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Lord Rayleigh O.M. F.R.S.

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- L. *On the Work done by Forces Operative at one or more Points of an Elastic Solid.* By Lord RAYLEIGH, O.M., F.R.S.*

AN investigation of the waves generated in an isotropic elastic solid by a periodic impressed force, localized in the neighbourhood of a point, was first given by Stokes†. A simpler treatment of the problem will be found in my paper "On the Light from the Sky, &c."‡, and more fully in 'Theory of Sound,' § 378. It will be desirable to recapitulate the principal steps.

If α, β, γ be the displacements at any point of the solid, X', Y', Z' the impressed forces reckoned per unit of *mass*, we have equations such as

$$\frac{d^2\alpha}{dt^2} = (a^2 - b^2) \frac{d\delta}{dx} + b^2 \nabla^2 \alpha + X', \quad \dots \quad (1)$$

in which a and b are the velocities of dilatational and transverse waves respectively, and δ represents the dilatation expressed by

$$\delta = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} \dots \dots \dots (2)$$

* Communicated by the Author.

† Camb. Phil. Trans. vol. ix. p. 1 (1849); Collected Works, vol. ii. p. 243.

‡ Phil. Mag. xli. pp. 107, 274 (1871); Scientific Papers, i. p. 96

If, as throughout the present paper, α, β, γ , &c. be proportional to e^{ipt} , $d^2\alpha/dt^2 = -p^2\alpha$, and (1) &c. become

$$(a^2 - b^2)d\delta/dx + b^2\nabla^2\alpha + p^2\alpha + X' = 0, \quad . \quad . \quad (3)$$

$$(a^2 - b^2)d\delta/dy + b^2\nabla^2\beta + p^2\beta + Y' = 0, \quad . \quad . \quad (4)$$

$$(a^2 - b^2)d\delta/dz + b^2\nabla^2\gamma + p^2\gamma + Z' = 0. \quad . \quad . \quad (5)$$

These are the fundamental equations. For our purpose we may suppose that X', Y' vanish throughout, and that Z' is finite only in the neighbourhood of the origin. It will be convenient to write

$$k = p/b, \quad h = p/a. \quad . \quad . \quad . \quad (6)$$

The *dilatation* δ is readily found. Differentiating (3), (4), (5) with respect to x, y, z and adding, we get

$$\nabla^2\delta + h^2\delta + a^{-2}dZ'/dz = 0. \quad . \quad . \quad (7)$$

The solution of (7) is

$$\delta = \frac{1}{4\pi a^2} \iiint \frac{dZ'}{dz} \frac{e^{-ikr}}{r} dx dy dz,$$

r denoting the distance between the element at x, y, z near the origin (O) and the point (P) under consideration. If we integrate partially with respect to z , we find

$$\delta = -\frac{1}{4\pi a^2} \iiint Z' \frac{d}{dz} \left(\frac{e^{-ikr}}{r} \right) dx dy dz, \quad . \quad . \quad (8)$$

the integrated term vanishing in virtue of the condition that Z' is finite only within a certain space T. Moreover, since the dimensions of T are supposed to be very small in comparison with the wave-length, $d(r^{-1} e^{-ikr})/dz$ may be removed from under the integral sign.

It will be convenient also to change the meaning of x, y, z , so that they shall represent as usual the coordinates of P relatively to O. Thus, if $Z_1 e^{ipt}$ denote the whole force applied at the origin, so that

$$Z_1 = \rho \iiint Z' dx dy dz,$$

in which ρ is the density,

$$\delta = \frac{Z_1}{4\pi a^2 \rho} \frac{d}{dz} \left(\frac{e^{-ikr}}{r} \right), \quad . \quad . \quad . \quad (9)$$

giving the dilatation at the point P.

In like manner we may find the rotations ϖ' , ϖ'' , ϖ''' , defined by

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 2\varpi', \quad \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 2\varpi'', \quad \frac{d\beta}{dx} - \frac{d\alpha}{dy} = 2\varpi'''. \quad (10)$$

For from (3), (4), (5) we have

$$\nabla^2 \varpi' + k^2 \varpi' + \frac{1}{2} b^{-2} dZ'/dy = 0, \quad . \quad . \quad . \quad (11)$$

$$\nabla^2 \varpi'' + k^2 \varpi'' - \frac{1}{2} b^{-2} dZ'/dx = 0, \quad . \quad . \quad . \quad (12)$$

$$\nabla^2 \varpi''' + k^2 \varpi''' = 0, \quad . \quad . \quad . \quad (13)$$

whence $\varpi''' = 0$, and

$$\varpi' = \frac{Z_1}{8\pi b^2 \rho} \frac{d}{dy} \left(\frac{e^{-ikr}}{r} \right), \quad \varpi'' = -\frac{Z_1}{8\pi b^2 \rho} \frac{d}{dx} \left(\frac{e^{-ikr}}{r} \right). \quad (14)$$

These are the results given in my paper of 1871.

