Every method that will produce this clearness of vision, this definiteness of outline, this certainty in reasoning, this usefulness of effort, this vividness of aim should be welcomed by the teacher of mathematics. Away with traditions; growth is always a breaker of traditions. The day is coming, it has already dawned in some schools, when the desire expressed by Professor Sylvester will be fulfilled. I believe that it is in the mathematical laboratory that these changes will be worked out. Here and only here can we succeed in answering the challenge. I firmly believe that the same kind of a transformation will be effected in the teaching of mathematics that has been made in physics, chemistry, biology and psychology. It means labor; it means successes and failures; but it means in the end a vivifying, an enriching, a focusing of meaning, which is foreign to us today, but which is essential. This is no vain prophecy; already have I seen the results of a beginning, a beginning made in a small way but nevertheless a beginning, and I cannot close my eyes to what they indicate.

ALGEBRA EVOLVED FROM THE LEARNER'S EXPERIENCE.

ARTHUR C. LUNN.

Associate in Applied Mathematics, University of Chicago.

The following paragraphs are intended to suggest the need of systematic development in courses of algebra in the high school, of a number of hints, due to different teachers, which have grown out of their efforts to make the transition from arithmetic to algebra more natural, and to relieve the latter subject of that abstractness in fundamental points which is apt to make it seem so remote from the other interests of teacher and pupil.

Probably most students obtain at last a more or less definite idea that algebra owes its extended interest and usefulness, as compared with arithmetic, to the introduction of certain new ideas with respect to the nature and representation of numbers, and to the persistent following out of certain other notions which have lain dormant, as it were, in the arithmetic already familiar. Chief among the notions of the latter class are the formal laws of

operation and the idea of the equation, borrowed from arithmetic so far as concerns the unsigned numbers; of the former, the use of letters as symbols of numbers, and the notion of the signed number, with the corresponding extensions of meaning of many important concepts.

But it may be doubted whether this sort of understanding of algebra, of the value of its characteristic ideas and methods, presupposing as it does a certain maturity of mind, is not reached by the student too tardily to relieve the dissatisfaction sure to be caused by the seemingly arbitrary character of the new concepts, as often presented; whether the great power and real simplicity of the new realm of ideas are not largely offset by their apparent artificiality.

Every earnest teacher of algebra has probably felt this difficulty and worked out some way of meeting it. It is therefore believed that an outline of some of the suggestions available may prove useful, by contributing to the interchange of ideas concerning problems in the teaching of subjects having abstract phases, of which the problem in hand may be taken as a type.

It is doubtful whether conscious attention to any great extent need be directed to the formal laws relative to the signless numbers of arithmetic until quite late in the student's career. The facts expressed by the laws of commutation, association, and distribution, as to whole numbers, are chiefly *elements of freedom in the process of counting*, and as such are unconsciously familiar very early; no one needs to be told that the total number of a set of things can be found in many different ways, the choice being guided chiefly by convenience. The mode of expression in speaking of *the* product or *the* sum of a set of numbers implies indifference as to the order of factors or terms; while the parentheses, whose insertion and removal according to the associative law, later complicated with the law of signs, are often employed in a tiresome sort of way for algebraic gymnastics, really stand for halting-places in what is intended to be an economic method of counting by groups; or perhaps they are simply signs which direct the patient reckoner to pause for a long breath before he pursues his weary counting way.

What is really required seems to be simply a sufficient reali-

zation of the freedom expressed in these conventional forms; to be gained best perhaps by counting in different ways, singly or in groups, a set of objects whose number is actually required to be known for other purposes. Thus, in Fig. 1 the area of the rectangle in terms of a small square as unit may be obtained in the forms:

FIG. 1.

66 (by direct counting), 6×11 , $6 \times (4+7)$, $11 \times (4+2)$, $(4+2) \times (4+7)$, and many others. Counting by groups serves to relieve the eye, while the geometric experiment indicated has another value in suggesting the general formula for the area of the rectangle.

The extension of this point of view to fractions is immediate, since the operations on fractions are reduced to those on whole numbers, in a way corresponding to a mode of counting with an aliquot part of the original single thing as unit. Illustrations are abundant in all kinds of denominate numbers. What is important, however, is that these concrete cases should be first considered, and the more abstract theory of fractional numbers left to be sifted out later on.

With the literal symbolism, the sign-bearing number, the widened meaning of the old words, the case is different; these are new ideas, not old ones more deeply realized. The most fruitful suggestion here seems to have been to have recourse to a method of development parallel with that of the original growth of algebra; to introduce each new concept *exactly when its need is felt* by the student as a result of varied experience in the employment of mathematical modes of thinking about objects and events in daily life which have a definite interest of their own, which are not merely hauled in by the ears to serve as unwilling witnesses to the divine right of certain vaguely seen and apparently usurping ideas to reign on the throne of abstract mathematics. From life to thought, from the varied details of every day's activity in seeing, hearing, feeling, to the powerful gen-

eralizations, abstract mental creations, of modern science, by a firmly trodden path, guided always by the desire to discover the most economical, most fruitful ways of thinking; such as this would seem to be the motto whose guidance has enabled many to release mathematical instruction at least in part from its isolation, and let it demonstrate the value of its functions in an organic educational system.

