The direction of motion is, however, indicated

Fig. 5. Isallobars of a normal cyclone moving unchanged for five hours at 25 m. p. h. Isobars at end of interval shown by pecked lines.

since it is at right angles to the isallobars and it would be theoretically possible to determine the velocity by measuring the isallobaric gradient, A being supposed known.

Fig. 5 illustrates the case where $A = .00032$, the mile being the unit of length. As in previous examples the speed of the cyclone is 25 m. p. h. and the time interval five hours. The distance between consecutive millibar isallobars is 12.5 miles.

7. Normal cyclone increasing in depth and intensity.

The general equation for the isallobars of a normal cyclone is obtained by substituting r^2 for $f(r)$ in equation (14). We obtain

$$c' = p_0' - p_0 + A' \cdot r_2^2 - A \cdot r_1^2 \dots \dots \dots (18)$$

For the case we are considering $p_0' - p_0$ is negative and $A' > A$. Putting $p_0' - p_0 = -n$, substituting for r_2 and r_1 , and reducing, (18) becomes

$$r^2 + \{x - a \cdot (A' + A)/(A' - A)\}^2 = \{(A' - A)(n + c') + 4a^2 A \cdot A'\}/(A' - A)^2 \dots (19)$$

which represents a circle of radius

$$\frac{1}{A' - A} \cdot \sqrt{(A' - A)(n + c') + 4a^2 A \cdot A'}$$

centred at the point

$$x = a \cdot (A' + A)/(A' - A)$$

The isallobars are therefore concentric circles. We observe that if c' is less than

$$-\left[n + 4a^2 A \cdot A' / (A' - A)\right] \dots \dots \dots (20)$$

the radius becomes imaginary. The expression (20) therefore gives the upper limit to the negative tendency. There is clearly no limit to the positive tendencies. Calling the minimum tendency c'_m the expression for the radius r' may be written

$$\sqrt{c' - c'_m} / \sqrt{A' - A}$$

whence

$$c' - c'_m = (A' - A) \cdot r'^2$$

Since this is an equation of the same form as that representing the original depression, differing from it only in the change of scale, the isallobaric gradient is not uniform but decreases towards the centre. We may note in passing that if the value of A is doubled during the time interval, i. e. $A' - A = A$, the isallobars are exactly similar to the isobars at the beginning of the interval.

It may be useful to summarize here the main points of difference between this case and the corresponding results for a uniform depression.

(a) The isallobaric gradient between A and B is not uniform.

(b) There is no discontinuity in the isallobaric gradient at A and B .

(c) The minimum tendency is found, not at the centre of the depression, but at a point some distance in advance of it.

(d) The isallobaric chart contains no features whereby the path of the centre of isobars is recognizable. The direction of movement is however that of a line joining the centres of isobars and isallobars.

(e) The isallobars do not all enclose the centre of the depression.

Fig. 6 illustrates the case of a normal cyclone moving from A to B (125 miles) in five hours, the pressure at the centre falling by 2 mb. while the coefficient A increases from .00024 to .00032. The centre of isallobars is 375 miles in ad-

Fig. 6. Isallobars of a normal cyclone moving for five hours at 25 m. p. h., pressure at the centre decreasing by 2 mb. and intensity increasing in ratio 4 : 3 during the movement. The isallobars are concentric circles, the common centre being 375 miles in advance of B . Isobars in final position shown by pecked lines.

vance of the centre of the depression, and the theoretical minimum tendency occurring at the centre of isallobars, is -20 mb. Only a portion of the isallobaric system is therefore shown.

Other cases of simultaneous variation of depth and intensity may be dealt with in a manner exactly analogous with the foregoing and need not to be separately considered.

8. Simple Vortex moving unchanged.

In a simple vortex the velocity, and therefore the pressure gradient, is inversely proportional to the radius. That is to say

$$dp/dr = K/r \quad \text{where } K \text{ is a constant.}$$

As it stands this expression leads to an infinite gradient at the centre. In nature, friction would fix a limiting value to the velocity, so that the relationship would not hold for values of r smaller than a certain limit. So far as the present investigation is concerned the only effect will be to produce a corresponding modification in the isallobaric distribution near the centre. The complete solution, in which account is taken of this modification, is beyond the scope of this enquiry. We shall therefore confine our attention to the simple case represented by the above equation, bearing in mind however that the solution is inapplicable close to the centre.

