VII.—*On the Motion of a Heavy Body along the Circumference of a Circle.* By
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In the year 1861 I laid before the Royal Society of Edinburgh a theorem concerning the time of descent in a circular arc, by help of which that time can be computed with great ease and rapidity. A concise statement of it is printed in the fourth volume of the Society's Proceedings at page 419.

The theorem in question was arrived at by the comparison of two formulæ, the one being the common series and the other an expression given in the "Edinburgh Philosophical Magazine" for November 1828, by a writer under the signature J. W. L. Each of these series is reached by a long train of transformations, developments, and integrations, which require great familiarity with the most advanced branches of the infinitesimal calculus; yet the theorem which results from their comparison has an aspect of extreme simplicity, and seems as if surely it might be attained to by a much shorter and less rugged road. For that reason I did not, at the time, give an account of the manner in which it was arrived at, intending to seek out a better proof. On comparing it with what is known in the theory of elliptic functions, its resemblance to the beautiful theorem of HALLE became obvious; but then the coefficients in HALLE's formulæ are necessarily less than unit, whereas for this theorem they are required to be greater than unit.

The search after the mutual relations of the two theorems has led me to the discovery of a few simple propositions which involve only the very first principles of the calculus, and the well-known law that the square of the velocity which a heavy body acquires in descending along a curve is proportional to the vertical distance, and to the intensity of gravitation jointly; and which, yet, contain the whole theory of motion in a circle whether that motion be oscillatory or continuous. I am thus enabled to present this hitherto intricate theory in a form which renders it intelligible to junior students of mechanical science.

By a well-known method of extension, the doctrine of the motion of a heavy physical point along the circumference of a circle can be made to include that of the rotation of any mass of matter on an axis not passing through its centre of gravity, whether that axis be horizontal or be inclined; hence, in the following investigation, I may confine my attention to the motion of a physical point in the circumference of a circle placed vertically.

2. Let N be the nadir and Z the zenith point of a circle, along the circumference of which a minute heavy body is free to move. If that body be projected

from N with some given velocity V, it will ascend along the circumference, losing velocity as it rises. When the initial velocity exceeds that which is due to a descent along the diameter ZN, the body will rise to the zenith-point Z, and will proceed onwards to descend along the other semicircumference; after that it will continue (all resistance being supposed away) to repeat revolution after revolution. But when the initial velocity is less than that which is due to a descent along ZN, the heavy body will have lost the whole of its velocity at some point below Z, from that point it will descend again to N, pass to the other side, on which it will reach to the same height, and thence descending it will continue to oscillate as in the familiar example of a pendulum.

There are thus two distinct cases of circular motion, viz., the *continuous* and the *oscillatory*.

3. These two cases may be connected in the following manner:—

Let us suppose that a heavy body α , has been projected at N, with a velocity due to its descent from some point A beyond Z, and that it has now reached to the point marked a . Having drawn the horizontal line aG , we see that its velocity at the point a is that which is due to a fall through the distance AG ; so that if we put V_A for the initial velocity, and v_a for the velocity at the point a , we must have the proportion

$$\begin{aligned} V_A^2 : v_a^2 &:: NA : AG :: NZ \cdot NA : NZ \cdot AG \\ &:: NZ \cdot NA : NZ \cdot NA - NZ \cdot NG. \end{aligned}$$

Through Z draw the horizontal line ZE, and make it a mean proportional between NZ and ZA; join NE, EA, then the trigons NZE, NEA are similar, so that NE is a mean proportional between NZ and NA, wherefore the above analogy may be written,—

$$V_A^2 : v_a^2 :: NE^2 : NE^2 - N\alpha^2.$$

4. F being the intersection of NE with the circumference of the circle, draw FB horizontally, then the five lines NA, NE, NZ, NF, and NB, are in continued proportion; so that $NA : NZ :: NZ : NB$.

If a second body β be projected from N, with a velocity due to a descent from B, it will rise along the curve only to the point F, its velocity there being entirely exhausted. The greatest distance, then, which β can reach from N, viz., NF, is to the greatest distance to which α can attain, viz., NZ, in the ratio of NE to NA. Let us take an intermediate point β to correspond with a , by making $N\alpha : N\beta$

$:: NA : NE :: NE : NZ :: \&c.$; and seek the ratio of the two velocities, viz., of α at the point α , and of β at the point β .