The values of δ , ϖ' , ϖ'' , ϖ''' determine those of α , β , γ . If we take

$$\alpha = \frac{d^2 \chi}{dx dz}, \quad \beta = \frac{d^2 \chi}{dy dz}, \quad \gamma = \frac{d^2 \chi}{dz^2} + w, \quad . \quad (15)$$

where

$$w = k^2 A \frac{e^{-ikr}}{r}, \quad \chi = A \left(\frac{e^{-ikr}}{r} - \frac{e^{-ikr}}{r} \right), \quad . \quad . \quad (16)$$

and

$$A = \frac{Z_1}{4\pi k^2 b^2 \rho}, \quad . \quad . \quad . \quad (17)$$

it is easy to verify that these forms give the correct values to δ , ϖ' , ϖ'' , ϖ''' . As regards the dilatation,

$$\delta = \frac{d}{dz} (\nabla^2 \chi + w),$$

in which

$$\nabla^2 \chi = A \left(-k^2 \frac{e^{-ikr}}{r} + k^2 \frac{e^{-ikr}}{r} \right).$$

This reproduces (9).

As regards the rotations, we see that χ does not influence them. In fact

$$\varpi''' = 0, \quad \varpi' = \frac{1}{2} \frac{dw}{dy}, \quad \varpi'' = -\frac{dw}{dx};$$

and these agree with (14). The solution expressed by (15), (16), (17) is thus verified, and it applies whether the solid be compressible or not.

In the case of incompressibility, $h=0$. If we restore the time-factor e^{ipt} and throw away the imaginary part of the solution, we get

$$z = \frac{Ak^2xz}{r^3} \left[\left(-1 + \frac{3}{k^2r^2} \right) \cos (pt - kr) - \frac{3}{kr} \sin (pt - kr) - \frac{3}{k^2r^2} \cos pt \right], \quad . \quad . \quad . \quad (18)$$

$$\gamma = \frac{Ak^2}{r} \left[\left(1 - \frac{z^2}{r^2} + \frac{3z^2}{k^2r^4} - \frac{1}{k^2r^2} \right) \cos (pt - kr) + \left(\frac{1}{kr} - \frac{3z^2}{kr^3} \right) \sin (pt - kr) - \left(\frac{3z^2}{k^2r^4} - \frac{1}{k^2r^2} \right) \cos pt \right], \quad (19)$$

the value of β differing from that of α merely by the substitution of y for x . The value of A is given by (17), and $Z_1 \cos pt$ is the whole force operative at the origin at time t .

At a great distance from the origin (18), (19) reduce to

$$\alpha = -\frac{Z_1}{4\pi h^2 \rho} \frac{xz}{r^2} \frac{\cos (pt - kr)}{r}, \quad . \quad . \quad . \quad (20)$$

$$\gamma = +\frac{Z_1}{4\pi h^2 \rho} \left(1 - \frac{z^2}{r^2} \right) \frac{\cos (pt - kr)}{r}. \quad . \quad . \quad (21)$$

Upon this ('Theory of Sound,' § 378) I commented:—"W. König (Wied. *Ann.* xxxvii. p. 651, 1889) has remarked upon the non-agreement of (18), (19), first given in a different form by Stokes, with the results of a somewhat similar investigation by Hertz (Wied. *Ann.* xxxvi. p. 1, 1889), in which the terms involving $\cos pt$, $\sin pt$ do not occur, and he seems disposed to regard Stokes's results as affected by error. But the fact is that the problems treated are essentially different, that of Hertz having no relation to elastic solids. The source of the discrepancy is in the first terms of (3) &c., which are omitted by Hertz in his theory of the æther. But assuredly in a theory of elastic solids these terms must be retained. Even when the material is supposed to be incompressible, so that δ vanishes, the retention is still necessary, because, as was fully explained by Stokes in the memoir referred to, the factor $(a^2 - b^2)$ is infinite at the same time."

