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XXVI. *On the General Laws of Brightness of Images.*
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I AM not acquainted with any optical Treatise which contains a general investigation of the brightness of an image seen by rays which have traversed a medium of continuously varying index. The only discussion of the subject that I have met with is contained in chapter xii. of Clausius's 'Mechanical Theory of Heat.' That chapter is entitled "On the Concentration of Rays of Light and Heat," and the main purpose of it is not optical. I have found that its optical results can be obtained by a much simpler method, as set forth in the present paper. The results themselves are probably familiar to leading physicists, but they certainly are not so well known as they deserve to be.

The particular case in which all the refractions and reflexions take place at coaxial spherical surfaces with nearly normal incidence is treated in my edition of 'Deschanel' (see especially footnote to § 768 of the earlier, or § 1037 of the later, editions), and had been previously treated by a different method in Helmholtz's 'Physiological Optics' (pp. 171-175).

The mode of investigation here adopted has been suggested partly by Clausius's discussion above referred to, partly by an investigation of Kirchhoff's in Poggendorff's *Annalen*, vol. cix. pp. 284-287, to which Clausius refers, and partly by § 334 (2nd edition) of Thomson and Tait's 'Natural Philosophy.'

We shall begin by establishing the following

Theorem on Apparent Size.

If two small equal plane areas A_1, A_2 are so placed that a ray from the centre of one to the centre of the other is perpendicular to them both, the two solid angles formed by rays at the centre of each which come from the circumference of the other are inversely as the squares of the absolute indices of refraction at the centres at which they are formed. That is, if $\mu_1 \mu_2$ be the indices at the centres of A_1, A_2 , ω_1 the angle formed at the centre of A_1 by rays from the circumference of A_2 , and ω_2 the solid angle formed at the centre of A_2 by rays from the circumference of A_1 , then will

$$\mu_1^2 A_1 \omega_1 = \mu_2^2 A_2 \omega_2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As a particular case, *if the two indices of refraction are equal, as well as the two areas, the two solid angles will be equal; in other words, the apparent size of the first area as seen from*

* Communicated by the Author.

the second is the same as that of the second as seen from the first.

Proof.

Let T denote the time of propagation of light from a point xyz to a point $\xi\eta\zeta$, or from the latter point to the former. If δs be an indefinitely short line drawn from the point xyz at an inclination θ to the forward direction of a ray from $\xi\eta\zeta$, the increase in T will obviously be the same as for a line $\delta s \cos \theta$ drawn along the ray. Hence, if v denote the velocity of light at xyz , and μ the absolute index of refraction at the same point, we have $dT = \frac{1}{v} \delta s \cos \theta$, or

$$\frac{dT}{ds} = \frac{\cos \theta}{v} = \frac{\mu \cos \theta}{V}, \quad (2)$$

where V denotes the velocity *in vacuo*.

Take the centre of the small area A_1 as origin of the rectangular coordinates xyz , and the centre of A_2 as origin of $\xi\eta\zeta$. Let the axes of z and ζ be tangential to the ray, and reckoned positive in the outward directions. The axes of xy and of $\xi\eta$ will therefore be in the planes of A_1 and A_2 respectively, and we shall choose them so as to satisfy the two conditions

$$\left[\frac{d^2 T}{dx d\eta} \right] = 0, \quad \left[\frac{d^2 T}{dy d\xi} \right] = 0, \quad (3)$$

the brackets signifying that the six variables $x, y, z, \xi, \eta, \zeta$ are to be put equal to zero after the performance of the differentiations.

To express the solid angle formed at the centre of A_1 by rays from the circumference of A_2 , suppose tangents of unit length, forming prolongations of the rays, to be drawn from the centre of A_1 . The solid angle (being small) will be numerically equal to the base of the cone or pyramid enclosed by the tangents, or to the projection of this base on the plane of xy . Let x' and y' be the coordinates of the projection of the extremity of one of these unit-tangents, then the solid angle will be equal to the area traced by the point $x'y'$ in the plane of A_1 , while the point $\xi\eta$, from which the ray proceeds, moves round the circumference of A_2 . The coordinates x' and y' are obviously equal to the cosines of the angles which the ray makes with the axes of x and y .

