THEORETICAL INVESTIGATION OF SOME OSCIL- LATIONS OF EXTREMELY HIGH POTENTIAL IN ALTERNATING HIGH POTENTIAL TRANSMISSIONS.

BY CHARLES PROTEUS STEINMETZ.

In the following I intend to give a section of a series of investigations extending over some years, on the effect of the exponential term in the general equation of alternating current:

If a sine wave of E. M. F. is impressed upon a general electric circuit containing resistance, inductance and capacity, the ultimate current is a sine wave also. In the moment of closing the circuit, however, or in any other way changing the circuit conditions, upon the sine wave of current an exponential term is superimposed, of the general character:

$$\sum A_i \varepsilon^{-a_i \varphi}$$

In a single circuit containing only resistance r and inductive reactance x , only one exponential term

$$A \varepsilon^{-\frac{r}{x} \varphi}$$

appears. In a system of such circuits, directly or inductively connected with each other, a series of exponential terms appear,

the exponents of which are determined as the roots of an equation of n th degree.

If, however, a circuit contains resistance, inductance and capacity in series, two exponential terms

$$A_1 \varepsilon^{-a_1 \varphi} + A_2 \varepsilon^{-a_2 \varphi}$$

exist, where a_1 and a_2 are the roots of a quadratic equation, and either real or complex imaginary. In the latter case the exponential term resolves into the form :

$$\varepsilon^{-b \varphi} \left\{ B_1 \cos \varphi + B_2 \sin \varphi \right\}$$

an electric oscillation.

With several circuits of general character connected with each other directly or inductively, a series of exponential terms

$$\Sigma A_i \varepsilon^{-a_i \varphi}$$

appear, in which the exponents a_i are the roots of an equation of $2n$ th degree, which, when complex imaginary, give electrical oscillations, and the coefficients A_i depend on the initial conditions of the circuits. Thus in such a system of circuits a number of superimposed oscillations may occur, of different frequencies and intensities.

Under certain conditions by such oscillations caused by a change of circuit conditions, destructive voltages may be produced.

The first time I noticed such a phenomenon was when making a series of observations on the effect of opening a high potential circuit with different types of circuit-breakers, on an artificial line consisting of inductive coils and about half a mile of high potential cable, with a spark gap between needle points connected across the system, to measure an instantaneous rise of voltage. If the needle points were brought so close together that a discharge occurred between them, invariably by the short-circuiting of the system across the needle points, an extremely high voltage was induced locally in the system, of a total striking distance reaching as high as 40'', that is between one-quarter and one-half

million volts, the normal voltage of the system being about 10,000.

The phenomenon is probably as follows: By striking between the needle points shunting the circuit, the system was short-circuited and thus instantly drained of its electro-static charge, so that the potential locally fell to practically nothing, while an excessive current passed between the needle points as arc. By the heating effect of this current the arc was instantly blown out explosively and thus the short-circuit current of the system ruptured. After the explosion the discharge between the needle points again short-circuited, interrupted itself, etc. The phenomenon is thus of similar character, but infinitely greater power, than the action of the Wehnelt interrupter, the latter depending upon the combined effect of self-induction and electrolytic polarization, while the phenomenon I observed occurred with self-induction and capacity.

Since that time in a number of instances of high potential circuits electrical discharges were noticed, of striking distances many times greater than that of the normal potential of the system, under circumstances where a short-circuiting arc interrupted itself in the open air. A short-circuit in a constrained space, as a buried cable, does not appear to give this effect.

From this and other reasons I was led to believe that a short-circuiting high potential arc in free space, in a system containing self-induction and capacity, is a self-interrupting phenomenon, and thus may cause very serious high potential effects. On any high potential limited power transformer this phenomenon can be observed by an entire change of the character of the arc between the high potential terminals when inserting capacity in shunt. From the flame of the steady and noiseless arc, which curves upward by its heat, the discharge changes to the glaring white, noisy, lightning-like, destructive high potential oscillation. This change can be shown photographically, the photographs giving in the former case, that is absence of capacity, the striations of the successive half waves of the impressed frequency, while in the latter case they show a system of successive electro-static sparks not followed by the striated alternating arc flame. As shown by Mr. W. S. Andrews, a number of oscillating discharges may occur during the same half wave of alternating primary E. M. F.

