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THE DEVELOPMENT OF THE FUNCTION CONCEPT¹.

BY L. L. DINES,

Department of Mathematics, University of Saskatchewan.

The history of any science in its larger aspect is a record of the development of unifying principles. The biologist, not content with studying the isolated forms of life in the vegetable and animal world, has sought a unifying principle in the cellular organism which is common to all material life. The physicist in his search for a unifying principle in material structure has traveled a road marked by the signs molecule, atom, electron—and he is still going. In the investigation of forces, unification was begun by Newton, when he pronounced with such marvelous precision the law which governs the phenomena of gravitational attraction. Later developments have led to similar unification in the cases of electric and magnetic forces, and it requires no great stretch of the imagination to foresee the time when a single unifying principle will be pronounced as the basis for all physical forces.

Mathematics, as a pure science, deals not primarily with matter, with material life, or with forces, but with abstract classes of elements, and with relations existing between the various classes and elements. To be sure, the elements may be specialized to represent material objects, living organisms, forces, or any entities whatever. It is thus that mathematics, while being first of all an abstract science, is of universal application.

Does the abstract science of mathematics with its multitude of applications admit of any unifying principle? It would be bold indeed to offer any concept as ultimate. Such is certainly not my purpose. What I purpose is to develop, in so far as time will allow, a concept which, arising in a small and very special

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connection, has gradually expanded in significance until at present it seems, more than any other, to be serviceable as a unifying concept in mathematics.

The term function in a mathematical setting greets the present-day student probably for the first time in his study of trigonometry. He learns that associated with every angle there are six algebraic ratios, each with a name for which the teacher is obliged to offer an excuse more or less plausible; and that the six ratios are collectively called the trigonometric functions of the angle. Or the student may first meet the term function in his study of algebraic equations, when he is told that the left side of an equation such as

$$x^2+3x-2=0$$

may be abbreviated by the notation $f(x)$, to be read function of x .

Neither of these uses of the term comes first chronologically in the historical development. It is said that the term function was first used as a synonym of power. Thus a function of x would be equivalent to x^n . Evidently the use above mentioned in connection with algebraic equations is a direct generalization of this usage, and one which probably followed close upon the introduction of the term as synonymous with power.

From the application of the name function of x to a polynomial in x , it was evidently a short step to the extension of the term to represent any mathematical expression involving x , which furnished a formula by means of which when x was known the value of the function could be determined. The name seems first to have been used with this generalized formula significance by the famous Swiss Bernoulli brothers, John and James. Certainly the term function of a variable was first defined explicitly by John Bernoulli in 1718 as a quantity composed in any manner of this variable and of constants. In 1730 John Bernoulli took another distinctive step when he classified functions into two classes, algebraic and transcendental, according to the nature of the formula. The present-day functional notation $f(x)$, $\phi(x)$, etc., is due to D'Alembert and Euler, and was used first about 1754. The essential feature of the notion of function which we have so far developed is the existence of a formula which serves to determine the value of the function, when the value of the variable is known. The record of the development of the function concept thus far may be called the first chapter of its history.

From the earliest times the properties of geometric curves engaged the interest of mathematicians. Many of our simpler

curves were well known to the Greeks; for example, the circle, ellipse, parabola, hyperbola, and cycloid. The interest in these and other curves persisted through the period of mathematical stagnation which ended with the beginning of the seventeenth century. It was the great philosopher and mathematician Descartes who in the seventeenth century prepared the way for the union of our formula function concept and the concept of the geometric curve. His method of procedure is of course well known. With reference to a pair of fixed rectangular axes, he established a correspondence between the points of a plane and the totality of pairs of real numbers. To every point in the plane corresponds a unique pair of numbers (x, y) , the coördinates of the point, and conversely to every pair of real numbers (x, y) there corresponds a unique point, the point of which these numbers are the coördinates.

The complete significance of Descartes' invention was of course not immediately grasped. It takes our present-day students at least a year to appreciate it, with the aid of what we hope are pedagogically correct methods of presentation. It was almost obvious, however, that in view of this invention the study of a geometric curve thought of as generated by a moving point was equivalent to the study of the simultaneous variation of the coordinates of a point which traversed it. And for most of the geometric curves then known, the ordinate y of a point traversing the curve could be expressed in terms of the abscissa x by means of a formula of a type then known; while on the other hand any formula $f(x)$ could be visualized by a graph, the locus of points whose coördinates satisfied the equation $y = f(x)$. It is of interest to note that this correlation of the formula function concept and the geometric curve concept took place before the name function had been attached to the former concept, at least by explicit definition.

