ON THE REPRESENTATION OF A GROUP OF FINITE ORDER
AS AN IRREDUCIBLE GROUP OF LINEAR SUBSTITU-
TIONS AND THE DIRECT ESTABLISHMENT OF THE
RELATIONS BETWEEN THE GROUP-CHARACTERISTICS

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It is hardly too much to say that the methods by which new ideas are introduced and new results obtained in any branch of mathematical science are in general indirect. In fact the historical development of a subject rarely follows the lines of the logical presentment that can be given when a sufficient number of results and their connexion with each other have become familiar.

A case in point is the theory of group-characteristics and the allied subject of the representation of a group of finite order as a group of linear substitutions. In the remarkable series of memoirs, published in the *Berliner Sitzungsberichte*, in which Herr Frobenius developed this theory, the actual starting point is an algebraical theorem connected with permutable matrices; and the methods made use of in the subsequent discussion turn largely on the properties of matrices. In the paper on "Group-Characteristics" * in which I have given an account of Herr Frobenius's theory the methods used, though quite distinct from his, are at least as indirect. They involve the consideration of a continuous group from whose properties those of the group-characteristics are deduced.

The present paper is an attempt to deal with the theory of the representation of a group of finite order as a group of linear substitutions and the properties of the group-characteristics from what may be described as a self-contained point of view. No considerations or ideas are introduced foreign to those involved in the conceptions of an abstract group of finite order and of a group of linear substitutions of finite order. Moreover, the theorems made use of are those which lie at the foundations of the subject. It will be seen that the order in which the various parts of the subject are considered differs materially from that in which they occur in Herr Frobenius's memoirs. First, the complete reducibility (defined below) of a group of linear substitutions of finite order is taken; secondly,

* *Proc. Lond. Math. Soc.*, Vol. xxxiii., pp. 146-162. This paper will be referred to as "G.-C."

the number of distinct irreducible representations ; thirdly, the composition of the irreducible representations ; and, lastly, the group-characteristics and their properties.

1. Let $S_1, S_2, \dots, S_i, \dots$ be the operations of an abstract group G , and let $s_1, s_2, \dots, s_i, \dots$ be a set of linear substitutions in n variables such that, for every pair of suffixes i and j , if $S_i S_j = S_k$, then $s_i s_j = s_k$. The set of linear substitutions then form a group g which is isomorphic with G . This group of linear substitutions is said to constitute a *representation* of G , the substitution s_i being the representative of the abstract operation S_i . One and the same group of linear substitutions may thus give two or more representations of a given abstract group.

Let $s'_1, s'_2, \dots, s'_i, \dots$ be a second set of linear substitutions, in the same number n of variables, which gives a representation of G , s'_i being the representative of S_i . If it is possible to find a linear substitution t in the n variables, such that, for each suffix i , $t^{-1} s_i t = s'_i$, the two representations of G given by the first and second sets of linear substitutions are said to be *equivalent*. If, on the other hand, no such substitution as t exists, the representations are called *distinct*.

A group of linear substitutions in n variables is called *reducible* when it is possible to find $n' (< n)$ linear functions of the variables which are transformed among themselves by every substitution of the group. I propose to call a group of linear substitutions *completely reducible* when it is possible to choose the variables in such a way that they fall into sets with the properties (i) that the variables of each set are transformed among themselves by every substitution of the group ; and (ii) that the group of linear substitutions in each separate set is irreducible.

2. In a memoir to appear shortly in the *Acta Mathematica* I have dealt with the questions of the reducibility of a group of linear substitutions of finite order, and of the number of distinct representations of a group of finite order as an irreducible group of linear substitutions. The discussion is based solely (i) on the possibility of representing any group of finite order as a regular group of permutations, and (ii) on the existence of at least one invariant Hermitian form for any group of linear substitutions of finite order.* These two theorems are fundamental in connexion with groups of finite order, and the results arrived at in the memoir referred to are deduced from them without introducing any extraneous considerations, and in particular without any reference to group-characteristics.

*Loewy, *Comptes Rendus*, Vol. cxxiii., pp. 168-171 (1896) ; and Moore, *Math. Ann.*, Vol. L., pp. 213-219 (1898).