The investigation of the character of electric oscillation pro-

duced by a short-circuiting and self-interrupting arc is the purpose of the following. The bearing of this phenomenon on high potential engineering is obvious.

If in a transmission line of impressed voltage $e = E \cos (\varphi - \omega)$ carrying current $I \cos (\varphi - \omega - \delta)$, the circuit is opened at the receiving end, there remain in circuit the line capacity in series with the resistance and inductance of step-up transformers, etc., and a part of the line.

Let, as an approximation, the line capacity be represented by a condenser shunted across the center of the transmission line, and let :

r = resistance,

L = inductance, thus : $x = 2 \pi N L$ = reactance of circuit between constant potential generator bus-bars and line capacity (reduced to the line voltage).

Let :

C = capacity, thus : $k = \frac{1}{2 \pi N C}$ = capacity reactance of line.

Let now the circuit be opened at the moment : $t = 0$, or : $\varphi = 0$, and at the moment when the circuit is opened :

e_0 = potential difference at line condenser,

i_0 = current in line.

If : i = instantaneous value of current at time t , it is :
E. M. F. consumed by self-induction :

$$L \frac{di}{dt}$$

Potential difference at condenser terminals :

$$e_1 = \frac{1}{C} \int i dt$$

Thus the differential equation of the phenomenon is :

$$e - ir - L \frac{di}{dt} - \frac{1}{C} \int i dt = 0$$

or, substituting:

$$\varphi = 2 \pi N t,$$

$$e = E \cos (\varphi - \omega)$$

it is :

$$E \cos (\varphi - \omega) - ir - x \frac{di}{d\varphi} - k \int i d\varphi = 0 \quad (1)$$

and :

$$e_1 = k \int i d\varphi = E \cos (\varphi - \omega) - ir - x \frac{di}{d\varphi} \quad (2)$$

Equation (1) differentiated :

$$E \sin (\varphi - \omega) + k i + r \frac{di}{d\varphi} + x \frac{d^2 i}{d\varphi^2} = 0 \quad (3)$$

This differential equation of second order is integrated by :

$$i = A \varepsilon^{-\alpha\varphi} + B \cos (\varphi - \sigma) \quad (4)$$

Substituting (4) in (3) gives :

$$\begin{aligned} (\alpha^2 x - \alpha r + k) A \varepsilon^{-\alpha\varphi} + \sin (\varphi + \sigma) [E \cos (\sigma - \omega) - r B] \\ + \cos (\varphi - \sigma) [E \sin (\sigma - \omega) - (x - k) B] \end{aligned} \quad (5)$$

Hence :

$$\left. \begin{aligned} \alpha^2 x - \alpha r + k &= 0 \\ E \cos (\sigma - \omega) - r B &= 0 \\ E \sin (\sigma - \omega) - (x - k) B &= 0 \end{aligned} \right\} \quad (6)$$

$$A = \text{indefinite,}$$

and herefrom :

$$\tan (\sigma - \omega) = \frac{x - k}{r}$$

or, by denoting :

$$\tan \alpha = \frac{k - x}{r} \quad (7)$$

where α is a constant of the circuit,

$$\sigma = \omega - \alpha \quad (8)$$

Further :

$$B = \frac{E}{\sqrt{r^2 + (x - k)^2}}$$

or, denoting :

$$z_0 = \sqrt{r^2 + (x - k)^2} \quad (9)$$

where z_0 is a constant of the circuit,

$$B = \frac{E}{z_0} \quad (10)$$

and α is, as root of a quadratic equation :

