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NOTE ON A METHOD OF PROOF IN THE THEORY OF FOURIER'S SERIES.

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It has been pointed out on various occasions* that if $f(x)$ is a continuous function of period 2π satisfying the Lipschitz-Dini condition, that is, if $\lim_{\delta \rightarrow 0} \omega(\delta) \log \delta = 0$, where $\omega(\delta)$ is the maximum of the oscillation of $f(x)$ in an interval of length δ , then the uniform convergence of the Fourier series for $f(x)$ can be inferred almost immediately from the following two propositions:

A.† If $f(x)$ satisfies the Lipschitz-Dini condition,‡ there exists for every positive integral value of n a finite trigonometric sum $\tau_n(x)$, of order n at most, such that $\lim_{n \rightarrow \infty} r_n \log n = 0$, where r_n is the maximum of $|f(x) - \tau_n(x)|$.

* Cf., e.g., Lebesgue, "Sur les intégrales singulières," *Annales de la Faculté de Toulouse*, series 3, vol. 1 (1909), pp. 25-117; pp. 116-117.

† Cf., e.g., Lebesgue, loc. cit., p. 116; D. Jackson, "On the approximate representation of an indefinite integral, etc.," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 343-364; p. 350.

‡ It is understood throughout the paper that every function considered has the period 2π .

B.* If $\varphi(x)$ is any continuous† function, and $S_n(x)$ the partial sum of the Fourier series for $\varphi(x)$ to terms of order n , then $|S_n(x)|$ can not exceed $KM \log n$, where M is the maximum of $|\varphi(x)|$, and K is an absolute constant.

The central point in the proof is the fact that $\tau_n(x)$ is identical with the partial sum of its own Fourier series to terms of order n . It is the purpose of this note to show that similar reasoning can be applied to the arithmetical mean of Fejér,‡ in spite of the fact that the Fejér mean formed for a finite trigonometric sum $\tau_n(x)$ is not the same as $\tau_n(x)$. It is necessary to change the argument somewhat, but there is no difficulty in making the required modification.

Let $f(x)$ be now an arbitrary continuous function, and let $\sigma_n(x)$ be the arithmetical mean of the partial sums of the Fourier series for $f(x)$, to terms of order n . The uniform convergence of $\sigma_n(x)$ to the value $f(x)$ is to be deduced from the propositions:

C. (Weierstrass's theorem.)§ If $f(x)$ is continuous, there exists for every positive integral value of n a finite trigonometric sum $\tau_n(x)$, of order n at most, such that $\lim_{n=\infty} r_n = 0$, where r_n is the maximum of $|f(x) - \tau_n(x)|$.

D.|| If $\varphi(x)$ is any continuous function (more generally, any integrable function), and $\sigma_n(x)$ the Fejér mean of the Fourier series for $\varphi(x)$ to terms of order n , then $|\sigma_n(x)|$ can not exceed M , where M is the maximum of $|\varphi(x)|$.

Let ϵ be any positive quantity. Let a finite trigonometric sum $\tau_p(x)$, of order p , be determined, according to Proposition C, so that

$$|f(x) - \tau_p(x)| < \frac{1}{3}\epsilon.$$

Let $\sigma_{n1}(x)$ be the Fejér mean, of order n , for the function $\tau_p(x)$, and $\sigma_{n2}(x)$ the corresponding mean for the function $f(x) - \tau_p(x)$.

* Cf., e.g., Lebesgue, loc. cit., p. 116; D. Jackson, "On approximation by trigonometric sums and polynomials," *Transactions Amer. Math. Society*, vol. 13 (1912), pp. 491-515; pp. 502, 512-515.

† It is sufficient for the truth of the statement that φ be integrable, but for present purposes there is no need of speaking of any but continuous functions.

‡ Fejér, "Untersuchungen über Fouriersche Reihen," *Mathematische Annalen*, vol. 58 (1904), pp. 51-69.

§ Weierstrass, "Ueber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen," *Berliner Sitzungsberichte*, 1885, pp. 633-639, 789-805; p. 801.

|| Fejér, loc. cit., p. 60.

Then

$$(1) \quad \sigma_n(x) = \sigma_{n1}(x) + \sigma_{n2}(x).$$

By Proposition *D*,

$$|\sigma_{n2}(x)| < \frac{1}{3}\epsilon,$$

$$(2) \quad |f(x) - \tau_p(x) - \sigma_{n2}(x)| < \frac{2}{3}\epsilon,$$

for all values of n . The quantity $\sigma_{n1}(x)$ is the arithmetical mean of $n + 1$ finite trigonometric sums, of which all from the $(p + 1)$ th on, if $n \geq p$, are identical with $\tau_p(x)$, while each of the first p is composed of a part of the terms of $\tau_p(x)$. Added together, the first p sums which enter into the mean give a finite trigonometric sum $\omega_{p-1}(x)$, which is of order $p - 1$ at most, and independent of n . So $\sigma_{n1}(x)$ can be written in the form

$$\begin{aligned} \sigma_{n1}(x) &= \frac{\omega_{p-1}(x) + (n + 1 - p)\tau_p(x)}{n + 1} \\ &= \tau_p(x) + \frac{\omega_{p-1}(x) - p\tau_p(x)}{n + 1}. \end{aligned}$$

As the last numerator is independent of n , $\sigma_{n1}(x)$ approaches $\tau_p(x)$ uniformly as n becomes infinite—a fact which is fairly obvious in the first place—and, if n is sufficiently large,

$$(3) \quad |\tau_p(x) - \sigma_{n1}(x)| < \frac{1}{3}\epsilon.$$

By combination of (1), (2), and (3), for values of n satisfying (3),

$$|f(x) - \sigma_n(x)| < \epsilon,$$

which completes the convergence proof.

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