By integration we obtain for the pressure distribution

$$p - p_0 = A \cdot \log r \dots \dots \dots (21)$$

and we will suppose that the constant A applies when logarithms are taken to base 10. Substituting for $f(r)$ in the general equation, the isallobar for a rise of pressure c' millibars is given by

$$c' = A(\log r_2 - \log r_1) \dots \dots \dots (22)$$

or

$$r_2/r_1 = 10^{c'/A}$$

The ratio r_2 to r_1 for a given value of c' is therefore constant. The locus satisfying this requirement is obtained by dividing AB internally and externally in this ratio and describing a circle having the points so obtained at the extremities of a diameter. For the positive isallobars $r_2 > r_1$ and the whole system lies on the negative side of the y -axis. The co-ordinates of the centres of the circular isallobars are given by

$$x_c = -a \cdot (\rho^2 + 1)/(\rho^2 - 1) \quad \text{where } \rho = 10^{c'/A}$$

and the radii by

$$R = 2a\rho/(\rho^2 - 1)$$

The zero isallobar is clearly coincident with the y -axis, and the negative isallobars are a symmetrical set lying on the positive side of the y -axis. As c' increases, the centre of isallobars approaches the value $x = -a$ reaching that value when c' becomes infinite. The centres of all positive isallobars therefore lie to left of A . Similarly the centres of negative isallobars lie to right of B .

The five-hour isallobars of a simple vortex moving at 25 m. p. h. are illustrated in Fig. 7. The isobars towards the centre are not continued beyond the point at which the gradient reaches

0.1 mb/mi, corresponding to a gradient wind of 100 m. p. h., pressure being supposed uniform and equal to 1,000 mb. within this area. It will be seen that the -4 mb. isallobar is almost coincident with the 1,000 mb. isobar. If an isobar of 999 mb. were drawn, there would be isallobars of $+5$ mb. and -5 mb., the latter being practically coincident with the 999 mb. isobar, and so on. Similarly the innermost positive isallobars are nearly coincident with the isobars at the beginning of the interval. The path of the centre of the depression is therefore very clearly shown on the isallobaric chart.

away from the centre. It is therefore very necessary to examine the effect on the isallobars of limiting the size of the systems.

For the present purpose the most convenient way of applying the limitation is to suppose that the depression is superposed on a area of uniform pressure, say pL . The boundary of the system is therefore the isobar $p = pL$ outside

Fig. 8. Isallobars of a small depression moving through a distance greater than its own diameter. Positive isallobars coincident with isobars at beginning of interval. Negative isallobars coincident with isobars at end of interval.

which pressure is everywhere equal to pL . This introduces a discontinuity in the pressure gradient at the boundary, the effect of which is seen on the isallobars. No convenient method of avoiding this however suggests itself and it is not of much consequence in the present connexion.

To lead up to the general case we will first of all consider the isallobars of a small depression, not necessarily circular, the time interval being such that the disturbance has moved through a distance greater than its own diameter. The isobars at the times of beginning and end

(t_A and t_B) if drawn on the same chart would therefore nowhere intersect. To fix our ideas we may suppose that the boundary is the 1,010 mb. isobar and need not necessarily be of the same shape at t_A and t_B .

Consider first the 1,010 isobar. It encloses the area in which pressure is below 1,010 mb. At the beginning of the interval this area was all at a pressure of 1,010 mb. Therefore, *all the negative isallobars must lie inside this isobar, which itself forms a zero isallobar.*

Consider next the 1,009 mb. isobar. At all points on it pressure has fallen from 1,010 mb. at time t_A to 1,009 mb. at time t_B . The 1,009 isobar is therefore also the -1 mb. isallobar. Similarly the -2 mb. isallobar is identical with the 1,008 mb. isobar, and so on. By a similar process of reasoning it could be shown that the positive isallobars are coincident with the isobars at time t_A , the 1,010 isobar forming another zero isallobar. In the case considered the isallobars therefore give an exact representation of the isobars at the beginning and end of the time interval. It is interesting to notice that the isallobars in this case represent a fall or rise of pressure not only through the same range but between exactly the same limits.