5. Putting V_B for the initial velocity of β , and v_β for its velocity at the point β we have,—

$$\begin{aligned} V_B^2 : v_\beta^2 &:: BN : BH :: BN : BN - NH \\ &:: NZ \cdot BN : NZ \cdot BN - NZ \cdot NH \\ &:: NF^2 : NF^2 - N\beta^2 . \end{aligned}$$

But we have also

$$V_A^2 : V_B^2 :: NE^2 : NF^2 ,$$

wherefore

$$v_\alpha^2 : v_\beta^2 :: NE^2 - N\alpha^2 : NF^2 - N\beta^2$$

Now, from our construction,

$$NE^2 : NZ^2 :: N\alpha^2 : N\beta^2$$

wherefore

$$NE^2 : NZ^2 :: NE^2 - N\alpha^2 : NZ^2 - N\beta^2$$

but

$$NZ^2 - N\beta^2 = Z\beta^2 ,$$

wherefore

$$NE^2 - N\alpha^2 : NE^2 :: Z\beta^2 : NZ^2$$

and similarly

$$NE^2 : NA^2 :: NF^2 - N\beta^2 : NZ^2 - N\alpha^2 ;$$

or,

$$NE^2 : NF^2 - N\beta^2 :: NA^2 : Z\alpha^2 .$$

Compounding these ratios we obtain,—

whence

$$\begin{aligned} NE^2 - N\alpha^2 : NF^2 - N\beta^2 &:: NA^2 : Z\beta^2 : NZ^2 : Z\alpha^2 , \\ v_\alpha : v_\beta &:: NA : Z\beta : NZ : Z\alpha . \end{aligned}$$

6. Let us now suppose that the body α moves through an exceedingly minute distance, represented by $\alpha\alpha'$, and let us make the proximate chord $N\beta'$, in the same ratio to Na' as before; then, since Na may be held equal to Na and Nb to $N\beta$, we have $\alpha\alpha' : b\beta :: Na : N\beta$.

The minute trigons $\alpha\alpha\alpha'$ and $b\beta\beta'$ are similar, respectively, to αNZ and βNZ , wherefore

$$\begin{aligned} \alpha\alpha' : \alpha\alpha' &:: NZ : Z\alpha \\ b\beta' : \beta\beta' &:: Z\beta : NZ . \end{aligned}$$

By compounding these three ratios we obtain

$$\begin{aligned} \alpha\alpha' : \beta\beta' &:: Na : Z\beta : N\beta : Z\alpha \\ &:: NZ : Z\beta : NF : Z\alpha . \end{aligned}$$

7. On comparing the lengths of the arcs $\alpha\alpha'$ and $\beta\beta'$, and also the velocities with which they are passed over, we find that the minute intervals of time are in the ratio

$$\text{time in } \alpha\alpha' : \text{time in } \beta\beta' :: \frac{NZ}{NA} : \frac{NF}{NZ} :: NZ : NE .$$

Now, if we suppose that the semicircumference NaZ is divided into a multitude of minute portions, of which aa may be taken as one; and if we divide the arc $N\beta F$ into as many corresponding portions by making the chords $N\beta$ always to the chords Na in the constant ratio NZ to NE ; the time of describing each element aa' of the semicircumference NZ is to that of describing the corresponding element $\beta\beta'$ of the arc NF in the constant ratio NZ to NE ; and, consequently, the time of describing any portion as Na must be to that of the describing the corresponding portion $N\beta$ in the same ratio; and so also must be the periodic times of the two motions.

Hence, if we can discover the law of the motion in the arc NF , we shall be able thence to deduce the law of the continuous motion due to the velocity obtained by descent from the point A ; and contrariwise.

For the sake of convenience, we shall call these two motions *conjugate* to each other.

8. It will conduce greatly to the clearness of our subsequent investigations to introduce here another consideration. The time of describing the arc NF is greater than that of describing the conjugate arc NZ in the ratio of NZ to NF ; the oscillatory motion will thus fall behind the continuous motion. Now, if we were to suppose that the body β is acted on by gravitation of an intensity greater than that which acts on α in the ratio duplicate of the ratio of the actual periodic times, the two motions would be rendered alike.