Perhaps no other union of two distinct fields of any science ever did so much to enrich both fields. It is indeed difficult to say which profited more. Let me, however, point out a certain lack of reciprocity in the situation which was destined to lead to developments of tremendous significance. Given any formula function $f(x)$, the equation

$$y = f(x)$$

determined a curve, the locus of the variable point (x, y) . By choosing arbitrary values of x as abscissas and computing from the formula the corresponding ordinates, any desired number of points could be plotted, and so the curve determined with an

accuracy limited only by the patience of the plotter. The existence of the locus which was being approximated was a logical certainty.

Consider now the converse problem. Given an arbitrary geometric curve; is there necessarily a function (formula) $f(x)$, such that the equation

$$y = f(x)$$

represents the dependence of the ordinate upon the abscissa? The question now before us came not from geometry but from the concept of physical law. Now the formulation of a physical law consists usually in the statement of the manner in which one measurable physical magnitude depends upon another. The fact of the dependence of one physical magnitude upon another is ascertained by experience. The general nature of the dependence is determined by repeated experiment and by induction. The question then arises: Can the dependence be expressed by a mathematical formula? This is the question which Newton answered in the case of the gravitational attraction between two material particles when he expressed its measure by the formula

$$\text{attraction} = c/x^2,$$

where c is a constant and x the distance between the particles.

Suppose now that it is desired to express by means of a formula the dependence of a physical quantity y upon a physical quantity x . Suppose that for a large number of values of x , the corresponding values of y have been determined (other factors being assumed constant), and that these correspondences have been represented by points in a Cartesian system. If then the physical quantities x and y are such that a gradual variation in x produces a gradual variation in y , it follows that the points which have been plotted are points of a continuous curve which represents the dependence of y upon x for all values of x . The next step is to formulate a relation

$$y = f(x)$$

which will represent this curve. It is not difficult to determine a function $f(x)$ which will correctly represent the correspondences for any finite number of points. This can, as a matter of fact, be done by means of a simple polynomial $f(x)$ whose degree is equal to the number of correspondences to be represented. However, for the vigorous mathematical physicist of the past two centuries, this is not a sufficient solution. He says: If x is a quantity susceptible of continuous variation, then no matter

how many correspondences of y and x you determine and comprehend in your formula, there still remains infinitely many more correspondences not considered. While our intuition tells us, and we are inclined to believe it, that if our formula is constructed so as to give correctly the value of y for a large number of values of x , it will then give y with close approximation for all values of x , we would like to have a definite "yes" or "no" answer as to whether there is a mathematical formula which will represent the correspondence between y and x accurately for all values of x at least within a finite interval. This question was first encountered by John Bernoulli in connection with the problem of the vibrating string.

Consider an elastic string, for instance a violin string, fastened at both ends. If it is initially distorted and then released, it will vibrate in the plane of its initial position. Bernoulli showed that an equation representing the form of the string at any instant of time t could be obtained if it was possible to represent by an equation, $y = f(x)$ of a certain type the form of the string at the initial instant of distortion. And then he showed that this latter representation was possible. There were, however, certain defects in Bernoulli's work and the most serious fault of all was the one of omission, in that he failed altogether to see the great import of his work. What he had really done when his problem was stripped of its physical setting was to show that any arbitrary continuous curve could be exactly represented by a formula. It was some years later, in 1812, that the French mathematician, Fourier, encountered the same question in a physical investigation. He was investigating the laws governing the conduction of heat, and he wished to express by means of a formula, temperature = $f(x)$, an arbitrary distribution of heat along a straight line. He found that whatever be the curve representing the temperature it could be represented by an equation of the form

$$y = a_0 + (a_1 \sin x + b_1 \cos x) + (a_2 \sin 2x + b_2 \cos 2x) + \dots,$$

the function $f(x)$ being an infinite series of trigonometric functions. Since his time series of this type have been called Fourier Series, and have occupied an extremely important place in the developments of mathematical physics and of pure mathematics.