These results are (i) that every group of linear substitutions of finite order is completely reducible,* and (ii) that the number of distinct representations of a group of finite order as an irreducible group of linear substitutions is equal to the number of sets of conjugate operations contained in the group.

I propose here, starting from these results, as acquired by direct and comparatively simple reasoning, to deduce by a further direct process the properties of the group-characteristics.

3. Let G be any abstract group of finite order n with r sets of conjugate operations, and denote by $g_1, g_2, \dots, g_r$ the r distinct representations of G as an irreducible group of linear substitutions. One of these, which will always be denoted by g_1 , necessarily operates on a single variable and consists of the identical operation only. If g is any representation of G as a group of linear substitutions, the complete reducibility of g will be represented by the equation

$$g = \sum_1^r a_i g_i,$$

where the a 's are positive integers or zeros; and the equation expresses that the variables in g may be so chosen that they fall into sets; there being, for each i , a_i sets each of which undergoes an irreducible group of linear substitutions equivalent to g_i .

If in each of the substitutions $s_1, s_2, \dots, s_i, \dots$ of g_i every coefficient is replaced by its conjugate imaginary, a new set of substitutions $s'_1, s'_2, \dots, s'_i, \dots$ arises for which $s'_i s' = s'_k$ if $S_i S_j = S_k$. This new set of substitutions therefore constitute a group, which is obviously irreducible, and gives rise to an irreducible representation of G , which must be included among $g_1, g_2, \dots, g_r$. If this representation is equivalent to g_i , then g_i is called *self-inverse*. If it is distinct, it may be denoted by g_x , and then g_i and g_x are called *inverse* representations.

Since g_i is a group of finite order, any one of its substitutions s_j is of finite order. Now a linear substitution of finite order can always be transformed so that, if m is the number of variables operated on, it takes the form †

$$x'_1 = \omega_1 x_1, \quad x'_2 = \omega_2 x_2, \quad \dots, \quad x'_m = \omega_m x_m,$$

* Independently of the investigations of Herr Frobenius, this result has been directly proved by Maschke, *Math. Ann.*, Vol. LII., pp. 363-368 (1899); and more recently, as the particular case of a more general theorem, by Loewy, *American Transactions*, Vol. IV., pp. 44-64 (1903).

† Many proofs have been given of this theorem; but, in the present connexion, I may refer to one I have given (*Messenger of Mathematics*, Vol. XXVIII., pp. 111-114) which involves no reference to the characteristic determinant, and depends directly and solely on s being of finite order.

where $\omega_1, \omega_2, \dots, \omega_m$ are roots of unity. The ω 's are called the *multipliers* of the substitution, and their sum $\omega_1 + \omega_2 + \dots + \omega_m$ is defined to be the *characteristic* of S_j in the irreducible representation g_i . Any other substitution s_k of g_i may similarly be transformed to the form

$$x'_1 = \varpi_1 x_1, \quad x'_2 = \varpi_2 x_2, \quad \dots, \quad x'_m = \varpi_m x_m.$$

If S_j and S_k are conjugate operations of G , then s and s_k are conjugate substitutions of g_i ; and therefore

$$x'_1 = \omega_1 x_1, \quad x'_2 = \omega_2 x_2, \quad \dots, \quad x'_m = \omega_m x_m$$

can be transformed into

$$x'_1 = \varpi_1 x_1, \quad x'_2 = \varpi_2 x_2, \quad \dots, \quad x'_m = \varpi_m x_m.$$

It may be immediately verified that this is only possible when the two sets of roots of unity $\omega_1, \omega_2, \dots, \omega_m$, and $\varpi_1, \varpi_2, \dots, \varpi_m$ differ only in the sequence in which they are written. Hence two conjugate substitutions of g_i have the same multipliers; and the characteristics of two conjugate operations of G in g_i are equal to each other.