The conventions of algebra were made to use; they exist in their present form as the result of successive improvements of a mental tool forged at a time when perhaps no other function was dreamed of for mathematics; and even today this view may well be the dominant one in early work, giving place to others also, as fast as the student can be led to realize the significance, rational, esthetic, ethical, of an ideal in the sense of a perfected logical system of thought, gradually unfolded, and still in process of growth. Most students, however, never reach this latter view in the high school, if at all, and to them an emphasis solely on the logical aspects of mathematics acts more as a constraint than as an inspiration.

The use of letters seems to grow naturally out of the use of abbreviated forms of writing. In fact, it is useful to write equations originally in words, to shorten the notation gradually, and only after a fair acquaintance is gained with the nature of the equation to proceed to the conventional literal symbolism, which is thus to be justified solely by its greater convenience. Even then the best letters to choose are usually the initials of words in the concrete statement which is translated into equational form. Variety, flexibility in the choice of notation is a great help toward the proper subordination of all symbolism to understanding.

The general concept of the signed number is better left to ripen gradually out of an abundant experience in thinking and speaking about concrete magnitudes admitting of an opposition in direction or sense. The earliest statements about such magnitudes may well be written out in words, but as soon as a certain familiarity with literal arithmetic is available, one might have such equations as the following, referring to North and South journeys:

Na miles and then Nb miles amounts to $N(a+b)$ miles and corresponding ones in shorter form:

$$Na \text{ th } Sb, \text{ or, } Sb \text{ th } Na, \text{ gives } \begin{cases} N (a-b) & \text{if } a > b \\ S (b-a) & \text{if } a < b \end{cases}$$

Similar statements would apply to the rising and falling of the thermometer, N and S being replaced by R and F . For the forward or backward motion of a train, and of a brakeman upon the train, the signs of quality might be F and B , and the symbol of connection "&," since the two motions, of train and brakeman, might be simultaneous; thus:

$$B12 \text{ \& } B4 \text{ } m \text{ } B16 \quad (m \text{ for makes})$$

The next step might be the use of the sign of equality to indicates "makes," "amounts to;" but this is an extension of meaning and should not be hastened. Following this is the recognition that the same two marks of quality might be used continually, to stand not only for "forward," "backward," but for any pair of contrasted directions, real or metaphorical. The two signs chosen should be suggestive; for instance, two opposite arrows, or the two dagger marks † ‡. *It is better not to use at first the signs of "plus" and "minus" as symbols of quality; their identification with the symbols of addition and subtraction is a fertile source of confusion, and is probably the chief trouble in the way of a common sense understanding of the addition and subtraction of directed magnitudes.*

With respect to the operations on what will later be considered as signed numbers, the same spirit of treatment can be main-

FIG 2.

tained; a single instance must suffice here. Fig. 2 gives the scheme of a lever pivoted at P , and a force is to be applied as a pull up or down on the right or left arm; the problem is to

determine the resultant force at the point F at the right, whose distance from the pivot is taken as the unit of length. Experiment gives: A force up of 3 pounds at a point 2 units to the right gives a resultant force up of 6 pounds:

$(Ua \text{ on } Rb) \text{ or } (Ub \text{ on } Ra) \text{ } g \text{ } Uab$

$(Ua \text{ on } Ll) \text{ or } (Ub \text{ on } La) \text{ } g \text{ } Dab$

$(Da \text{ on } Rb) \text{ or } (Db \text{ on } Ra) \text{ } g \text{ } Dab$

$(Da \text{ on } Ll) \text{ or } (Db \text{ on } La) \text{ } g \text{ } Vab$

which furnish one concrete suggestion of the need of the commutative and sign laws in the multiplication of signed numbers. *These laws are established by convention* precisely to make it possible to include all special cases in one general statement such as this: "The resulting moment is the product of force and lever arm." It is a matter of experiment to show that the same conventions are suggested wherever the product of two magnitudes occurs in the study of concrete phenomena.

But the abstract formulation of these conventions had better be postponed until most of the notions of elementary algebra are thoroughly familiar in this naïve, non-technical way.

PURE AND APPLIED MATHEMATICS.*

BY ELIAKIM HASTINGS MOORE.

Professor and Head of the Mathematical Department, University of Chicago.

In the ultimate analysis for any epoch, we have general logic, the mathematical sciences, that is, all special formally and abstractly deductive self-consistent sciences, and the natural sciences, which are inductive and informally deductive. While this classification may be satisfactory as an ideal one, it fails to recognize the fact that in mathematical research one by no means confines himself to processes which are mathematical according to

* This paper and two others which are to follow are extracts from Mr. Moore's presidential address delivered before the American Mathematical Society at its ninth annual meeting, December 29, 1902, *On the Foundations of Mathematics*. In the papers to appear in our next two issues Mr. Moore considers carefully and sympathetically the weaknesses of current mathematical teaching in elementary and secondary schools, blazes out lines along which effort at improvement should be directed in these schools as also in colleges, adduces reasons for the wide adoption of laboratory methods in mathematics classes, and calls upon friends of progress in mathematical methodology everywhere to join in a concerted movement for the energizing of mathematical teaching. The cause, as also its advocacy in these three papers, are so worthy and so timely, that our readers can profitably spend time for our next two issues, studying them.