$$\alpha = \frac{r \pm \sqrt{r^2 - 4 x k}}{2 x}$$

Since the inductive reactance x is of the same magnitude as the resistance r , the capacity reactance k , however, that is the reactance of the line as condenser, of higher magnitude than r or x , it is always :

$$4 x k > r^2$$

that is,

$$\sqrt{r^2 - 4 x k} \text{ is imaginary}$$

Substituting :

$$q = \sqrt{4 x k - r^2} \quad (11)$$

it is :

$$\alpha = \frac{r \pm j q}{2 x} \quad (12)$$

and the integral equation of the problem is :

$$i = A_1 \epsilon^{-\frac{r+jq}{2x}\varphi} + A_2 \epsilon^{-\frac{r-jq}{2x}\varphi} + \frac{E}{z_0} \cos(\varphi - \omega + \alpha) \quad (13)$$

that is :

$$i = \epsilon^{-\frac{r}{2x}\varphi} \left\{ A_1 \epsilon^{-j\frac{q}{2x}\varphi} + A_2 \epsilon^{+j\frac{q}{2x}\varphi} \right\} + \frac{E}{z_0} \cos(\varphi - \omega + \alpha)$$

Substituting the trigonometric expressions :

$$\left. \begin{aligned} \epsilon^{+j\frac{q}{2x}\varphi} &= \cos \frac{q}{2x} \varphi + j \sin \frac{q}{2x} \varphi \\ \epsilon^{-j\frac{q}{2x}\varphi} &= \cos \frac{q}{2x} \varphi - j \sin \frac{q}{2x} \varphi \end{aligned} \right\}$$

and rearranging, it is :

$$i = \varepsilon^{-\frac{r}{2x}} \left\{ (A_1 + A_2) \cos \frac{q}{2x} \varphi - j (A_1 - A_2) \sin \frac{q}{2x} \varphi \right\} \\ + \frac{E}{z_0} \cos (\varphi - \omega + \alpha)$$

or, substituting :

$$\left. \begin{aligned} A_1 + A_2 &= C_1 \\ j (A_1 - A_2) &= C_2 \end{aligned} \right\}$$

it is :

$$i = \varepsilon^{-\frac{r}{2x}\varphi} \left(C_1 \cos \frac{q}{2x} \varphi - C_2 \sin \frac{q}{2x} \varphi \right) + \frac{E}{z_0} \cos (\varphi - \omega + \alpha) \quad (14)$$

where

C_1 and C_2 are determined by the circuit conditions at $t = 0$.

The potential difference at the condenser terminals is, by substituting (14) in (2) :

$$e_1 = \varepsilon^{-\frac{r}{2x}\varphi} \left\{ C_1 \left(\frac{q}{2} \sin \frac{q}{2x} \varphi - \frac{r}{2} \cos \frac{q}{2x} \varphi \right) + C_2 \left(\frac{q}{2} \cos \frac{q}{2x} \varphi \right. \right. \\ \left. \left. + \frac{r}{2} \sin \frac{q}{2x} \varphi \right) \right\} \\ + E \left\{ \cos (\varphi - \omega) - \frac{r}{z_0} \cos (\varphi - \omega + \alpha) + \frac{x}{z_0} \sin (\varphi - \omega + \alpha) \right\}$$

or, substituting :

$$\frac{r}{z_0} = \cos \alpha \quad \frac{x}{z_0} = \frac{k}{z_0} + \sin \alpha :$$

$$e_1 = \varepsilon^{-\frac{r}{2x}\varphi} \left\{ C_1 \left(\frac{q}{2} \sin \frac{q}{2x} \varphi - \frac{r}{2} \cos \frac{q}{2x} \varphi \right) \right. \\ \left. + C_2 \left(\frac{q}{2} \cos \frac{q}{2x} \varphi + \frac{r}{2} \sin \frac{q}{2x} \varphi \right) \right\}$$

$$+ \frac{k}{z_0} E \sin (\varphi - \omega + \alpha) \quad (15)$$

At time: $t = 0$, or: $\varphi = 0$, it is, substituting in (14) and (15)

