*On the Construction and Use of Commutation Tables for Calculating the Values of Benefits depending on Life Contingencies. By PETER GRAY, Esq., F.R.A.S.**

(Continued from page 104.)

PROBLEM VI.—To find the value of a life annuity on (x) , of which the first payment is to be £1; the second, £2; the third, £3; and so on, increasing annually, by the amount of the first payment, to the end of life.

This benefit may be conceived to consist of a series of annuities of £1, of which the first is to be entered upon now; the second, one year hence; the third, two years hence; and so on, an additional annuity being entered upon every year, till the end of life. The present values of these annuities are, by Problems II. and III.,

$$\frac{N_x}{D_x}, \frac{N_{x+1}}{D_x}, \frac{N_{x+2}}{D_x}, \text{ \&c.};$$

and the sum of these expressions, which will be the required present value, is, by (2),

$$\frac{N_x + N_{x+1} + N_{x+2} + \text{\&c.}}{D_x} = \frac{S_x}{D_x}.$$

The same result will be obtained by viewing the benefit in another light. Thus, the first payment is £1, to be received a year hence; its present value, therefore, is $\frac{D_{x+1}}{D_x}$. The second payment is £2, to be received two years hence; its present value, therefore,

* Extracted from the *Mechanics' Magazine* for 1842.

is $\frac{2D_{x+2}}{D_x}$. In like manner, the present values of the third, fourth, &c., payments will be respectively

$$\frac{3D_{x+3}}{D_x}, \frac{4D_{x+4}}{D_x}, \text{ \&c.}$$

and the sum of these expressions, which will be the present value of the benefit, is by (11), $\frac{S_x}{D_x}$, as before.

Example.—Required the present value of an increasing annuity of £1, £2, £3, &c., on (30).

$$\text{Answer. } \frac{S_{30}}{D_{30}} = \frac{539239.0044}{2257.6521} = 238.8494 = \text{£}238.17s.$$

PROBLEM VII.—To find the present value of a deferred increasing annuity on (x), that is, of an annuity to be entered upon n years hence, and whose first payment is to be £1; the second, £2; the third, £3; and so on, increasing every year, by the amount of the first payment, to the end of life.

This benefit may be considered as a series of deferred annuities of £1, of which the first is to be entered upon n years hence; the second, $n+1$ years hence; the third, $n+2$ years hence; and so on, an additional annuity being entered upon every year, till the end of life. The present values of these annuities are, by Problem III.,

$$\frac{N_{x+n}}{D_x}, \frac{N_{x+n+1}}{D_x}, \frac{N_{x+n+2}}{D_x},$$

and so on; and the sum of these expressions, which will be the present value required, will be, by (4), $\frac{S_{x+n}}{D_x}$.

Example.—An annuity, whose successive payments are to be £1, £2, £3, &c., is to be entered upon by (20), 10 years hence, that is, when he attains the age of 30. Required its present value.

$$\text{Answer. } \frac{S_{30}}{D_{20}} = \frac{539239.0044}{3818.9594} = 141.2005 = \text{£}141.4s.$$

PROBLEM VIII.—To find the present value of a temporary increasing annuity for n years on (x), the first payment being £1; the second, £2; and so on, increasing each year by the amount of the first payment, till the end of the term, when all payments cease.

This benefit consists of a series of temporary annuities of £1, of which the first is to be entered upon now, to continue n years, and then to cease; the second is to be entered upon one year hence, to continue $n-1$ years, and then to cease; and so on to the

n th annuity, which will be entered upon $n-1$ years hence, continue one year, and then cease—that is, will make but a single payment.