To the r conjugate sets of operations in G there will therefore correspond r characteristics in g_i which may be denoted by $\chi_1^i, \chi_2^i, \dots, \chi_j^i, \dots, \chi_r^i$. Taking, as will be done throughout, the first conjugate set to be that which consists of the identical operation only, χ_1^i is simply the number of variables operated on by g_i , and, if p is the order of any operation of the j -th set, χ_j^i is the sum of χ_1^i p -th roots of unity. To each irreducible representation of G there belongs such a set of characteristics, the characteristics in the set belonging to g_1 each being unity. Further, it follows at once from the definition of inverse representations that, if g_i and g_j are inverse representations, then χ_j^i and χ_i^j are conjugate imaginaries for each j ; while, if g_i is a self-inverse representation, χ_j^i is real for each j . So also, if the j -th and the j' -th are inverse conjugate sets (*i.e.*, if S belongs to one when S^{-1} belongs to the other), then χ_j^i and $\chi_{j'}^i$ are conjugate imaginaries (or are real and equal); while, if the j -th is a self-inverse set, χ_j^i is real for each i .

If the two irreducible representations g_i and g_j are set up in two distinct sets of χ_1^i and χ_1^j variables, the $\chi_1^i \chi_1^j$ products of the variables undergo a linear substitution for every operation of G . This group may be represented by $g_i g_j$, and the equation which expresses its complete reduction is

$$g_i g_j = \sum_1^r d_{ijk} g_k.$$

This equation implies ("G.-C.," p. 161) that for the characteristics of each

conjugate set the equation

$$\chi_s^i \chi_s^j = \sum_1^r d_{ijk} \chi_s^k$$

holds.

If d_{i1} is not zero, the above equation implies that there are d_{i1} invariant bilinear functions of the two sets of χ_1^i and χ_1^j variables. Denote these variables by $x_1, x_2, \dots, x_{\chi_1^i}$ and $y_1, y_2, \dots, y_{\chi_1^j}$, and let χ_1^i be not greater than χ_1^j . Any bilinear function of the x 's and y 's can then be expressed in the form $x_1 Y_1 + x_2 Y_2 + \dots + x_{\chi_1^i} Y_{\chi_1^i}$. If this is invariant when the x 's and y 's undergo simultaneously the corresponding substitutions of g_i and g_j , then $Y_1, Y_2, \dots, Y_{\chi_1^i}$ must be transformed among themselves by every operation of g_j . Since g_j is irreducible, this is only possible if $\chi_1^j = \chi_1^i$; and when this condition is satisfied there can only be one such invariant function. In fact, if both

$$x_1 y_1 + x_2 y_2 + \dots + x_{\chi_1^i} y_{\chi_1^i} \quad \text{and} \quad x_1 Y_1 + x_2 Y_2 + \dots + x_{\chi_1^i} Y_{\chi_1^i}$$

were invariant, then $y'_1 = Y_1, y'_2 = Y_2, \dots, y'_{\chi_1^i} = Y_{\chi_1^i}$ would be permutable with every operation of g_j , which is impossible since g_j is irreducible. When g_i is assigned the corresponding substitutions of g_j are completely defined by the condition that $x_1 y_1 + x_2 y_2 + \dots + x_{\chi_1^i} y_{\chi_1^i}$ shall be invariant; so that corresponding to a given i there is just one j such that $d_{i1} = 1$.

On the other hand, the theorem of the existence of an invariant Hermitian form for g_i is equivalent to the assertion that $g_i g_v$ has an invariant bilinear form. Hence

$$d_{ii'1} = 1, \quad d_{ij1} = 0 \quad (j \neq i').$$

4. Let now

$$x_s = \sum_t a_{stl_a} x_t \quad (s, t = 1, 2, \dots, \chi_1^i)$$

be the representative in g_i of one of the operations of G belonging to the j -th conjugate set, the h_j different operations of the set being given by the values 1, 2, ..., h_j of the suffix a . When transformed by any substitution of g_i , these h_j substitutions are permuted among themselves. Hence the linear substitution

$$x'_s = \sum_t x_t \sum_a a_{stl_a} \quad (s, t = 1, 2, \dots, \chi_1^i)$$

is permutable with every substitution of g_i . But, since g_i is irreducible, the only substitutions permutable with every substitution of g_i are those which multiply each variable by the same number. Hence

$$\sum_a a_{stl_a} = 0 \quad (s \neq t) \quad \text{and} \quad \sum_a a_{ssl_a} = \lambda.$$