Hence, putting $\cos \theta$ in (2) successively equal to x' and y' , we have

$$x' = \frac{V}{\mu_1} \frac{dT}{dx}, \quad y' = \frac{V}{\mu_1} \frac{dT}{dy}, \quad (4)$$

μ_1 denoting the value of μ at the centre of A_1 .

Again, since x' and y' are functions of the two independent variables ξ and η , and vanish with them, we have, to the first order of small quantities,

$$x' = \left[\frac{dx'}{d\xi} \right] \xi + \left[\frac{dx'}{d\eta} \right] \eta, \quad y' = \left[\frac{dy'}{d\xi} \right] \xi + \left[\frac{dy'}{d\eta} \right] \eta. \quad (5)$$

But by (4),

$$\frac{dx'}{d\xi} = \frac{V}{\mu_1} \frac{d^2 T}{dx d\xi}, \quad \frac{dx'}{d\eta} = \frac{V}{\mu_1} \frac{d^2 T}{dx d\eta}. \quad (6)$$

Thus equations (5) reduce, by the help of (3), to

$$x' = \frac{V}{\mu_1} \left[\frac{d^2 T}{dx d\xi} \right] \xi, \quad y' = \frac{V}{\mu_1} \left[\frac{d^2 T}{dy d\eta} \right] \eta. \quad (7)$$

Hence x' has a constant ratio to ξ , and y' a constant ratio to η . The product of these two ratios will be the ratio of the area traced by the point $x'y'$ to the area traced by $\xi\eta$, that is, it will be the ratio of the solid angle ω_1 to the area A_2 . We have accordingly

$$\omega_1 = \left(\frac{V}{\mu_1} \right)^2 \left[\frac{d^2 T}{dx d\xi} \right] \left[\frac{d^2 T}{dy d\eta} \right] A_2. \quad (8)$$

By applying similar reasoning to rays from the perimeter of A_1 to the centre of A_2 , we shall find (with similar notation)

$$\xi' = \frac{V}{\mu_2} \left[\frac{d^2 T}{dx d\xi} \right] x, \quad \eta' = \frac{V}{\mu_2} \left[\frac{d^2 T}{dy d\eta} \right] y, \quad (9)$$

$$\omega_2 = \left(\frac{V}{\mu_2} \right)^2 \left[\frac{d^2 T}{dx d\xi} \right] \left[\frac{d^2 T}{dy d\eta} \right] A_1. \quad (10)$$

By comparison of (8) and (10) we have

$$\mu_1^2 A_1 \omega_1 = \mu_2^2 A_2 \omega_2. \quad (1)$$

The rays considered in the foregoing proof may undergo any amount of either gradual or abrupt bending by refraction, and may be reflected any number of times; but there must be no abrupt difference between the histories of rays which come from consecutive points.

We now proceed to apply this theorem to the investigation of brightness.

The brightness of an object as seen from any point is measured by

$$\frac{q}{A\omega}, \quad (11)$$

A denoting a small area at the point, sensibly perpendicular to the rays which reach it from the object, ω a small solid angle of arbitrary magnitude, and q the quantity of light from the object which converges to A (and also diverges

from A if not stopped) in pencils of magnitude ω . The justness of this measure is obvious from the fact that if we halve either A or ω we shall halve the quantity of light that will come into consideration.

When the small area A is taken on the surface of the object itself, the value of the expression (11) is called the *intrinsic brightness* of the object.

In the case of the two small areas which we have been discussing, let q denote the quantity of light sent by A_2 on the road to A_1 . If none of it is lost on the road, it all reaches A_1 ; thus the same quantity of light which leaves A_2 in pencils of angle ω_2 reaches A_1 in pencils of angle ω_1 . Let I_2 denote the intrinsic brightness of A_2 , and I its apparent brightness as seen from A_1 ; then

$$I = \frac{q}{A_1 \omega_1} = \frac{A_2 \omega_2}{A_1 \omega_1} \frac{q}{A_2 \omega_2} = \left(\frac{\mu_1}{\mu_2} \right)^2 \frac{q}{A_2 \omega_2} = \left(\frac{\mu_1}{\mu_2} \right)^2 I_2. \quad (12)$$

By supposing μ_2 equal to μ_1 we obtain the following theorem. *If the eye and the object are in media of the same index, and no light is stopped on its way from the object to the eye, the apparent brightness is equal to the intrinsic brightness, notwithstanding any refractions or total reflexions that the rays may have undergone between the object and the eye.*