Now, the present values of these annuities are (omitting, for brevity, the common divisor D_x), by Problems IV. and V.,

$$\begin{aligned} &\text{Of the 1st, } N_x - N_{x+n} \\ &\quad \text{2nd, } N_{x+1} - N_{x+n} \\ &\quad \text{3rd, } N_{x+2} - N_{x+n} \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\text{Of the } n\text{th, } N_{x+n-1} - N_{x+n}. \end{aligned}$$

The sum of these is, (7),

$$S_x - S_{x+n} - n N_{x+n};$$

and, inserting the divisor, we have for the required present value,

$$\frac{S_x - S_{x+n} - n N_{x+n}}{D_x}.$$

Example.—Required the present value of an increasing annuity of £1, £2, £3, &c., on (55) for the next 10 years.

$$\begin{aligned} \text{Answer. } \frac{S_{55} - S_{65} - 10 N_{65}}{D_{55}} &= \frac{60976.4989 - (17008.1044 + 23769.292)}{587.3514} \\ &= \frac{20199.1025}{587.3514} = 34.390 = \text{£}34. 7s. 10d. \end{aligned}$$

PROBLEM IX.—To find the present value of an arrested increasing annuity on (x), that is, of an annuity whose successive payments are to be £1, £2, £3, and so on, increasing £1 every year till n payments have been made, the n th payment being thus £ n , which is also to be the amount of each subsequent payment till the end of life.

The difference between this benefit and that which formed the subject of the last problem is, that while, in the case of the benefit referred to, all payments were to cease at the end of n years, in the present case the increase only is arrested, and the annuity continues to pay £ n a year during the remainder of life. If, therefore, to the present value of the benefit in the last problem we add the present value of an annuity of £ n , deferred for n years, the sum will be the present value required. That is—

$$\frac{S_x - S_{x+n} - n N_{x+n}}{D_x} + \frac{n N_{x+n}}{D_x} = \frac{S_x - S_{x+n}}{D_x}.$$

Example.—Required the present value of an annuity of £1, £2, &c., on (55), the increase of which is to be arrested after 10 years.

$$\text{Answer. } \frac{S_{55} - S_{65}}{D_{55}} = \frac{60976 \cdot 4989 - 17008 \cdot 1044}{587 \cdot 3514} = \frac{43968 \cdot 3945}{587 \cdot 3514} \\ = 74 \cdot 8589 = \text{£}74. 17s. 2d.$$

We have now to deduce the expressions for the present values of the assurance benefits.

PROBLEM X.—To find the present value of an endowment assurance of £1 on (x) ; that is, of £1 to be received n years hence, provided (x) shall have died in the preceding year, viz., in the n th year from the present time.

Of the l_x individuals, whose present age is x years, represented by the mortality table to be now alive, l_{x+n-1} survive $n-1$ years, and l_{x+n} survive n years; consequently, $l_{x+n-1} - l_{x+n}$ is the number who die in their n th year; and it is also the number of pounds which will have to be paid, at the end of n years, to the representatives of those who thus die. The present value of this sum, therefore—that is, the sum which, put out at interest now, would in n years just amount to the first-named sum, is what must be advanced now to provide for this payment. This present value is $(l_{x+n-1} - l_{x+n})v^n$. And since all the l_x individuals now alive are equally interested, all of them contribute equally to this amount. The contribution of each will be, therefore—

$$\frac{(l_{x+n-1} - l_{x+n})v^n}{l_x},$$

and this is the required present value.

As in Problem I., multiply the numerator and the denominator of this expression by v^x , and it becomes

$$\frac{(l_{x+n-1} - l_{x+n})v^{x+n}}{l_x v^x}.$$

But the numerator of this expression is equal to C_{x+n-1} , and the denominator to D_x , by (1) and (3). Hence, the expression for the required present value, by the Commutation Table, is

$$\frac{C_{x+n-1}}{D_x}.$$

If $n=1$, that is, if the assurance is to be received a year hence, provided the death of x take place in the present year, the expression becomes $\frac{C_x}{D_x}$. If $n=2$, it is $\frac{C_{x+1}}{D_x}$, and so on.

Example.—Required the present value of an endowment assurance of £1 on (30), provided he die in his 40th year.

Here $x=30$, and $n=10$. Hence, the present value is $\frac{C_{39}}{D_{30}}$.