But

$$\sum_s a_{ssl:a} = \chi_j^i,$$

and therefore

$$\lambda = \frac{h_j \chi_j^i}{\chi_1^i}.$$

Hence, if C_j is the sum of the operations of G which belong to the j -th conjugate set, then, in the representation of G given by g_i , C_j is represented by the substitution

$$x'_s = \frac{h_j \chi_j^i}{\chi_1^i} x_s \quad (s = 1, 2, \dots, \chi_1^i).$$

Now the sums of the operations of G which belong to the different conjugate sets satisfy the equations ("G.-C.," p. 148)

$$C_j C_k = \sum_s c_{jks} C_s,$$

where c_{jks} are positive constants (or zeros) which can be calculated from the multiplication table of the group. These relations, being in fact identities, must hold between the expressions which correspond to the C 's in any concrete representation. But in the concrete representation here considered

$C_j C_k$ is the substitution which multiplies each variable by $\frac{h_j h_k \chi_j^i \chi_k^i}{(\chi_1^i)^2}$;

while $\sum_s c_{jks} C_s$ is the substitution which multiplies each variable by

$\sum_s c_{jks} \frac{h_s \chi_s^i}{\chi_1^i}$. Hence, for all values of i, j, k ,

$$h_j h_k \chi_j^i \chi_k^i = \chi_1^i \sum_s c_{jks} h_s \chi_s^i. \quad (i)$$

From this system of relations, and those given by $d_{ii} = 1$, $d_{ij} = 0$, the complete system may be deduced algebraically.

If each side of (i) is multiplied by $\chi_{i'}^l$, while $c_{jks} h_s$ is replaced by $c_{j's'k'} h_{i'}$ ("G.-C.," p. 149), it becomes

$$h_j \chi_j^i h_k \chi_k^i \chi_{i'}^l = \chi_1^i \sum_s \chi_s^i c_{j's'k'} h_{i'} \chi_{i'}^l.$$

Summing each side with respect to k ,

$$h_j \chi_j^i \sum_k h_k \chi_k^i \chi_{i'}^l = \chi_1^i \sum_s \chi_s^i \frac{h_j h_{i'} \chi_j^i \chi_{i'}^l}{\chi_1^i}.$$

Hence

$$(\chi_1^l \chi_j^i - \chi_1^i \chi_j^l) \sum_k h_k \chi_k^i \chi_{i'}^l = 0$$

for all values of i, l, j .

In this put $l = 1$. Then, since χ_1^i and χ_j^i cannot be equal for all values of j ,

$$\sum_k h_k \chi_k^i = 0$$

for all values of i except unity.

Now in the equation $\chi_k^i \chi_k^{i'} = 1 + \sum_s d_{is} \chi_k^s$,

Σ' extending to all values of s except unity, multiply each side by h_k , and sum with respect to k . Then

$$\sum_k h_k \chi_k^i \chi_k^{i'} = \sum_k h_k + \sum_s d_{is} \sum_k h_k \chi_k^s = n.$$

Similarly, from $\chi_k^i \chi_k^j = \sum_s d_{is} \chi_k^s$, ($j \neq i'$),

it follows that $\sum_k h_k \chi_k^i \chi_k^j = 0$ ($j \neq i'$).

The two sets of relations $\sum_k h_k \chi_k^i \chi_k^{i'} = n$, (ii)

$$\sum_k h_k \chi_k^i \chi_k^j = 0 \quad (j \neq i') \quad \text{(iii)}$$

give the ratio of the first minors in the determinant of the χ 's; and thence the two remaining sets of relations

$$\sum_i \chi_k^i \chi_{k'}^i = \frac{n}{h_k}, \quad \text{(iv)}$$

$$\sum_i \chi_k^i \chi_l^i = 0 \quad (l \neq k') \quad \text{(v)}$$

immediately follow. The relations (i), (ii), (iii), (iv), and (v) are, in fact, the relations which connect the group-characteristics ("G.-C.," pp. 151, 153).