$$\left. \begin{aligned} i_0 &= C_1 + \frac{E}{z_0} \cos (\omega + \alpha) \\ e_0 &= -\frac{r}{2} C_1 + \frac{q}{2} C_2 - \frac{k}{z_0} E \sin (\omega + \alpha) \end{aligned} \right\}$$

and herefrom:

$$\left. \begin{aligned} C_1 &= i_0 - \frac{E}{z_0} \cos (\omega + \alpha) \\ C_2 &= \frac{2 e_0 + r i_0}{q} + \frac{E}{q z_0} [2 k \sin (\omega + \alpha) - r \cos (\omega + \alpha)] \end{aligned} \right\} \quad (16)$$

Substituting (16) in (14) and (15), gives:

Current:

$$\begin{aligned} i &= \frac{E}{z_0} \cos (\varphi - \omega + \alpha) + \varepsilon^{-\frac{r}{2x}} \left\{ \left[i_0 - \frac{E}{z_0} \cos (\omega + \alpha) \right] \cos \frac{q}{2x} \varphi \right. \\ &\quad \left. - \left[\frac{2 e_0 + r i_0}{q} + \frac{E}{q z_0} (2 k \sin (\omega + \alpha) - r \cos (\omega + \alpha)) \right] \sin \frac{q}{2x} \varphi \right\} \end{aligned} \quad (17)$$

Potential Difference at Condenser:

$$\begin{aligned} e_1 &= \frac{k}{z_0} E \sin (\varphi - \omega + \alpha) + \varepsilon^{-\frac{r}{2x}} \left\{ \left[e_0 + \frac{k}{z_0} E \sin (\omega - \alpha) \right] \cos \frac{q}{2x} \varphi \right. \\ &\quad \left. + \left[\frac{2 r e_0 + (r^2 + q^2) i_0}{2 q} + \frac{E}{2 q z_0} (2 k r \sin (\omega - \alpha) \right. \right. \\ &\quad \left. \left. - (r^2 + q^2) \cos (\omega - \alpha) \right] \sin \frac{q}{2x} \varphi \right\} \end{aligned}$$

These equations (17) can be essentially simplified by neglecting terms of secondary magnitude.

k is in high potential transmission lines or cables always very large compared with r and x .

The full load resistance and reactance voltage may vary from less than five per cent. to about 20 per cent. of impressed voltage, the charging current of the line from five per cent. to about 20 per cent. of full load current, at normal voltage and frequency.

In this case, k is from 25 to more than 400 times as large as r or x , and r and x thus negligible compared with k .

It is then, in close approximation :

$$\left. \begin{aligned} z_0 &= k \\ q &= 2 \sqrt{x k} \\ B &= \frac{E}{k} \\ \alpha &= \frac{\pi}{2} = 90^\circ \end{aligned} \right\} \quad (18)$$

Substituting these values in equations (17), gives:

Current:

$$\begin{aligned} i = & -\frac{E}{k} \sin(\varphi - \alpha) + \varepsilon^{-\frac{r}{2x}\varphi} \left\{ \left[i_0 - \frac{E}{k} \sin \alpha \right] \cos \sqrt{\frac{k}{x}} \varphi \right. \\ & \left. - \left[\frac{2 e_0 + r i_0}{2 \sqrt{x k}} - \frac{E}{2 \sqrt{x k}} (2 \cos \omega + \frac{r}{k} \sin \omega) \right] \sin \sqrt{\frac{k}{x}} \varphi \right\} \end{aligned} \quad (19)$$

Potential Difference at Condenser:

$$\begin{aligned} e_1 = & E \cos(\varphi - \omega) + \varepsilon^{-\frac{r}{2x}\varphi} \left\{ [e_0 - E \cos \omega] \cos \sqrt{\frac{k}{x}} \varphi \right. \\ & + \left[\frac{2 r e_0 + (r^2 + 4 x k) i_0}{4 \sqrt{x k}} - \frac{E}{4 \sqrt{x k}} (2 r \cos \omega \right. \\ & \left. \left. + \frac{r^2 + 4 x k}{k} \sin \varepsilon) \right] \sin \sqrt{\frac{k}{x}} \varphi \right\} \end{aligned}$$