We shall then suppose that the gravitation acting on α , which we may designate by G_α , is proportional to NZ , while the intensity of the gravitation acting on β , appropriately denoted by G_β , is proportional to NA . And, as we have to do with the subduplicate of the ratio of these intensities, we shall, for the sake of additional convenience, put

$$G_\alpha = NZ \cdot NZ; \quad G_\beta = NZ \cdot NA.$$

9. The height from which a body has fallen being denoted by h , and the intensity of gravitation being G , the velocity acquired is, according to the well-known law of motion, proportional to $\sqrt{Gh}$; we shall, therefore, put the general formula for that velocity thus:—

$$v = \sqrt{(G \cdot NZ \cdot h)}.$$

10. On inserting the above value of G_α in this general formula, and at the same time making $h = AG = NA - NG$ we have

$$v_\alpha = \sqrt{\{NZ \cdot NZ \cdot NZ(NA - NG)\}} = NZ \cdot \sqrt{(NE^2 - Na^2)}.$$

And similarly for the body β

$$v_\beta = \sqrt{\{NZ \cdot NA \cdot NZ(NB - NH)\}} = NE \cdot \sqrt{(NF^2 - N\beta^2)}.$$

But since

$$NE : NZ :: NZ : NF :: N\alpha : N\beta$$

$$NZ \cdot \sqrt{(NE^2 - N\alpha^2)} = NE \cdot \sqrt{(NZ^2 - N\beta^2)} = NE \cdot Z\beta,$$

and

$$\mathbf{NE} \cdot \sqrt{(\mathbf{NF}^2 - \mathbf{N}\beta^2)} = \mathbf{NZ} \cdot \sqrt{(\mathbf{NZ}^2 - \mathbf{N}\alpha^2)} = \mathbf{NZ} \cdot \mathbf{Z}\alpha$$

wherefore

$$v_\alpha = \text{NE} \cdot Z\beta; \quad v_\beta = \text{NZ} \cdot Z\alpha.$$

Hence, under this supposition of two distinct intensities of gravitation, we have

$$v_\alpha : v_\beta :: \text{NE. Z}\beta : \text{NZ. Z}\alpha,$$

but we have shown in Article 6 that the minute distance $\alpha\alpha'$ is to the corresponding distance $\beta\beta'$ in the very same ratio, wherefore the time in which the body α describes the distance $\alpha\alpha'$ is now equal to that in which β describes the distance $\beta\beta'$. And consequently if α and β start at the same instant from N, they will reach the points α and β simultaneously; and just when α has reached the highest point Z, β will have reached its highest point F; so that the periodic times of the two conjugate motions have been made alike.

11. In the figure 1 hitherto referred to, the points α and β have been placed on opposite sides of the diameter NZ for the sake of perspicuity. We shall now, in figure 2, suppose that they are both projected in the same direction from N and at the same instant, so that when α has reached the point α , β has reached β . Proceeding onwards, when α comes to Z, β arrives at F, the velocity of α being then that which is due to a fall from A to Z, and the velocity of β being zero. Subsequently, while α returns to N along the other semicircumference, β returns to N by retracing its previous path FN. In this way both bodies arrive at N at the same instant, but moving in opposite directions. While α , for the second time, describes the entire circumference of the circle, β ascends to L and thence returns to N at the same instant that α reaches that point. The two bodies are now moving in the same direction as at first, and these phases, all resistance being set aside, are again and again reproduced.

12. Let us now imagine that the arc $\alpha\beta$ is continually bisected in γ , and let us trace the motion of this middle point.

When α has reached Z and β has come to F, the point γ must be at M the middle of the arc FZ: when α has passed Z and β is descending from F towards

N, the point γ must be approaching to Z, and it must reach Z just when α and β have met at N. Thus the time in which γ describes the arc MZ is just equal to that in which it passes from N to M. By the time that α has reached Z again, β has reached L the extreme point of its motion on the other side, and therefore γ is at P the middle of ZL; lastly, when α has once more descended along ZPLN to N, β has descended along LN to the same point, and so γ also has come to N.