With Fourier, then, our function concept reached a stage in which it represented a correspondence between two simultaneously variable quantities, of a type which could be represented by the simultaneous variation of the ordinate and abscissa of an

arbitrary curve. The formula concept was not lost but was extended by the introduction of Fourier Series, so as to include this geometric concept. We may consider this as closing the second chapter of the history of the function concept.

The third stage of the development of the function concept begins with the work of Dirichlet in the first half of the nineteenth century. He was engaged in perfecting the work of Fourier on the representation of a function by means of a series whose terms were trigonometric functions. His work was of a kind more vigorous and critical than that of his predecessors, and he sought to make precise by an explicit definition what he meant by the term function of a variable. I will give the definition, not in the exact form in which it was given by Dirichlet, but in a form slightly more general, which is, however, usually associated with the name of Dirichlet.

Dirichlet's Definition of a Function: Let a and b be two fixed real numbers, and x a variable which gradually assumes all values between a and b . If y is a second variable whose value depends upon the value of x in such a way that to every value of x between a and b there corresponds a definite value of y , then y is said to be a function of x on the interval a to b .

How is this definition related to the two which we have previously considered? In the first place, nothing is said about the existence of a formula. If the dependence of y upon x in the interval a to b is expressible by means of an equation $y = f(x)$, where $f(x)$ is a formula, then y is a function of x in the Bernoulli sense. That is, the Bernoulli formula function concept is included in the Dirichlet definition. Likewise, nothing is said about the correspondence between y and x being representable by means of a curve. If, however, it is thus representable, then y is a function of x in the Fourier geometric curve sense. That is, the Dirichlet definition also includes the arbitrary geometric curve function concept. It is instructive to note some examples of functions in the Dirichlet sense which would not be included under the earlier definitions. For definiteness let the fixed numbers a and b be given by $a = 1$, $b = 4$.

First example: For any value of x , let y be the smallest integer greater than x .

Second example: When x is rational, let y be 1; when x is irrational, let y be $1/2$.

The peculiarities of these functions are best seen if one attempts to construct their graphs. For the first example, the

graph consists of a sequence of parts of straight lines, each having a definite left extremity but no definite right extremity. For the second example the graph consists of two sets of points, one on the line $y = 1$, the other on the line $y = 1/2$. Each set contains an unlimited number of points. Furthermore, if x increases gradually from any initial value, by any amount however small, the point of which x is the abscissa oscillates an unlimited number of times between the two lines.

The consideration of functions presenting such novel characteristics has received a great deal of attention during the last half century. Persons who are unsympathetic with the critical tendency of modern mathematics are apt to look with disfavor upon such investigations. It cannot be insisted too strongly that such a view is unwarranted and unjust. These investigations have contributed very greatly to the building of the powerful machinery of modern analysis. In particular, our real number system was evolved from an unsatisfactory condition admitting contradictions and paradoxes to its present state of consistency and efficiency by just such considerations on the part of the mathematicians Dedekind and Cantor.

We now come to the latest extension of the function concept, a precise formulation of which is a product of the last ten years, though the essence of the idea is much older. I warn you in advance that it is abstract. But I also remind you that it is the abstractness of mathematics which makes it of universal application. I will also endeavor to forestall possible adverse criticism by stating in advance that this generalization has already justified itself by unifying fields previously considered distinct, thus mutually enriching them as did the discovery of Descartes in the seventeenth century. Let me also encourage you by saying that it is quite as simple as the idea of Descartes. It is nothing more nor less than the Dirichlet definition generalized by the removal of the restriction that x and y must represent real numbers. To get the thing before us clearly, I first define what we shall mean by a variable, and then define the expression function of a variable.

Definition of Variable: A variable is simply a symbol, say p , which in a given discussion may be used to denote any one of a given set of objects. The set of objects, any one of which may be represented by p , is called the *range* of the variable p .

Definition of Function of a Variable: If p is a variable whose range is a class of objects P , and q is a second variable whose

range is a class of objects Q , and if the variable q is so related to the variable p that when p is given a definite value on the range P , q has a definite value on the range Q , then q is said to be a function of p , or more explicitly, a function on the range P , to the range Q . We may express the functional relation notationally by the equation

$$q = f(p).$$

If in particular the range P of p is the class of real numbers between two limits a and b , and the range Q of q is the class of real numbers, this definition reduces to that of Dirichlet. The generality of our present definition lies solely in the generality of the ranges of the two variables.