These equations consist of three terms:

$$\left. \begin{aligned} i &= i' + i'' + i''' \\ e_1 &= e_1' + e_1'' + e_1''' \end{aligned} \right\} \quad (20)$$

$$\left. \begin{aligned} i' &= -\frac{E}{k} \sin (\varphi - \omega) \\ e_1' &= E \cos (\varphi - \omega) \end{aligned} \right\} \quad (21)$$

$$\left. \begin{aligned} i'' &= -\frac{E}{k} \varepsilon^{-\frac{r}{2x}\varphi} \left\{ \sin \omega \cos \sqrt{\frac{k}{x}} \varphi - \left[\sqrt{\frac{k}{x}} \cos \omega \right. \right. \\ &\quad \left. \left. + \frac{r}{2\sqrt{xk}} \sin \omega \right] \sin \sqrt{\frac{k}{x}} \varphi \right\} \\ e_1'' &= -E \varepsilon^{-\frac{r}{2x}\varphi} \left\{ \cos \omega \cos \sqrt{\frac{k}{x}} \varphi + \left[\frac{r}{2\sqrt{xk}} \cos \omega \right. \right. \\ &\quad \left. \left. + \frac{r^2 + 4xk}{4k\sqrt{xk}} \sin \omega \right] \sin \sqrt{\frac{k}{x}} \varphi \right\} \end{aligned} \right\} \quad (22)$$

or, by dropping terms of secondary order :

$$\left. \begin{aligned} i'' &= \frac{E}{\sqrt{xk}} \varepsilon^{-\frac{r}{2x}\varphi} \cos \omega \sin \sqrt{\frac{k}{x}} \varphi \\ e_1'' &= -E \varepsilon^{-\frac{r}{2x}\varphi} \cos \omega \cos \sqrt{\frac{k}{x}} \varphi \end{aligned} \right\} \quad (23)$$

$$\left. \begin{aligned} i''' &= \varepsilon^{-\frac{r}{2x}\varphi} \left\{ i_0 \cos \sqrt{\frac{k}{x}} \varphi - \frac{2e_0 + r i_0}{2\sqrt{xk}} \sin \sqrt{\frac{k}{x}} \varphi \right\} \\ e_1''' &= \varepsilon^{-\frac{r}{2x}\varphi} \left\{ e_0 \cos \sqrt{\frac{k}{x}} \varphi + \frac{2re_0 + (r^2 + 4xk)i_0}{4\sqrt{xk}} \sin \sqrt{\frac{k}{x}} \varphi \right\} \end{aligned} \right\} \quad (24)$$

or, by dropping terms of secondary order :

$$\left. \begin{aligned} i''' &= \varepsilon^{-\frac{r}{2x}\varphi} \left\{ i_0 \cos \sqrt{\frac{k}{x}} \varphi - \frac{e_0}{\sqrt{xk}} \sin \sqrt{\frac{k}{x}} \varphi \right\} \\ e_1''' &= \varepsilon^{-\frac{r}{2x}\varphi} \left\{ e_0 \cos \sqrt{\frac{k}{x}} \varphi + i_0 \sqrt{xk} \sin \sqrt{\frac{k}{x}} \varphi \right\} \end{aligned} \right\} \quad (25)$$

Thus :

Total Current, approx :

$$i = -\frac{E}{k} \sin(\varphi - \omega) + \varepsilon^{-\frac{r}{2x}\varphi} \left\{ i_0 \cos \sqrt{\frac{k}{x}} \varphi - \frac{e_0 - E \cos \omega}{\sqrt{xk}} \sin \sqrt{\frac{k}{x}} \varphi \right\}$$

Difference of Potential at Condenser :

$$e_1 = E \cos(\varphi - \omega) + \varepsilon^{-\frac{r}{2x}\varphi} \left\{ (e_0 - E \cos \omega) \cos \sqrt{\frac{k}{x}} \varphi + i_0 \sqrt{xk} \sin \sqrt{\frac{k}{x}} \varphi \right\}$$

(26)

Of the three terms: i' , e_1' ; i'' , e_1'' ; i''' , e_1''' , the first obviously represents the stationary condition of charging current and condenser potential, since the two other terms disappear for $t = \infty$.