It thus appears that while the point α makes two complete revolutions, the point γ makes only one. In the progress of this revolution the velocity of γ varies; when at N it is half the sum of the initial velocities V_A and V_B , and when at Z it is reduced to be their difference; so that the motion of γ has the general characteristic of one due to the action of gravity upon a heavy body.

13. If the motion of γ can be truly represented by the action of gravitation upon a heavy body, we may determine the point from which γ may be supposed to have descended, and the intensity of the gravitation which must act upon it, by comparing the velocities at the lowest and highest points of its path. Let C be the point from which γ must have descended in order to acquire at N the velocity $\frac{1}{2}(V_A + V_B)$, or at Z the velocity $\frac{1}{2}(V_A - V_B)$, and put G_γ for the intensity of the gravitation to which it is subjected; then

$$\sqrt{(G_\gamma \cdot NZ \cdot CN)} = \frac{1}{2}(V_A + V_B); \quad \sqrt{(G_\gamma \cdot NZ \cdot CZ)} = \frac{1}{2}(V_A - V_B).$$

Now according to Article 10,

$$V_A = NE \cdot NZ; \quad V_B = NE \cdot NF$$

while if we make ZI a mean proportional between CZ and ZN, and join NI, we have

$$NZ \cdot CN = NI^2, \quad NZ \cdot CZ = IZ^2,$$

so that

$$\sqrt{(G_\gamma) \cdot NI} = \frac{1}{2} NE (NZ + NF); \quad \sqrt{(G_\gamma) \cdot IZ} = \frac{1}{2} NE (NZ - NF),$$

whence $NI : IZ :: NZ + NF : NZ - NF$; a proportion which enables us to determine the position of the point I, and, consequently, that of C.

Taking the square of each term of that proportion we have

$$NI^2 : IZ^2 :: NZ^2 + 2NZ \cdot NF + NF^2 : NZ^2 - 2NZ \cdot NF + NF^2$$

whence

$$NI^2 : NZ^2 :: (NZ + NF)^2 : 4NZ \cdot NF$$

and consequently

$$NI^2 = \frac{NZ^2 (NZ + NF)^2}{4NZ \cdot NF}; \quad NI = NZ \frac{NZ + NF}{2\sqrt{(NZ \cdot NF)}} = NZ \frac{NE + NZ}{2\sqrt{(NE \cdot NZ)}},$$

whence also

$$NC = \frac{(NZ + NF)^2}{4NF}; \quad 4NF \cdot NC = (NZ + NF)^2.$$

14. Having thus determined the position of the point C, we can determine also the intensity of the gravitation for, putting for NI the value just found,

$$\sqrt{G_\gamma} \cdot NZ \cdot \frac{NF + NZ}{2\sqrt{(NZ \cdot NF)}} = \frac{1}{2} NE (NZ + NF)$$

whence $G_\gamma = NZ \cdot NE$; so that the intensity of the gravitation for γ must be a mean proportional between those for α and β .

15. A descent from C under the influence of gravitation of the intensity $NZ \cdot ZE$ would cause a heavy body to have at Z and at N the very velocities which the moveable point γ has at those places; and we have now to inquire whether the same influence would give to that body, when at any intermediate point, the corresponding velocity. Before treating of this matter generally, it may be instructive to inquire into the velocity of the point γ when it is at M the middle of FZ; the moveable point γ is at M when α is at Z and β at F, now at that instant the velocity of β is zero, while the velocity of α , proportional to the square root of $NZ \cdot ZA$, is represented by $NZ \cdot ZE$, so that the velocity of γ must then be $\frac{1}{2} NZ \cdot ZE$.