10. General case of a small depression.

If the isobars of a small depression at two different epochs are drawn on the same chart they will in general overlap. Within the overlap the conditions

are precisely the same as if the depression were of indefinite extent, consequently the isallobars assume the forms we have discussed in Part I of this paper. Within the non-overlapping portions the course of the isallobars is determinable by reasoning on the lines of the previous section. That is to say the portions of the isallobars which lie within those areas are coincident with the isobars, at the beginning of the interval in the case of positive tendencies and at the end of the interval in the case of negative tendencies. The zero isallobar is the boundary of the area which has been influenced by the depression, with a transverse line dividing the positive from the negative tendencies.

Fig. 9 illustrates the case of a small normal cyclone moving unchanged, all conditions being the same as in fig. 5 except that the system is limited by the 1,004 mb. isobar.

It will be seen that the boundaries of the overlap are the loci of discontinuities in the isallobars. This is of course due to the fact that there is a discontinuity in the pressure gradient at the boundary of the depression. In nature such discontinuities would not exist and the corners would be rounded off. The peculiar "bivalve" shape of the zero isallobar is however an essential characteristic of the system. Given the initial condition that the depression must have moved through a distance less than its own diameter there seems to be no reason why the zero line should not actually be so drawn on a chart.

Fig. 9. Isallobars of a small normal cyclone moving through a distance less than its own diameter. The circular portions of the positive isallobars are coincident with the isobars at beginning of interval and those of the negative isallobars with the isobars at end of interval.

11. Superposed distributions — Secondary depressions moving in uniform field of pressure.

Hitherto we have confined our attention to simple systems capable of mathematical representation by compact expressions. The results however may be made applicable over a much wider range of possible pressure-distributions since they are evidently additive. For example if we are given a pressure distribution consisting of a large depression moving along a certain path, with a secondary moving and developing in a specified manner we can construct the isallobaric chart due to the whole system by superposing the isallobars due to the separate constituents. A formal proof of this proposition could of course be given but is hardly necessary.

A special case of considerable interest is that of a small normal cyclone moving in a permanent uniform field of pressure with a velocity equal to that due to the uniform gradient. Since the uniform field is not changing, it contributes nothing to the isallobars and the latter are therefore the same as those due to the cyclone alone. We have seen that the isallobars reproduce to some extent the isobars of such a system. Consequently the isallobars provide a means of separating the cyclone or "revolving fluid" from the uniform field of pressure upon which it is superposed. This leads to an important dynamical result. Sir Napier Shaw¹ has shown that in such a system moving from West to East the centre of isobars is at a point North of the true centre of revolving fluid or "tornado centre" by an amount given by the expression

$$2\omega V \sin \Phi / \zeta (2\omega \sin \Phi + \zeta)$$

where ω is the angular velocity of rotation of the earth, V the velocity of the depression, Φ the latitude, and ζ the uniform vorticity in the cyclone. From the isallobaric chart the movement of the centre of revolving fluid or tornado centre can be followed. Knowing the position of the centre of isobars at all points of its path, it should be possible, in a well-developed case, to measure the distance between the two centres directly. A method of checking Sir Napier Shaw's result therefore appears to be available.

A common feature of weather charts is the secondary depression which appears merely as a "bulge" in the isobars. Such a distribution is obtained by superposing a small feeble depression on a strong uniform field or upon the field due to a large depression centred at a point relatively remote. As such systems are frequently very persistent the isallobars will usually be chiefly due to the small superposed disturbance and may be of considerable service in showing its form, velocity and direction of movement.

12. Concluding Remarks.

We may now attempt some estimate of the value of isallobaric charts in forecasting. The subject presents two distinct aspects according as it is proposed to draw the charts

- (a) from pressure differences reported at time t_B covering the interval $t_B - t_A$; or
- (b) from pressure differences obtained by subtracting known values of the pressure at time t_A from those at time t_B .

¹ "The Travel of Circular Depressions and Tornadoes". Meteorological Office Geophysical Memoirs, No. 12. p. 40.