The second term, i'' , e_1'' , represents that component of oscillation which depends upon the phase of impressed E. M. F., or the point of the impressed E. M. F. wave, at which the oscillation begins, while the third term, i''' , e_1''' , represents the component of oscillation which depends upon the instantaneous values of current resp. E. M. F. at the moment at which the oscillation begins.

$\varepsilon^{-(r/2x)\varphi}$ is the decrement of the oscillation.

The frequency of oscillation is :

$$N_o = \sqrt{\frac{k}{x}} N$$

where N is the impressed frequency. That is, the frequency of oscillation equals the impressed frequency times the square root of the ratio of capacity reactance and inductive reactance of the circuit, or is the impressed frequency divided by the square root of inductance voltage and capacity current, as fraction of impressed voltage and full load current.

Since :

$$k = \frac{1}{2\pi N C}, \quad x = 2\pi N L,$$

the frequency of oscillation is:

$$N_0 = \frac{1}{2 \pi \sqrt{C L}}$$

that is, independent of the frequency of the impressed E. M. F.

Substituting:

$$\varphi = 2 \pi N t, k = \frac{1}{2 \pi N C}, x = 2 \pi N L$$

in equations (23), (25), and (26), it is:

$$\left. \begin{aligned} i'' &= \sqrt{\frac{C}{L}} E \varepsilon^{-(r/2L)t} \cos \omega \sin \frac{t}{\sqrt{C L}} \\ e_1'' &= -E \varepsilon^{-(r/2L)t} \cos \omega \cos \frac{t}{\sqrt{C L}} \end{aligned} \right\} \quad (27)$$

$$\left. \begin{aligned} i''' &= \varepsilon^{-(r/2L)t} \left\{ i_0 \cos \frac{t}{\sqrt{C L}} - e_0 \sqrt{\frac{C}{L}} \sin \frac{t}{\sqrt{C L}} \right\} \\ e_1''' &= \varepsilon^{-(r/2L)t} \left\{ e_0 \cos \frac{t}{\sqrt{C L}} + i_0 \sqrt{\frac{L}{C}} \sin \frac{t}{\sqrt{C L}} \right\} \end{aligned} \right\} \quad (28)$$

$$\left. \begin{aligned} i &= -2 \pi N C E' \sin (\varphi - \omega) + \varepsilon^{-(r/2L)t} \left\{ i_0 \cos \frac{t}{\sqrt{C L}} \right. \\ &\quad \left. - (e_0 - E \cos \omega) \sqrt{\frac{C}{L}} \sin \frac{t}{\sqrt{C L}} \right\} \\ e_1 &= E \cos (\varphi - \omega) + \varepsilon^{-(r/2L)t} \left\{ (e_0 - E \cos \omega) \cos \frac{t}{\sqrt{C L}} \right. \\ &\quad \left. + i_0 \sqrt{\frac{L}{C}} \sin \frac{t}{\sqrt{C L}} \right\} \end{aligned} \right\} \quad (29)$$

The oscillating terms of these equations is independent of the impressed frequency. That is, the oscillating current and potential difference, caused by a change of circuit conditions (as starting, change of load or opening circuit), are independent of the impressed frequency, and thus also of the wave shape of the im-

pressed E. M. F., or its higher harmonics (except as regards to terms of secondary order).

The first component of oscillation, equations (22) and (23), besides upon the line constants and upon the impressed E. M. F., depends only upon the phase, or the point of the impressed E. M. F. wave, at which the oscillation starts, but does not depend upon the previous condition of the circuit.