But the velocity of the heavy body when at M is given by the general formula,

$$\sqrt{(G_\gamma \cdot NZ \cdot QC)} = \sqrt{\{G_\gamma \cdot NZ (NC - NQ)\}}.$$

Substituting for G_γ the value above found we have

$$v_M^2 = NZ \cdot NE (NI^2 - NM^2)$$

but since M is the middle of the arc FZ

$$2NZ : NZ + NF \text{ or } 2NE : NE + NZ :: NZ^2 : NM^2$$

wherefore

$$NM^2 = NZ^2 \frac{NE + NZ}{2NE}; \text{ but } NI^2 = NZ^2 \frac{(NE + NZ)^2}{4NE \cdot NZ}$$

wherefore

$$\begin{aligned} NI^2 - NM^2 &= NZ^2 \left\{ \frac{NE^2 + 2NE \cdot NZ + NZ^2}{4NE \cdot NZ} - \frac{2NE \cdot NZ + 2NZ^2}{4NE \cdot NZ} \right\} \\ &= NZ^2 \frac{NE^2 - NZ^2}{4NE \cdot NZ} = NZ^2 \frac{ZE^2}{4NE \cdot NZ} \end{aligned}$$

and consequently

$$v_M^2 = \frac{1}{4} \cdot NZ^2 \cdot ZE$$

or

$$v_M = \frac{1}{2} NZ \cdot ZE$$

and thus the velocity at M due to a descent from the level of C is exactly that which the moveable point γ has at the same place.

16. We may now examine the velocity which this same heavy body would have at any intermediate point as γ . The general expression for that velocity is

$$v_\gamma = \sqrt{\{G_\gamma (NI^2 - N\gamma^2)\}} = \sqrt{\{NZ \cdot NE (NI^2 - N\gamma^2)\}}$$

but since γ is the middle of the arc $a\beta$,

$$N\gamma^2 = \frac{1}{2} \{NZ^2 + Na \cdot N\beta - Za \cdot Z\beta\}$$

subtracting this from the value of NI^2 and simplifying

$$NI^2 - N\gamma^2 = \frac{NE^2 \cdot NZ^2 + NZ^2 \cdot NZ^2 + 2NE \cdot NZ \cdot Za \cdot Z\beta - 2NE \cdot NZ \cdot Na \cdot N\beta}{4NE \cdot NZ}.$$

Now $NE \cdot Na = NZ \cdot N\beta$ so that the continued product $NE \cdot NZ \cdot Na \cdot N\beta$ may be written either $NE^2 \cdot Na^2$ or $NZ^2 \cdot N\beta^2$; writing it once each way we obtain

$$\begin{aligned} NI^2 - N\gamma^2 &= \frac{NE^2 \cdot Za^2 + 2NE \cdot NZ \cdot Za \cdot Z\beta + NZ^2 \cdot Z\beta^2}{4NE \cdot NZ} \\ &= \frac{(NE \cdot Za + NZ \cdot Z\beta)^2}{4NE \cdot NZ} \end{aligned}$$

wherefore

$$v_\gamma = \sqrt{\left\{ NE \cdot NZ \frac{(NE \cdot Za + NZ \cdot Z\beta)^2}{4NE \cdot NZ} \right\}} = \frac{1}{2} (NE \cdot Za + NZ \cdot Z\beta);$$

but we have seen that $NE \cdot Za$ is the velocity of β at the point β , $NZ \cdot Z\beta$ that of a at the point a , so that

$$v_\gamma = \frac{1}{2} \{v_a + v_\beta\}$$

and thus, at every point of the circumference, the velocity of a body projected from N with a velocity due to a descent from Z, and acted on by a gravitation having its intensity represented by $NZ \cdot NE$, is equal to the velocity of the middle of the arc $a\beta$.

17. The motion of the body γ round the circumference has for its *conjugate* that of a fourth, which we may name δ ascending from N to K, and thence returning to N, while γ rises from N to Z, and proceeding onwards, returns to N; the conjugation being analogous to that which connects the motions of a and β .

Hence, if we inflect the chord $N\delta$, a fourth proportional to NZ , NK , and $N\gamma$, we shall obtain the point at which the body δ is found when γ is at γ ; and if we make G_δ a fourth proportional to NZ , NC , and G_γ , we shall obtain for the intensity of the gravitation to which δ must be subjected

$$NZ : NC :: NZ \cdot NE : NC \cdot NE = G_\delta.$$

18. In this way we have obtained two pair of conjugate motions, the periodic time of the second pair being double of that of the first, and the intensities of gravitation being

$$\begin{aligned} G_a &= NZ \cdot NZ \\ G_\beta &= NZ \cdot NA \\ G_\gamma &= NZ \cdot NE \\ G_\delta &= NC \cdot NE \end{aligned}$$

such that any one of the four motions being known the other three may be found.