Let us see what are some of the functions relative to ranges other than the class of real numbers which have appeared as important in various branches of mathematics.

First Example:

P = [all complex numbers],

Q = [all complex numbers].

The study of functions $q = f(p)$ relative to these ranges constitutes the theory of functions of a complex variable.

Second Example:

P = [all points in a closed region of a plane],

Q = [all real numbers].

A function $q = f(p)$ relative to these ranges is sometimes called a numerical function of a point. Such functions are most common in mathematical physics, the numerical function representing for example the temperature at a variable point in a thin conductor, or the potential at a variable point in a plane field of force. Since the points in a plane can be put into one-to-one correspondence with the pairs of real numbers (x, y) by the method of Descartes, it is obvious that the study of the numerical functions of a point is logically equivalent to the study of the functions $z = f(x, y)$ of two real variables.

Third Example:

P = [all points in a closed region of a plane],

Q = [all vectors in a plane].

A function $q = f(p)$ relative to these ranges is sometimes called a vector function of a point. Such a function would represent for example the direction and rate of the flow of heat in a thin conductor, or the direction and intensity of the force active at any point in a plane field of force.

Fourth Example:

P = [all vectors in a plane and radiating from a fixed point],

Q = [all vectors in a plane].

A function $q = f(p)$ relative to these ranges is a vector function of a vector. The study of such functions is called *vector analysis*.

Fifth Example:

P = [a sequence of integers],

Q = [all real numbers].

Functions relative to these ranges are furnished by most tables of data. For example, suppose the range of p to be the sequence of days in a month numbered from 1 to 30, and the corresponding values of q to be the corresponding mean temperatures.

Sixth Example:

P = [all curves in a plane and joining two fixed points],

Q = [all real numbers].

A function $q = f(p)$ relative to these ranges is called a numerical function of a curve. One example of such a function would be the length of a curve. Since we have seen that every such curve can be represented by a formula function, the study of functions of a curve is logically equivalent to the study of numerical functions of a formula function. A small part of the theory of numerical functions of a variable curve, having to do with the maxima and minima of certain of these functions, has long been a subject of study under the name Calculus of Variations. It is only very recently (1913) that a work dealing with the general theory of functions of curves has appeared. It is by the eminent Italian mathematical physicist, Volterra.

Seventh Example:

P = [an absolutely arbitrary set of elements],

Q = [all real numbers].

The study of functions $q = f(p)$ relative to these ranges constitutes a branch of mathematics recently introduced by Professor E. H. Moore, and called General Analysis. It is of course very abstract, but for that reason its generality is great. It includes several of our earlier examples as special cases.

You will recall that my undertaking was to present a concept which to some extent at least forms a unifying concept for mathematics. Perhaps the numerous examples I have presented have persuaded you at least that the modern function concept is of very wide application. Before closing, I think it may be not without interest to examine those elementary branches of mathematics with which we are all as teachers or students of mathematics in vital contact, and to see if the function concept plays a fundamental role in them. I hope that in this investi-

gation I may not be misunderstood. I am not recommending the putting forward of this general and abstract concept as a pedagogical medium of instruction, at least in the elementary subjects; but I use these as familiar subjects for the testing as to whether the concept as a logical concept has a fundamental role in our mathematical structure as a whole. If it has, then the teacher should have the concept in the back of his head when he teaches algebra and geometry even though he brings it not out of his mouth.

Consider first algebra. Strip it of its applications, and look at the skeleton to see if we can discover a class of elements P , a class Q and a type of function $q = f(p)$ which is fundamental to algebra. I think we will all agree that the fundamental elements studied in algebra are numbers—let us say complex numbers. I now suggest for the classes P and Q :

P = [all *pairs* of complex numbers (a, b)],

Q = [all single complex numbers].

I then submit that the bulk of the logical structure of algebra is the study of the two functions

$q = f(p) = f(a, b) = a + b$,

$q = \phi(p) = \phi(a, b) = a \times b$.

There is a third function which plays a role, namely a function on the range

$P' =$ [all pairs of numbers (a, n) of which a may be complex but n is a real rational number],

$q = \psi(p') = a^n$.

The logical skeleton of algebra is the study of the three functions f , ϕ , and ψ .