If light is lost on the way, the above expression for the apparent brightness must be multiplied by a coefficient k less than unity, which, by well-established laws of Optics, is the same for rays going from A_2 to A_1 as for those going from A_1 to A_2 (for the same kind of light). If I_2 and I_1 are the intrinsic brightnesses of the two areas, their apparent brightnesses when each is seen from the other will be

$$k \left(\frac{\mu_1}{\mu_2} \right) I_2, \quad \text{and} \quad k \left(\frac{\mu_2}{\mu_1} \right) I_1. \quad \dots \quad (13)$$

When an observer sees the area A_2 , the pupil of his eye may be taken as the area A_1 ; the foregoing computations of apparent brightness accordingly assume that all parts of the pupil receive rays from every part of A_2 . If part of the pupil be covered, the apparent brightness of the object will be diminished; and a similar diminution will occur when the rays from the object are collected into a beam of such small section as not to fill the pupil. This is the cause of the falling off of light which is observed in the use of high magnifying-powers with optical instruments.

In the case of light sent out obliquely from the surface of an object, the above investigations will remain applicable if

we understand by A_2 the projection of this surface on a normal section of the beam.

When A_1 is the image of A_2 the "theorem on apparent size" requires modification, for the centre of A_1 will no longer receive one ray from each point of A_2 , but will receive all its rays from one point only, which we shall regard as the centre of A_2 . Let a section orthogonal to all the rays which go from the centre of A_2 to the centre of A_1 be taken anywhere in the intermediate region, but not so as to coincide with a surface at which there is an abrupt change of direction in any of the rays. For instance, if the image is formed by a lens, the section may be taken either in the substance of the lens or in the external medium, but must not coincide with a face of the lens. Divide this section into parts comparable in size with A_1 or A_2 , and let A_3 denote the area of one of these parts. Also let μ_3 denote the index of refraction at the centre of A_3 . Since A_1 is the image of A_2 , the rays which go from the perimeter of A_2 to a fixed point of A_3 will go on to the perimeter of A_1 ; thus A_2 and A_1 will subtend the same solid angle as seen from A_3 , call it ω_3 . Hence, by applying (1) first to A_1 and A_3 and then to A_3 and A_2 , we have

$$\mu_1^2 A_1 \omega_1 = \mu_3^2 A_3 \omega_3 = \mu_2^2 A_2 \omega_2, \quad . \quad . \quad . \quad (14)$$

ω_1 and ω_2 denoting the solid angles formed at A_1 and A_2 by rays which traverse the perimeter of A_3 .

Equations (14) show that the ratio of ω_1 to ω_2 is the same for all the portions such as A_3 into which the intermediate section is divided. Let $\Omega_1 \Omega_2$ be the whole solid angles at $A_1 A_2$ formed by the aggregate of all the rays which go from A_2 to A_1 . Ω_1 is the sum of all the partial angles ω_1 , and Ω_2 is the sum of all the partial angles ω_2 ; hence we have

$$\mu_1^2 A_1 \Omega_1 = \mu_2^2 A_2 \Omega_2, \quad . \quad . \quad . \quad . \quad (15)$$

a result which is similar in form to (1).

The rays which go from A_2 to A_1 diverge from A_2 in the solid angle Ω_2 , and converge to A_1 in the solid angle Ω_1 . Let Q be the whole light sent from A_2 on the way to A_1 , and kQ the portion of it which actually reaches A_1 , then the intrinsic brightness I of the real image A_1 will be

$$I = \frac{kQ}{A_1 \Omega_1} = \frac{kA_2 \Omega_2}{A_1 \Omega_1} \frac{Q}{A_2 \Omega_2} = k \left(\frac{\mu_1}{\mu_2} \right)^2 \frac{Q}{A_2 \Omega_2} = k \left(\frac{\mu_1}{\mu_2} \right)^2 I_2, \quad (16)$$

as in (13). When the pupil of the eye is placed at the image, the brightness of the field of view will be identical with this intrinsic brightness of the image, if the pupil is filled. The apparent brightness of A_2 as seen from A_1 will then be unaffected by the circumstance that A_1 is the image of A_2 .