19. Here it is to be observed, that the periodic time of γ and δ is double that of α and β ; in order to bring it to be the same, we must quadruple the intensities of the gravitation acting on these bodies, so that for all the

Periodic Times alike

$$\begin{aligned} G_a &= NZ \cdot NZ, & G_\beta &= NZ \cdot NA \\ G_\gamma &= 4NZ \cdot NE, & G_\delta &= 4NC \cdot NE. \end{aligned}$$

20. And if the intensities of gravitation be supposed the same for all the four bodies, their periodic times will then be proportional to the square roots of the preceding intensities: so that if we put $T_a, T_\beta, T_\gamma, T_\delta$ for the periodic times on the supposition of one gravitation, we have

Gravitations alike

$$\begin{aligned} T_a &= NZ; & T_\beta &= NE \\ T_\gamma &= 2\sqrt{(NZ \cdot NE)}; & T_\delta &= 2\sqrt{(NC \cdot NE)}. \end{aligned}$$

21. These two pairs of conjugate motions are so connected, that from one of them the other can be found, the law of connection being contained in the proportion

$$NI : IZ :: NE + NZ :: NE - NZ$$

or in

$$NI + IZ : NI - IZ :: NE : NZ.$$

If then from the pair α, β , we deduce the pair γ, δ ; we may again from this latter deduce another pair of conjugate motions which we may mark γ_1, δ_1 ; and from this again another pair γ_2, δ_2 , and so on without end. Or if we regard γ, δ , as the original pair, and deduce α, β , from it, we may thence deduce a new pair α_1, β_1 , and from that again, another α_2, β_2 ; and so on, so that we have a progression of conjugate motions extending indefinitely each way, and such that any one of the series being known, all the others can be thence deduced. The latter branch of the progression, viz., that from γ, δ , to α, β , and thence onwards is that which is available in our research.

22. The periodic time of a body descending from C is deducible from that of one which has the velocity belonging to a descent from A ; that again is deducible from the motion of a body supposed to have fallen from A₁, and so on. Now, the distances NA, NA₁, NA₂, &c, increase with greater and greater rapidity as we proceed, so that after a few terms NA may become enormously great as compared with NZ. But EZ is proportional to the velocity at Z, while NE is proportional to that at N; and the ratio of NE to EZ must approach nearer and nearer to a ratio of equality as the point A rises; so that when NA is very great, the velocity becomes almost uniform, and the periodic time of the motion becomes the quotient of the circumference by that velocity. In this way, the study of the law of this progression may conduct us to a knowledge of the periodic time of the motion of γ .

23. Analogously the distance NB is deduced from NC, from NB again we may deduce NB₁, thence NB₂, and so on; and in this manner, we may reduce the question of the time of descent in the arc KN, to that of the time of ascent in an excessively minute arc.

24. Attending first to the case of oscillatory motion, let it be proposed to compute the periodic time of a body having its velocity due to a descent from the level of D.

For this purpose, let us put B₀ for the angle NZK measured by half of the extreme arc NK, and B₁ for the angle NZF measured by half of the arc NF; then

$$NK = NZ \cdot \sin B_0, \quad KZ = NZ \cdot \cos B_0;$$

but

$$NZ + ZK : NZ - ZK :: NZ : NF,$$

wherefore

$$1 + \cos B_0 : 1 - \cos B_0 :: (\cos \frac{1}{2} B_0)^2 : (\sin \frac{1}{2} B_0)^2 :: 1 : \sin B_1$$

and

$$\sin B_1 = (\tan \frac{1}{2} B_0)^2.$$

But it has been shown in article 20 that the times of descent from K and from F, there marked by the symbols T _{δ} and T _{β} are in the ratio $2\sqrt{NC \cdot NE} : NE$ or of $2\sqrt{NC} : \sqrt{NE}$. It is now convenient to indicate these times by the characters T sin. B₀ and time B₁, so that the above proportion may be written