Therefore this component of oscillation is the same as the oscillation produced in starting the transmission line, that is, connecting it, unexcited, to the generator terminals.

There exists no point of the impressed E. M. F. wave, where no oscillation occurs (while, when starting a circuit containing resistance and induction only, at the point of the impressed E. M. F. wave, where the final current passes zero, the stationary condition is instantly reached).

With capacity in circuit, any change of circuit conditions involves an electric oscillation.

The maximum intensity of starting oscillation occurs near the value: $a=0$, and is:

$$\left. \begin{aligned} i'' &= \frac{E}{\sqrt{xk}} \varepsilon^{-(r/2x)\varphi} \sin \sqrt{\frac{k}{x}} \varphi \\ e_1'' &= -E \varepsilon^{-(r/2x)\varphi} \cos \sqrt{\frac{k}{x}} \varphi \end{aligned} \right\} (30)$$

Since:

$$i' = -\frac{E}{k} \sin (\varphi - \omega)$$

is the stationary value of charging current, it follows:

The maximum intensity which the oscillating current produced in starting a transmission line may reach, is $\sqrt{\frac{k}{x}}$ times the stationary charging current, or the initial rush of current bears to the stationary value the same ratio as the frequency of oscillation to the impressed frequency.

The maximum oscillating E. M. F. induced in starting a transmission line is of the same value as the impressed E. M. F. Thus the maximum value of potential difference occurring in a transmission line at starting is less than twice the impressed E. M. F. and no excessive voltages can be induced in starting a circuit.

The minimum value of starting oscillation occurs near: $\omega = 90^\circ$, and is, from equations (22):

$$\left. \begin{aligned} i'' &= -\frac{E}{k} \varepsilon^{-(r/2x)\phi} \cos \sqrt{\frac{k}{x}} \phi \\ e_1'' &= \sqrt{\frac{x}{k}} E \varepsilon^{-(r/2x)\phi} \sin \sqrt{\frac{k}{x}} \phi \end{aligned} \right\} (31)$$

That is, the oscillating current is of the same intensity as the charging current, and the maximum rush of current thus is less than twice the stationary value. The potential difference in the circuit rises only little above the impressed E. M. F.

The second component of oscillation, equations (24.) (25.), does not depend upon the point of the impressed E. M. F. wave, at which the oscillation starts, ω , nor upon the impressed E. M. F. as a whole, E , but, besides upon the constants of the circuit, it only depends upon the instantaneous values of current and of potential difference in the circuit at the moment where the oscillation starts, i_0 and e_0 .

Thus, if: $i_0 = 0$, $e_0 = 0$, or in starting a transmission line, unexcited, by connecting it to the impressed E. M. F., this term disappears.

It is this component which may cause excessive potential differences.

Two cases shall more fully be discussed.

(a.) Opening the circuit of a transmission line under load.

(b.) Rupturing a short-circuit on the transmission line.

A.

If i_0 is the instantaneous value of full load current, e_0 the instantaneous value of difference of potential at the condenser, it is:

ri small compared with e_0 :

$\sqrt{x/k} i_0$ of the same magnitude as e_0 .

Denoting:

$$\tan \delta = \frac{e_0}{i_0 \sqrt{x/k}}$$

it is, substituting in equations (24):

$$\left. \begin{aligned} i''' &= \sqrt{i_0^2 + \frac{e_0^2}{xk}} \varepsilon^{-(r/2x)\phi} \cos\left(\sqrt{\frac{k}{x}}\phi + \delta\right) \\ e_1''' &= \sqrt{e_0^2 + i_0^2 xk} \varepsilon^{-(r/2x)\phi} \sin\left(\sqrt{\frac{k}{x}}\phi + \delta\right) \end{aligned} \right\} \quad (32)$$

That is, the amplitude of oscillation is

$$\sqrt{i_0^2 + \frac{e_0^2}{xk}}$$

for the current, $\sqrt{e_0^2 + i_0^2 xk}$ for the E. M. F. Thus the induced E. M. F. can be larger than the impressed E. M. F., but is, as a rule, still of the same magnitude, except if k is very large.

