WHAT IS A RATIO?¹

BY PROFESSOR JULIAN L. COOLIDGE.

Harvard University, Cambridge.

I do not know what a ratio is, but I want to learn. I asked my pupils in solid geometry, but their replies were too contradictory to be of service to me. In view of this condition, it is interesting to note that Laisant's book, "*Les Mathématiques*," recommends that ratio should be introduced as soon as number. This view has also the sanction of Newton.

When we begin the study of algebra, we take up monomials and polynomials, addition, subtraction, multiplication, and division, easy equations, elimination, and so on; and we come to fractions, and fractions are hard. But though they are hard, there is no metaphysical difficulty. We understand what fractions are, even though we find them hard to manipulate.

On the other hand, when studying geometry, after passing through the first two books and dealing with matter that is apparently germane to the subject, we come, in the third book, upon a long interlude regarding proportion, full of long words. This takes a lot of time. Is it algebra? If it is algebra, why this hard way of doing easy problems in fractions? I remember when I was studying the subject myself, that I was much puzzled by this question, and I concluded that the mysterious thing about it was the knowledge of ratio which the writer of the book seemed to possess, but which had eluded me. Years later, inspired by Hamilton, I adopted the definition that ratio was the operation which would convert the second term into the first, and I even went so far as to write this sample of a ratio:

Chicken : egg = hatch.

Our text-books are prone to say that ratio may be expressed as a fraction. Why not say that a ratio is a fraction? There are some complications that seem to prevent that definition. In the first place, our inheritance from Euclid, who uses the concept of equal ratios in his treatment of similar figures. As he uses the word, it does not seem to mean a fraction. I have the greatest respect for Euclid's treatment of proportion; it is a monument of human reason, and furnishes rigorous proofs of the theorems with which it deals, without resorting to the incommensurable hypothesis. Our experience, however, has conclusively shown that Euclid's treatment is not teachable.

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The second impediment that complicates our use of the word ratio is confusion as to the meaning of the things that we are comparing. In geometry we continually use the same word in different senses. For example, the word "line" may mean a curved or a straight line; and if we restrict it to "straight line," it may mean the points in an indefinite straight line extending without stop in both directions; or it may mean the points that are in a line beginning at a certain point and extending indefinitely in one direction; or it may mean the points that are included between two end points; or it may mean the number of length units that are contained in a certain specified straight line. This ambiguity occurs in our use of letters and of symbols. For example, we have two rectangles with altitudes a and a' and with bases b and b' , and the rectangle could be represented by R and R' . Then we proceed into an argument in regard to these rectangles, in which we use a and a' , b and b' , etc., as if they were algebraic letters representing numbers. I suppose that our first intention in giving a name to these lines was just the same as when we give a name to a person. We might say, for example, that a and a' would stand for Araminta and Arabella, and b and b' would stand for Belial and Beelzebub, or any other names that we might find appropriate. With that hypothesis, it would be a little difficult for the pupil to guess just what was meant when we multiply Araminta by Belial, or Arabella by Beelzebub, and why any of them, or all of them, should cancel.

Of course you understand that if we were pupils of Halsted's, we should deal with this matter in a strictly logical way, under the regulations of the sect calculus. I do not know whether it would be quite proper, in these liberal days, to have anything to do with the sect calculus in the public schools. It might be disapproved by the school board.

Now that I have made my diagnosis complete by specifying these two causes of confusion, I will make my own suggestion as to the remedy. It seems to me that the pupils should have a clear notion of the process of measurement. If they have observational geometry in the preparatory work, that would be a part of their experience on which they could rely in geometrical study; but anyhow, it is not a difficult matter. With this premise, the ratio of two lines is the fraction formed by taking the length number of one line and dividing it by the length number of another line, and so for the ratio of any two comparable objects. It would be worth while to say at this point that there are two

or three algebraic manipulations that would be useful, and show them in half a page, and then use them; and it would be instructive to consider the effect of changing the unit of measurement, and note how that will effect ratio. In particular, take the second term as unit; the ratio is then the measure of the first in terms of the second.

The pupil cannot deal with irrational ratios until you have told him how to measure a quantity incommensurable with the unit. I think it would pay to have algebra and geometry converge to this point at the same time, so that the irrational numbers—the square root of two, for example—could appear in geometry at the same time that the algebra class was trying to deal with such numbers in its domain. Let us say that a pupil makes a succession of guesses at the square root of 2. The first guess would be 1—too small; the next would be 1.4, the next 1.41, the next 1.414, and so on. I think for a class beginning this subject it would be desirable to express the situation somewhat like this. These are numbers struggling toward an ideal. This ideal is a number whose square is two. They never get there. Illustrate this geometrically, by the attempt to measure the diagonal of a square in terms of a side. If you ask me what that ideal is, I can only reply that it is what these fallible numbers are striving for; that is to say, if you corner me arithmetically, I cannot tell you. All we have, as a matter of fact, is the series of numbers. You will find Hadamard's *Geometry* interesting on this point, especially in its reference to Tannery, who has the easiest discussion of this difficult subject.

We shall have to take on faith that we can combine these incommensurable numbers as we do ordinary numbers. We must also accept without proof that the ideal sought by a series of fractions is the quotient of the ideals of numerators and denominators.

How does all this apply to ratios? Let us try to show that two rectangles with equal bases are to each other as their altitudes. Take the second altitude as the unit of length, the second area as the unit of area; this will not affect either ratio. Then we have a series of numbers which at the same time represent our attempts to measure one altitude in terms of the other, and our attempt to measure one area in terms of the other. The ideal which these numbers are striving for will be the ratio of the altitudes, and also the ratio of the areas.

This method, without being absolutely rigorous, is well in advance of the traditional one, and does not seem to be any more difficult. Moreover, is it not a real gain to be free from all the troubles attendant on that ridiculous old theorem, "If two variables are constantly equal as each approaches its limit, the limits are equal"?

DISCUSSION.

Professor Osgood said that he could not agree with Professor Coolidge in the view that the existence of incommensurable numbers in arithmetic should be brought into connection with the incommensurable case in geometry. The idea of the scale, of a complete set of numbers, in terms of which continuous quantities like arcs and angles and areas can be measured, is one with which the pupil has long since been familiar, and this idea should be presupposed in geometry. Thus each of the quantities whose ratios are to be treated has a numerical measure, and the geometric proposition relates to the equality of two pairs of ratios of numbers and hence asserts that one number is equal to another. In the equation or proportion,

$$\frac{\text{arc } AB}{\text{arc } AC} = \frac{\text{angle } AOB}{\text{angle } AOC}$$

that we find in the geometries each one of the four members is a number, the numerical measure of the geometrical quantity in question, and the equation asserts the equality of the two numbers, namely, the ratio on the left and the ratio on the right.

The incommensurable case is a fact of mathematics. Its treatment need not, however, be made burdensome by introducing difficulties that the pupil would never be troubled with if they were not forced on him in the instruction.

An experiment which is being tried on the Hudson and Manhattan Railroad tunnel system beneath the Hudson River will be watched with much interest by both the railroads and the public. It consists in illuminated station signs, placed inside the cars, which are so arranged that the guard, by pressing a button when the train starts, rings a bell and causes the sign to display the name of the next station. This sign continues to be displayed until the train leaves the station designated. A simple device, this, whose utility is so obvious that it should have been in use from the very commencement of electrically operated rapid transit.