$$\begin{aligned} \text{time } B_0 : \text{time } B_1 &:: \sqrt{NC} : \sqrt{NE} \\ &:: 2NI : NI + IZ \\ &:: 2NZ : NZ + ZK \\ &:: 2 : 1 + \cos B_0 \\ &:: 1 : (\cos \frac{1}{2} B_0)^2 \\ &:: (\sec \frac{1}{2} B_0)^2 : 1 \end{aligned}$$

so that the time of descent from K is

$$\text{time } B_0 = \text{time } B_1 \cdot (\sec \frac{1}{2} B_0)^2.$$

Hence the following very simple construction :—Having made NS the fourth part of the arc NK and drawn a horizontal line through N, join OS and produce it to meet the horizontal tangent at T; draw TU perpendicular to OT, meeting ON produced in U; lastly inflect NF double of NU; then the time of descent along KN is to that of descent from F to N in the ratio of OU to ON.

By repeating this operation in regard to the arc NF, we should obtain a much smaller arc, and thence again one still smaller, and so on; the periodic times in these successive arcs bearing known ratios to each other. Now it is obvious, from a glance at the figure, that these arcs, which we may denote by $2B_0$, $2B_1$, $2B_2$, &c., form a very rapidly decreasing progression; and that the ratios of which $ON : OU$ is the first, approach at the same time to a ratio of equality; hence the time of descent along KN may be deduced from the time of oscillation in an exceedingly minute arc. A very familiar investigation shows that the time of oscillation in a small arc is almost independent of the extent of the arc; but instead of founding on this well-known proposition, I prefer to deduce the truth of it from our present considerations.

Fig. 3

25. If we suppose the arc NF of figure 2 to be very minute, the height NA, which is inversely proportional to NZ, must become very great in proportion to the diameter NZ, and hence the velocity at the point Z must be nearly equal to that at N, the two being in the ratio of ZE to EN; and the time in which a body descending from A describes the circumference must always be greater than that in which another moving with a uniform velocity equal to that at N would describe the same circumference. Putting g for the actual intensity of gravitation, the velocity acquired by falling from A to N is

$$V_A = \sqrt{(2g \cdot AN)}$$

so that, as πNZ is the length of the circumference, the value of T_A must be greater than

$$\frac{\pi NZ}{\sqrt{2g \cdot AN}} = \pi \sqrt{\left(\frac{NZ}{2g}\right)} \cdot \sqrt{\frac{NZ}{AN}} = \pi \sqrt{\left(\frac{NZ}{2g}\right)} \cdot \frac{NF}{NZ}$$

while, since the velocity at Z is to that at N as ZE to NE, the same time must be less than

$$\pi \sqrt{\left(\frac{NZ}{2g}\right)} \cdot \frac{NF}{NZ} \cdot \sec NZF.$$

26. Now we have seen that

$$T_a : T_\beta :: NF : NZ$$

wherefore the time of descending from F to N, and thence rising to L on the other side, is between the limits

$$\pi \sqrt{\left(\frac{NZ}{2g}\right)} \text{ and } \pi \sqrt{\left(\frac{NZ}{2g}\right)} \cdot \sec NZF,$$

which limits approach closer to each other when the arc NF is made smaller.

27. Resuming now the investigation as in article 24, we find that the time of oscillation from K is

$$\begin{aligned} \text{time } B_0 &\succ (\sec \tfrac{1}{2} B_0)^2 \pi \sqrt{\left(\frac{NZ}{2g}\right)} \\ &\prec (\sec \tfrac{1}{2} B_0)^2 \pi \sqrt{\left(\frac{NZ}{2g}\right)} \cdot \sec B_1. \end{aligned}$$

Continuing the same progression another step by making

$$\sin B_2 = (\tan \tfrac{1}{2} B_1)^2$$

we find

$$\begin{aligned} \text{time } B_0 &\succ (\sec \tfrac{1}{2} B_0)^2 (\sec \tfrac{1}{2} B_1)^2 \pi \sqrt{\left(\frac{NZ}{2g}\right)} \\ &\prec (\sec \tfrac{1}{2} B_0)^2 (\sec \tfrac{1}{2} B_1)^2 \pi \sqrt{\left(\frac{NZ}{2g}\right)} \cdot \sec B_2; \end{aligned}$$

these limits being now closer to each other, since B_2 is a smaller angle than B_1 .