Since C is not exhibited, we may use for it either of the expressions (10) or (12). Taking the first, the formula becomes

$$\frac{M_{39}-M_{40}}{D_{30}} = \frac{540.81079-522.65022}{2257.6521} = \frac{18.16057}{2257.6521} = .008244 = 2d.$$

If the sum to be received be £100, its present value will be
 $.008244 \times 100 = .8244 = 16s. 6d.$

We have seen that when $n=1$, that is, when the sum assured is to be received a year hence, provided (x) be then dead, the formula for the present value becomes $\frac{C_x}{D_x}$. But, by (12), $C_x = vD_x - D_{x+1}$. Hence, by substitution, the formula in this case becomes

$$\frac{vD_x - D_{x+1}}{D_x} = v - \frac{D_{x+1}}{D_x};$$

and this expression affords a proof of the correctness of the formula in this problem. For, since v is the present value of £1, to be certainly received a year hence, and $\frac{D_{x+1}}{D_x}$ is (by Problem I.) the present value of £1 to be received at the same time provided (x) be then alive, the difference between these two is evidently the present value of £1 to be received a year hence if (x) be then dead.

PROBLEM XI.—To find the present value of a life assurance of £1 on (x) ; that is, of £1 to be received at the end of the year in which (x) dies, whensoever that event may happen.

We saw, by the last problem, that $\frac{C_x}{D_x}$ is the sum which must be contributed now by each of the l_x individuals to provide for the payment of £1 to the representatives of each of those of their number who die in the first year; and that $\frac{C_{x+1}}{D_x}$ is the sum each must contribute to provide for a like payment to the representatives of each of those who die in the second year. And, in like manner, by making n successively equal to 3, 4, &c., to the end of life, in the formula deduced in last problem, we should obtain, for the amounts to be contributed now to provide for the deaths that take place in the 3rd, 4th, &c., years respectively,

$$\frac{C_{x+2}}{D_x}, \frac{C_{x+3}}{D_x}, \&c.$$

And the sum of all these expressions is evidently the total sum which each of the l_x individuals must now pay to entitle his representatives to £1 at the end of the year in which he dies.

This sum is

$$\frac{C_x + C_{x+1} + C_{x+2} + \&c.}{D_x},$$

which, by (2), is equal to $\frac{M_x}{D_x}$; and this is, therefore, the required present value.

Example.—Required the present value of a life assurance of £1 on (20).

$$\text{Answer. } \frac{M_{20}}{D_{20}} = \frac{1131.76185}{3818.9594} = .296353 = 5s. 11d.$$

If the amount assured be £100, its present value will be

$$.296353 \times 100 = 29.6353 = \text{£}29. 12s. 8d.$$

PROBLEM XII.—To find the present value of a deferred assurance of £1 on (x); that is, of £1 to be received at the end of the year in which (x) dies, provided that event take place after n years.

The present value of this benefit, similar to that of the benefit in Problem III., is evidently equal to the sum of all the terms after the n th in the series of last problem, which expresses the present value of the same benefit for the whole life. This sum is, by (4), $\frac{M_{x+n}}{D_x}$; which is, therefore, the present value required.

Example.—Required the present value of an assurance of £1, deferred for 10 years, on (20).

$$\text{Answer. } \frac{M_{30}}{D_{20}} = \frac{742.95673}{3818.9594} = .194544 = 3s. 11d.$$

If the assurance be £100, its present value will be

$$.194544 \times 100 = 19.4544 = \text{£}19. 9s. 1d.$$

PROBLEM XIII.—To find the present value of a temporary assurance of £1 for n years on (x); that is, of £1 to be received at the end of the year in which (x) dies, provided that event take place during the next n years.

The present value here is evidently the sum of the first n terms in the general series of Problem XI., which sum is, by (7),

$$\frac{M_x - M_{x+n}}{D_x};$$

and this is, therefore, the present value required.

Example.—Required the present value of a temporary assurance of £1 for the next 10 years on (20).

$$\text{Answer. } \frac{M_{20} - M_{30}}{D_{20}} = \frac{1131.76185 - 742.95673}{3818.9594} = \frac{388.80512}{3818.9594} = .101809 = 2s.$$

If the sum assured be £100, its present value is

$$\cdot 101809 \times 100 = 10 \cdot 1809 = \text{£}10. \text{ 3s. } 8d.$$

If the present value just found be added to that found by the last problem, their sum, $\cdot 296353$, is the present value of an assurance for the whole life, as found by Problem XI., which it evidently ought to be.