Although the substance of the above comment appears to be justified, I went too far in saying that Hertz's solution has no relation to elastic solids. It is indeed not permissible to omit the first terms of (3) &c. merely because the solid is incompressible; but if, though the solid is compressible, it be in fact not compressed, these terms disappear. Now

Hertz's solution, corresponding to the omission of the second part of χ in (16), makes

$$\alpha = \frac{12}{dx \, dz} \left(\frac{e^{-ikr}}{r} \right), \quad \beta = \frac{d^2}{dy \, dz} \left(\frac{e^{-ikr}}{r} \right), \quad \gamma = \left(\frac{d^2}{dz^2} + k^2 \right) \left(\frac{e^{-ikr}}{r} \right),$$

. . . . (22)

and accordingly

$$\delta = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0 :$$

values which satisfy

$$(\nabla^2 + k^2) (\alpha, \beta, \gamma) = 0. \quad (23)$$

Thus (3), (4), (5) are satisfied, and the solution applies to an elastic solid upon which no forces act except at the origin. The only question remaining open is as to the character of the forces which must be supposed to act at that place. This is rather a delicate matter ; but it is evident at any rate that the forces are not of the simple character contemplated in the preceding investigation. It would appear that they must be double or multiple, and have components parallel to x and y as well as z . By a *double* force is meant the limit of a *couple* of given moment when the components increase and their mutual distance decreases, analogous to the double source of acoustics.

I now propose to calculate the work done by the force Z_1 at the origin as it generates the waves represented by (18), (19). For this purpose we require the part of γ in the neighbourhood of the origin which is in quadrature with the force, *i. e.* is proportional to $\sin pt$. From (19) we get

$$\frac{Ak^2 \sin pt}{r} \left[\left(1 - \frac{z^2}{r^2} + \frac{3z^2}{k^2 r^4} - \frac{1}{k^2 r^2} \right) \sin kr + \left(\frac{1}{kr} - \frac{3z^2}{kr^3} \right) \cos kr \right],$$

. . . (24)

the last term (in $\cos pt$) not contributing. Expanding $\sin kr$, $\cos kr$ and retaining the terms of order kr , we get for the square bracket

$$\frac{2}{3} kr + \frac{z^2}{r^2} (0).$$

Thus

$$(24) = \frac{2}{3} k^3 A \sin pt,$$

when r is small, so that the part proportional to $\sin pt$ is in

the limit finite and independent of z/r . If W be the work done in time dt ,

$$\frac{dW}{dt} = Z_1 \cos pt \cdot \frac{2}{3} k^3 p A \cos pt;$$

and by (6), (17)

$$\text{mean } \frac{dW}{dt} = \frac{k^2 Z_1^2}{12\pi b \rho} \dots \dots \dots (25)$$

The right-hand member of (25) is thus the work done (on the average) in unit of time.

This result may be confirmed by a calculation of the energy radiated away in unit time, for which purpose we may employ the formulæ (20), (21) applicable when r is great. The energy in question is the double of the kinetic energy to be found in a spherical shell whose thickness $(r_2 - r_1)$ is the distance travelled by transverse waves in the unit of time, viz. b . In the expression for the kinetic energy the resultant (velocity)² at any point x, y, z is by (20), (21) proportional to

$$\frac{x^2 z^2}{r^4} + \frac{y^2 z^2}{r^4} + \frac{(r^2 - z^2)^2}{r^4} = 1 - \frac{z^2}{r^2} = 1 - \mu^2,$$

a quantity symmetrical with respect to the axis. Also $\sin^2(pt - kr)$ is to be replaced by its mean value, viz. $\frac{1}{2}$. Thus the kinetic energy is

$$\frac{1}{2} \rho \left(\frac{Z_1}{4\pi b^2 \rho} \right)^2 \int_{-1}^{+1} \int_{r_1}^{r_2} \frac{1}{2} p^2 (1 - \mu^2) 2\pi d\mu dr,$$

the double of which is identical with (25).

We will now form the expression for the resolved displacement at P due to $Z_1 \cos pt$ acting at O (parallel to OZ), the displacement being resolved in a direction PT in the plane

Fig. 1.

ZOP making an angle θ' with OP (fig. 1). The angle between PT and OZ is denoted by ϕ , so that $\phi = \theta + \theta'$.

The resolved displacement is

$$\gamma \cos \phi + \alpha \sin \phi, \dots \dots \dots (26)$$

α and γ being given by (18), (19), in which we write

$$z/r = \cos \theta, \quad x/r = \sin \theta.$$

We find

$$\begin{aligned} & \frac{\gamma \cos \phi + \alpha \sin \phi}{Ak^2/r} \\ &= \cos (pt - kr) \left[-\sin \theta \sin \theta' - \frac{\cos \phi}{k^2 r^2} + \frac{3}{k^2 r^2} \cos \theta \cos \theta' \right] \\ &+ \frac{\sin (pt - kr)}{kr} [\cos \phi - 3 \cos \theta \cos \theta'] \\ &+ \frac{\cos pt}{k^2 r^2} [\cos \phi - 3 \cos \theta \cos \theta'], \end{aligned}$$