The first method is that used for drawing the isallobaric chart which appears daily in the British Section of the Daily Weather Report, the data being the 3-hour barometric tendencies reported at 7 h. The second method is that mentioned in the first section of this paper in connection with the prediction of the future pressure-distribution.

In the first case the pressure distribution at time t_A is not explicitly known. In ordinary practice the forecaster attempts to estimate the effect of the barometric changes in progress by taking cognizance of the distribution of the tendencies plotted on the chart. The point at issue is therefore, merely, whether he would be materially aided in this endeavour by looking at a chart in which the tendencies were represented by continuous lines, instead of individual values at the stations. The results obtained in previous sections point strongly to the conclusion that the isallobars would often be of very material assistance, and would give a good deal of information about the changes of position and intensity of a depression during the period; information which could certainly not be so readily deduced from scattered individual observations.

It is clear that the value of the isallobars is largely dependent upon the accuracy with which they can be drawn. According to present practice in England the unit for barometric tendency is the half-millibar, consequently the values to be reported are frequently but small multiples of the unit and the uncertainty in drawing the isallobars is relatively great. It is equivalent to drawing isobars for each millibar among barometric readings expressed only to the nearest half millibar. Even with this rough method of procedure the isallobaric chart cannot fail to be of material assistance, and rough approximations to the forms investigated in this paper are frequently to be seen on the charts in the Daily Weather Report:

When the pressure distribution is accurately known both at the beginning and end of the time interval, the point at issue is whether the isallobaric chart is capable of affording any additional information, or of displaying the available information in a more effective way. We have found in the case of small depressions uncomplicated by other systems that the isallobars are merely a sort of composite picture of the two sets of isobars and no additional information is imparted by them. The same is true of large depressions. In most cases we found in fact that the isallobars tended to be more complex than the pressure distributions giving rise to them. No additional information as to the course taken by the depression in moving from its initial to its final position is afforded by the isallobaric chart, since, as has previously been pointed out, the isallobars depend only on the initial and final conditions and not on intermediate stages.

When the distribution is complex, however, the isallobaric chart is of value since it performs the function of separating the active from the quiescent features

of the pressure distribution. Since complexity is the normal condition of isobaric charts, any method whereby certain outstanding features could be disentangled from their surroundings would seem to be worthy of attention. An isallobaric chart is clearly capable of presenting to the eye a picture of the state of affairs which would not be readily perceived on the ordinary chart. A forecaster, seeing an isallobaric distribution resembling that shown in Fig. 8 would have his attention drawn to the presence of a small depression moving across the chart at a rapid rate. On the ordinary isobaric chart such a system might easily be more or less obscured by being superposed upon a large stationary depression of considerable intensity, such as for example one of the Icelandic depressions usually associated with spells of the Southwesterly type of weather in winter. In special investigations it might often be of material assistance to supplement a series of isobaric charts by a corresponding series of isallobaric maps for similar reasons.

No attempt has been made in this paper to investigate the relationship between the movements of pressure systems and those of the resultant isallobars. It is obvious however that when a depression moves regularly without change of depth or intensity the isallobars move with the depression and maintain the same relationships at any stage of the movement. When changes of intensity occur or the depression pursues a curved path the problem may become somewhat complex. Consider for example the case of a normal cyclone increasing in intensity (Section 6). We found that the centre of isallobars is in advance of the centre of isobars by an amount which depends upon the intensity of the depression at the beginning and end of the time interval. If the depression continues to move with uniform velocity in a straight line but the rate of change of intensity increases, the centre of isallobars will be nearer the centre of isobars and *vice-versa*. It is quite possible therefore for the rate of movement of the isallobaric systems to differ from that of the pressure systems. A complexity may in fact be imparted to the movements of isallobars by internal changes in the pressure systems. It is not proposed to go further into the matter here, beyond drawing attention to the fact that there are theoretical reasons against the supposition that the movements of isallobars are necessarily simpler than or as simple as, the movements of depressions. From the practical standpoint the matter has been investigated in some detail by S. Hanzlik.¹

¹ "Relations between velocities of progression of lows and the areas of rising and falling pressure accompanying them." Stanislav Hanzlik, Ph. D.; U. S. Monthly Weather Review 34 May 1906 p. 205.