PROBLEM XIV.—To find the present value of a deferred temporary assurance of £1 on (x); that is, of £1 to be received at the end of the year in which (x) dies, provided that event take place in the n years following the next k years.

The present value required in this problem is evidently the sum of n terms of the general series of Problem XI., beginning with the $(x+k)$ th; which sum is, by (9), equal to

$$\frac{M_{x+k} - M_{x+k+n}}{D_x};$$

and this is, therefore, the present value required.

Example.—Required the present value of an assurance of £1 on (30), to be received at the end of the year of death, provided that event take place between the ages of 40 and 50.

Here $x=30$, $k=10$, and $n=10$.

$$\therefore \frac{M_{x+k} - M_{x+k+n}}{D_x} = \frac{M_{40} - M_{50}}{D_{30}} = \frac{742 \cdot 05673 - 522 \cdot 65027}{2257 \cdot 6521} = \frac{220 \cdot 30646}{2257 \cdot 6521} \\ = \cdot 097582 = 1s. \text{ } 11\frac{1}{2}d.$$

If the sum assured be £100, its present value will be

$$\cdot 097582 \times 100 = 9 \cdot 7582 = \text{£}9. \text{ 15s. } 2d.$$

PROBLEM XV.—To find the present value of an increasing life assurance on (x), which is to be £1 if death take place in the first year, £2 if in the second, £3 if in the third, and so on—the assurance increasing each year that the payment is deferred, by the amount of the first year's assurance, till the end of life.

The present value here is found exactly in the same way as that of the corresponding annuity benefit in Problem VI.; for this assurance may be conceived to be made up of a series of assurances of £1, of which the first is to be entered upon immediately, and the others at intervals of one year, until the life shall have ceased to exist. The sum of the present values of all these assurances will evidently be the present value required. Now, the present value of the first is, by Problem XI., $\frac{M_x}{D_x}$; and of the second, third, fourth, &c., by Problem XII.,

$$\frac{M_{x+1}}{D_x}, \frac{M_{x+2}}{D_x}, \frac{M_{x+3}}{D_x}, \text{ \&c.};$$

and the sum of all these—that is, the present value required in the problem—is, by (2),

$$\frac{M_x + M_{x+1} + M_{x+2} + M_{x+3} + \text{\&c.}}{D_x} = \frac{R_x}{D_x}.$$

The same expression would be obtained by adding together the present values of the assurances for the separate years. Thus, the present value of £1 to be received at the end of one year, if death take place in that year, is, by Problem X., $\frac{C_x}{D_x}$; of £2 to be received at the end of two years, if death take place in the second year, $\frac{2C_{x+1}}{D_x}$; of £3 if death take place in the third year, $\frac{3C_{x+2}}{D_x}$, and so on. And the sum of these is, by (11), $\frac{R_x}{D_x}$, as before.

Example.—Required the present value of an increasing assurance of £1, £2, £3, &c., on (30).

$$\text{Answer. } \frac{R_{30}}{D_{30}} = \frac{17127 \cdot 42459}{2257 \cdot 6521} = 7 \cdot 5866 = \text{£}17. 11s. 9d.$$

PROBLEM XVI.—To find the present value of an increasing assurance, deferred for n years, on (x) , which is to pay £1, £2, £3, &c., at the end of the year in which (x) dies, according as that event shall take place in the $(n+1)$ th, $(n+2)$ th, $(n+3)$ th, &c., year from the present time.

By reasoning precisely similar to that employed in Problem VII. for finding the present value of the corresponding annuity benefit, and which it seems, therefore, unnecessary to repeat, the expression for the present value of the benefit in the present problem is found to be $\frac{R_{x+n}}{D_x}$.