If, then, we continue the progression indefinitely, by making

$$\sin B_3 = (\tan \tfrac{1}{2} B_2)^2; \sin B_4 = (\tan \tfrac{1}{2} B_3)^2; \&c.$$

we shall obtain for the entire time of an oscillation in the arc $4B_0$,

$$\text{time } B_0 = \pi \sqrt{\left(\frac{NZ}{2g}\right)} \cdot (\sec \tfrac{1}{2} B_0)^2 \cdot (\sec \tfrac{1}{2} B_1)^2 \cdot (\sec \tfrac{1}{2} B_2)^2 \cdot \&c.$$

28. As an example of the calculation we may require the time of oscillation over an arc of 180° , which is the extreme limit of a pendulum with a flexible thread. In this case $B_0 = 45^\circ$, whence $\log \tan \tfrac{1}{2} B_0 = \log \tan 22^\circ 30' = 9.617\ 2243$; wherefore $\log \sin B_1 = 9.234\ 4486$; $B_1 = 9^\circ 52' 45''.42$; from this again we have $\log \tan \tfrac{1}{2} B_1 = \log \tan 4^\circ 56' 22''.71 = 8.936\ 6506$; $\log \sin B_2 = 7.873\ 3012$, giving $B_2 = 0^\circ 25' 40''.74$. And once again, $\log \tan \tfrac{1}{2} B_2 = \log \tan 0^\circ 12' 50''.37 = 7.572\ 2861$; $\log \sin B_3 = 5.144\ 5722$, $B_3 = 0^\circ 0' 02''.88$. Here we observe that the log-secant of B_3 does not differ from zero by unit in the seventh decimal place; and that, therefore, we have brought our limits so close together that the difference cannot be appreciated by help of the ordinary logarithmic tables. The

logarithm of the ratio of the time of oscillation in a semicircumference to that in a small arc is thus

$$\begin{array}{r} \text{Log sec } \frac{1}{2} B_0 = 0.034 \ 3847 \\ \text{Log sec } \frac{1}{2} B_1 = \quad 1 \ 6160 \\ \text{Log sec } \frac{1}{2} B_2 = \quad \quad 30 \\ \hline 0.036 \ 0037 \times 2 \\ \hline 0.072 \ 0074 \\ 1.180341 \end{array}$$

In other words, the time of oscillation in a semi-circumference is to that for a minute arc as 72 to 61 very nearly.

When the arc is nearly a whole circumference, the steps of the progression are more numerous, and the computation may, with advantage, be systematically arranged. I subjoin the work for an arc of 320° , with logarithms to ten places.

n	$\frac{1}{2} B_n$	$2 \log \sec \frac{1}{2} B_n$	$2 \log \tan \frac{1}{2} B_n$	B_{n+1}
0	$40^\circ 00' 00.000''$	23149 20670	9.84762 70684	$44^\circ 45' 21.339''$
1	22 22 40.669	6800 53042	9.22920 47378	9 45 34.359
2	4 52 47.180	315 40158	7.86265 75152	25 03.441
3	12 31.721	57684	5.16071 41918	2.986
4	1.493	0		
		$.30265 \ 71554 = \log 2.00750740$		

Thus, even in this extreme case, four terms of the progression are sufficient.

These examples show that for all cases in ordinary clock-work, or in experiments on the length of the pendulum, the time of oscillation is sufficiently well represented by the formula

$$\text{time } B = \pi \sqrt{\left(\frac{l}{g}\right) \cdot (\sec \frac{1}{2} B)^2}$$

B being one-fourth part of the entire arc of oscillation. Hence the number of beats per day is proportional to the square of the cosine of the eighth part of the arc, and the daily retardation to the square of the sine of the same eighth part.

29. From the periodic time of an oscillation that of the conjugate continuous motion may be readily deduced; and thus, so far as the entire motions are concerned, the theory may be said to be complete. The investigation of the time at which the moving body arrives at any proposed point in its path is carried on by the application of the principle just explained; but as it is of comparatively little interest, and, at the same time, more tedious, I shall not, for the present, go into its details.