Next consider geometry. What are the elements studied in plane geometry? Perhaps we will agree they are points and lines. Can we think of any functional relation existing between these elements? Within the first few pages of almost any geometry text will appear the statement "Two points determine a line." This is a statement that there exists a function on the class of pairs of points to the class of lines. Let us then take

P = [all pairs of points (A, B) in the plane],

Q = [all lines in the plane],

and for the function on P to Q , take

$q = J(p) = J(AB) =$ junction line of A and B .

But we might equally well be guided by the statement "Two lines determine a point" (the point determined by two parallel lines being an ideal point at infinity), in which case our ranges would be

P = [all pairs of lines (a, b) in the plane],
 Q = [all points in the plane (including ideal points at infinity)],
 and

$q = J(p) = J(a, b)$ = junction point of a and b .

The study of this junction function, the range of whose variable consists of either pairs of points or pairs of lines, leads to that part of geometry known as the Geometry of Position or Projective Geometry. The interchangeability of the roles of points and lines in the functional relation leads to the beautiful theory of duality, which is so unhappily lost in our usual treatment of Euclidian geometry.

Much of the *Euclidian* geometry is based on a function relative to the range

$Q' =$ [all real numbers],

the class P being either of the classes defined above. This function is

$q' = M(p) = M(AB)$ = measure of segment AB ,
 or $q' = M(p) = M(ab)$ = measure of angle ab

The part of geometry depending upon this measure function is sometimes called metric geometry.

The remaining subjects of our curriculum: trigonometry, analytic geometry, and calculus, require little comment. It will suffice to point out the ranges and the nature of the functions of which they treat.

Trigonometry: P = [all angles],

Q = [all real numbers].

Trigonometry studies six well defined functions on P to Q .

Analytic Geometry: P = [all points P in a plane],

Q = [all pairs of real numbers].

The analytic geometry differs from the geometry we have already considered in its dependence upon the functional relation of Descartes,

$q = f(p) = f(P) = (x, y)$ = the coördinates of P .

Calculus: P = [all real numbers],

Q = [all real numbers].

Elementary calculus may be defined as a method of investigating those functions

$$q = f(p)$$

which can be represented by formulas of certain simple types.

We have traced the development of the function concept from its unpretentious origin as a power of a variable to its present very general signification. I think we have verified by a sufficient

number of examples that to some extent at least it is a unifying concept. Is it indeed characteristic of mathematics? Many attempts have been made to answer the question "What is mathematics?" Two different types of definition have been attempted, one aiming to define the science by stating the *methods* used, the other aiming to define it by stating the kind of *material* studied. Few definitions of the first type excel for concentrated suggestiveness that of Benjamin Peirce in which he defined mathematics as "the science which draws necessary conclusions." I suggest for your consideration a definition of mathematics in terms of the kind of material studied as "the science which treats of functional relations."

PERIODIC DECIMAL FRACTIONS.

BY M. O. TRIPP,

University of Maine.

The object of this article is to show in a concrete way the relation existing between the number of digits in the period of a repetend and the theory of number congruences. An attempt has been made to explain this relation in such a way that the reader who has no previous knowledge of number theory may readily follow the discussion.

Let us start by taking a common fraction m/n , in which m and n are positive integers relatively prime to each other. We assume that the denominator does not contain either the factor 2, or the factor 5; for if 2 or 5, or both of them, were factors of n , we can multiply the fraction by 10^k , k being a properly chosen positive integer, and thus make the factor 2, or the factor 5, cancel from the denominator; then in our resulting decimal we may divide by 10^k , merely by moving the decimal point k places to the left, giving us a decimal equal to the original fraction. As a concrete illustration of the removal of such factors, let us consider the fraction $1/12$. We multiply this fraction by 10^2 , since 12 contains the factor 2^2 . Therefore,

$$(1/12)10^2 = 8\frac{1}{3} = 8.333 \dots$$

Dividing 8.333 . . . by 10^2 , we have

$$1/12 = .08333 \dots$$

In general, if $n = l 2^a 5^b$, we must take k , the exponent of 10, equal to the largest of the numbers a and b , or if $a = b$, k must be taken equal to their common value. If $l = 1$, that is, if the denominator contains only the factors 2 and 5, or either of them, then the fraction m/n is equal to a finite decimal.