Example.—Required the present value of an increasing assurance of £1, £2, £3, &c., deferred 15 years, on (30).

$$\text{Answer. } \frac{R_{x+n}}{D_x} = \frac{R_{45}}{D_{30}} = \frac{8326 \cdot 12414}{2257 \cdot 6521} = 3 \cdot 6890 = \text{£}3. 13s. 9d.$$

PROBLEM XVII.—To find the present value of a temporary increasing assurance of £1, £2, £3, &c., for n years, on (x) .

The solution of this problem is strictly analogous to that of Problem VIII., and the required present value is found to be

$$\frac{R_x - R_{x+n} - nM_{x+n}}{D_x}.$$

Example. $x=30, n=5$.

$$\text{Ans. } \frac{R_{30}-R_{35}-5M_{35}}{D_{30}} = \frac{17127.42459-(13672.73323+3099.47060)}{2257.6521} \\ = \frac{355.22076}{2257.6521} = .15734 = 3s. 2d.$$

PROBLEM XVIII.—To find the present value of an arrested increasing assurance of £1, £2, £3, on (x) ; that is, of an assurance which is to be £1 if death take place in the first year, £2 if in the second year, £3 if in the third, and so on to the n th year, when the assurance will be £ n , at which amount it is to continue for the remainder of life.

The present value here, by a mode of proceeding similar to that in Problem IX., will be found to be

$$\frac{R_x - R_{x+n}}{D_x}.$$

Example. $x=30, n=5$.

$$\text{Answer. } \frac{R_x - R_{x+n}}{D_x} = \frac{R_{30} - R_{35}}{D_{30}} = \frac{17127.42459 - 13672.73323}{2257.6521} \\ = \frac{3454.69136}{2257.6521} = 1.5302 = £1. 10s. 7d.$$

The foregoing comprise all of what may be called the *simple* benefits, and some (we mean the increasing benefits) which perhaps do not, in strictness, belong to that category. Previous to making a few general and miscellaneous remarks upon them, we present the following synoptical table of the results we have obtained. We omit, for brevity and distinctness, the common divisor D_x .

The present Value of an	To be received in n years, is	For Life, is	For the first n years, is	After the first n years, is	For n years after k years, is	Uniform after n years, is
Endowment of £1 on (x)	D_{x+n}					
Endowment assurance of £1 on (x) . .	C_{x+n-1}					
Annuity of £1 on (x)		N_x	$\left\{ \begin{array}{l} N_x - \\ N_{x+n} \end{array} \right\}$	N_{x+n}	$\left\{ \begin{array}{l} N_{x+k} - \\ N_{x+k+n} \end{array} \right\}$	
Assurance of £1 on (x)		M_x	$\left\{ \begin{array}{l} M_x - \\ M_{x+n} \end{array} \right\}$	M_{x+n}	$\left\{ \begin{array}{l} M_{x+k} - \\ M_{x+k+n} \end{array} \right\}$	
Increasing annuity of £1, £2, &c., on (x)	..	S_x	$\left\{ \begin{array}{l} S_x - \\ S_{x+n} - \\ nN_{x+n} \end{array} \right\}$	S_{x+n}	$\left\{ \begin{array}{l} S_{x+k} - \\ S_{x+k+n} - \\ nN_{x+k+n} \end{array} \right\}$	$\left\{ \begin{array}{l} S_x - \\ S_{x+n} \end{array} \right\}$
Increasing assurance of £1, £2, &c., on (x)		R_x	$\left\{ \begin{array}{l} R_x - \\ R_{x+n} - \\ nM_{x+n} \end{array} \right\}$	R_{x+n}	$\left\{ \begin{array}{l} R_{x+k} - \\ R_{x+k+n} - \\ nM_{x+k+n} \end{array} \right\}$	$\left\{ \begin{array}{l} R_x - \\ R_{x+n} \end{array} \right\}$

On looking at the expressions in the above table, the following analogies present themselves, attention to which will prevent mis-

takes in practice. We confine our remarks to the numerators, and it must be remembered that the denominator *in each case* is the *D* corresponding to the present age.

1. When a benefit for the whole life is to be entered upon immediately, the quantities in the numerator have for their signature the present age.