a symmetrical function of θ and θ' as required by the general principle of reciprocity ('Theory of Sound,' § 108). The value of A is given by (17) in which, however, we will now write $\mathfrak{F}$ for Z_1 , so that

$$A = \frac{\mathfrak{F}}{4\pi k^2 b^2 \rho} \dots \dots \dots (27)$$

The above equation gives the resolved displacement at P in a direction making an angle θ' with r due to a force $\mathfrak{F} \cos pt$ at O acting in a direction inclined to r at angle θ . If we suppose that a force $\mathfrak{F}' \cos pt$ acts at P in direction PT and inquire as to the work done by this force upon the motion due to $\mathfrak{F}$, we have to retain that part of the resolved displacement due to $\mathfrak{F}$ which is in quadrature with $\mathfrak{F}' \cos pt$, viz. the part proportional to $\sin pt$. The mean work is given by the symmetrical expression

$$\begin{aligned} & \frac{k^2 \mathfrak{F} \mathfrak{F}'}{8\pi b \rho} \left\{ \frac{\cos (\theta + \theta') - 3 \cos \theta \cos \theta'}{k^2 r^2} \left(\cos kr - \frac{\sin kr}{kr} \right) \right. \\ & \quad \left. - \frac{\sin kr}{kr} \sin \theta \sin \theta' \right\} \dots \dots \dots (28) \end{aligned}$$

If the forces are parallel, $\phi = 0$, $\theta' = -\theta$, and (28) becomes

$$\frac{k^2 \mathfrak{F} \mathfrak{F}'}{8\pi b \rho} \left[\sin^2 \theta \frac{\sin kr}{kr} + \frac{1 - 3 \cos^2 \theta}{k^2 r^2} \left(\cos kr - \frac{\sin kr}{kr} \right) \right]. \quad (29)$$

If we further suppose that kr is very small, the square bracket reduces to the value $\frac{2}{3}$, and we get

$$\frac{k^2 \mathfrak{F} \mathfrak{F}'}{12\pi b \rho} \dots \dots \dots (30)$$

Comparing with (25) we see that the work done by $\mathfrak{F}'$ on the motion due to an equal $\mathfrak{F}$ is the same as that done by $\mathfrak{F}$ itself, as should evidently be.

If in (29) $\theta=90^\circ$, so that the forces are perpendicular to the line joining the two points of application, we get

$$\frac{k^2 \mathfrak{F} \mathfrak{F}'}{8\pi b\rho} \left[\frac{\sin kr}{kr} + \frac{1}{k^2 r^2} \left(\cos kr - \frac{\sin kr}{kr} \right) \right]. \quad (31)$$

As we have seen, when kr is small (31) is finite and positive. It vanishes when

$$\tan kr = \frac{kr}{1 - k^2 r^2}, \quad (32)$$

and this occurs first in the second quadrant.

In general, when there are a number of forces acting at detached points, the whole work done must be obtained by a double summation of (28). If the forces are continuously distributed, the sum becomes a double integral. A particular case, of interest in connexion with the problem of electrical vibrations along a circular wire*, occurs when the forces act tangentially at the various points of a circular arc. Here $\theta'=\theta$, and (28) becomes

$$\frac{k^2 \mathfrak{F} \mathfrak{F}'}{8\pi b\rho} \left\{ \frac{1 + \cos^2 \theta}{k^2 r^2} \left(\frac{\sin kr}{kr} - \cos kr \right) - \frac{\sin kr}{kr} \sin^2 \theta \right\}. \quad (33)$$

Aug. 2, 1903.

LI. *Change of Length of Ferromagnetic Substances under High and Low Temperatures by Magnetization.* By K. HONDA, *Rigakuhakushi*, and S. SHIMIZU, *Rigakushi* †.

[Plate XV.]

THE interesting experiments of Hopkinson on the magnetization at high temperatures promise us some important results in the magnetic change of length at such temperatures. On account, however, of the experimental difficulties, this interesting subject has as yet scarcely been investigated. Barrett‡ was the first to touch the subject: with a rise of temperature of about 50° C. he observed no effect on the magnetic change of length in iron and cobalt,

* Compare Pocklington, *Proc. Camb. Phil. Soc.* ix. p. 324 (1897); 'Nature,' lxii. p. 486 (1903). It would seem that (33) must lead to a more complicated expression for the energy radiated than that in Dr. Pocklington's investigation.

† Communicated by the Authors.

‡ Barrett, *Phil. Mag.* [4] xlvii. p. 51 (1874); 'Nature,' xxvi. pp. 515 586 (1882); *Beibl.* vii. p. 201.