2. When the benefit is not to be entered upon immediately, the quantities in the numerator (which, as regards the columns made use of, are the same as in the expressions for the corresponding benefits to last the whole life), have for their signature the age at which the benefit is to be entered upon.

3. When the benefit is to be entered upon immediately, and to last for a term of years only—for example, n years—the expressions, as regards the uniform benefits, are derived from those for the same benefits to last the whole life, by writing $N_x - N_{x+n}$ and $M_x - M_{x+n}$ for N_x and M_x respectively; and as regards the increasing benefits, by writing $S_x - S_{x+n} - nN_{x+n}$ and $R_x - R_{x+n} - nM_{x+n}$ for S_x and R_x respectively. If we write $S_x - S_{x+n}$ and $R_x - R_{x+n}$ for S_x and R_x , we adapt the formulæ to the case in which the increase only is arrested at the end of n years.

4. The expression for an annuity benefit is changed into that for the corresponding assurance benefit, by writing C , M , and R , for D , N , and S respectively—the signatures remaining the same, except in the case of C , the signature of which must be diminished by unity.

Since the expressions we have deduced are fractions, the terms of which consist of numbers in the Commutation Table all of the same dimension, none of them rising above the first degree, it follows that, if all the numbers in the table be either multiplied or divided by any number whatsoever, the numerical values of the expressions will not be affected.

Hence the truth of the remark in the Note, page 93, is established.

And hence, also, it follows, that the position of the decimal point in the Commutation Tables is perfectly arbitrary. The position assigned to it in our table is that corresponding to a radix of the mortality table of 10,000, although the mortality actually made use of corresponds to a much higher radix, as stated in the previous part of this paper, p. 87.

The general problem solved in the preceding pages is, What is the *present value* of a benefit whose *amount* is £1? Suppose the problem reversed, and that it is required to find what *amount* of

benefit is equivalent to a *present value* of £1. The following proportion evidently subsists in all cases :—The present value of a benefit whose amount is £1 is to the present value of a like benefit of any other amount, as the amount of the first to the amount of the second. Hence, calling a the present value of the first benefit, and £1 that of the second, we have $a : 1 :: 1 : \frac{1}{a}$. That is, we should be furnished in each case with an expression which is the reciprocal of that which we have obtained for the present value of the same benefit when its amount is £1. Thus, $\frac{D_x}{N_x}$ will be the amount, or annual rent, of a life annuity to be entered upon immediately, and $\frac{D_x}{M_x}$ the sum to be received at the end of the year of death, in consideration of a present payment of £1, by an individual now aged x years. So, also, $\frac{D_x}{S_x}$ will be the first, and $\frac{2D_x}{S_x}$, $\frac{3D_x}{S_x}$, &c., the second, third, &c., annual payments of an increasing life annuity whose present value is £1.

Hence the table on page 177 furnishes us also with a solution to the general problem in this form ; for, to apply it to any particular case, we have only to form a fraction having for its *numerator* D_x , and for its *denominator* the expression in the table corresponding to that case. But this belongs more properly to another branch of the subject.

Before concluding this portion of the present paper, we make a remark, which perhaps might with equal propriety, have been introduced elsewhere. It is as to the principle on which the preceding problems have been solved.

Each benefit consists of a payment or payments, to be made at a certain specified time or times, provided a state of things, also specified, shall then have place ; such state of things being, where a single life is concerned, either the existence or the non-existence of that life, coupled, in the latter case, with the further condition, that its failure shall have taken place within a certain specified period. Problems I. and X. (using the former or the latter according as the first or the second of the states of things mentioned is that on which the receipt of the benefit is made to depend) enable us to find the present value of the payment for any or each year of existence ; and the sum of the present values for each year that the benefit is to last is evidently the whole present value of the benefit. Thus, the present value of a life annuity of £1 is the

sum of a series, whose terms express respectively the present value of £1 to be received at the end of one year, of two years, of three years, &c., from the present age to the end of life. The present value of an annuity of £1 to last n years, is the sum of the first n terms of the same series; and that of an annuity of the same amount, to be entered upon n years hence, and to last during the remainder of life, is the sum of all the terms after the n th.

(To be continued.)