In the expression of the total current resp. potential difference at condenser, in equations (26.), $(e_0 - E \cos \omega)$ is the difference between the potential difference at the condenser and the impressed E. M. F., at the instant of starting of the oscillation, or the voltage consumed by the line impedance, and this is small, if the current is not excessive. Thus, neglecting the terms with $(e_0 - E \cos \omega)$, equations (26.) assume the form :

$$\left. \begin{aligned} i &= -\frac{E}{k} \sin(\phi - \omega) + i_0 \varepsilon^{-(r/2x)\phi} \cos \sqrt{\frac{k}{x}}\phi \\ e_1 &= E \cos(\phi - \omega) + i_0 \sqrt{xk} \varepsilon^{-(r/2x)\phi} \sin \sqrt{\frac{k}{x}}\phi \end{aligned} \right\} \quad (33)$$

That is, the oscillation of current is of the amplitude of full load current, and the oscillation of condenser potential is of the amplitude $i_0 \sqrt{xk}$.

xk is the ratio of inductance voltage to condenser current, in fractions of full load voltage and current. It is also :

$$i_0 \sqrt{xk} = i_0 \sqrt{\frac{L}{C}}$$

Thus in circuits of very high inductance L and relatively low capacity C , $i_0 \sqrt{xk}$ may be much higher than the impressed E. M. F., and a serious rise of potential occur in opening the circuit under load, while in low inductance cables of high capacity,

$i_0 \sqrt{xk}$ is moderate. That is, the inductance, by tending to maintain the current, induces a rise of potential, while capacity exerts a cushioning effect. Low inductance and high capacity thus are of advantage when breaking full load current in a circuit.

B.

If a transmission line containing resistance, inductance and capacity is short-circuited, and the short-circuit suddenly opened at time $t = 0$, it is, for: $t < 0$:

$$\left. \begin{aligned} e_0 &= 0 \\ i &= \frac{E}{z} \cos (\varphi - \omega - j) \\ \text{where: } z &= \sqrt{r^2 + x^2} \\ \tan j &= \frac{x}{r} \end{aligned} \right\} (34)$$

thus, at time:

$$t = 0 \quad i_0 = \frac{E}{z} \cos (\omega + j) \quad (35)$$

Substituting these values of e_0 and i_0 in equations (24), gives:

$$i''' = \frac{E}{z} \cos (\omega + j) \varepsilon^{-(r/2x)\phi} \left\{ \cos \sqrt{\frac{k}{x}} \varphi - \frac{r}{2\sqrt{xk}} \sin \sqrt{\frac{k}{x}} \varphi \right\}$$

$$e_1''' = \frac{E}{z} \cos (\omega + j) \varepsilon^{-(r/2x)\phi} \frac{r^2 + 4xk}{4\sqrt{xk}} \sin \sqrt{\frac{k}{x}} \varphi$$

or, neglecting terms of secondary magnitude:

$$\left. \begin{aligned} i''' &= \frac{E}{z} \varepsilon^{-(r/2x)\phi} \cos (\omega + j) \cos \sqrt{\frac{k}{x}} \varphi \\ e_1''' &= \frac{E\sqrt{xk}}{z} \varepsilon^{-(r/2x)\phi} \cos (\omega + j) \sin \sqrt{\frac{k}{x}} \varphi \end{aligned} \right\} (36)$$

That is, i''' is of the magnitude of short-circuit current, and e_1''' of higher magnitude than the impressed E. M. F., since z is small compared with $\sqrt{xk}$.

The total values of current and condenser potential difference are, from equation (26):

$$\left. \begin{aligned} i &= -\frac{E}{k} \sin(\varphi - \omega) + E \varepsilon^{-(r/2x)\phi} \left\{ \left[\frac{\cos(\omega + j)}{z} \right. \right. \\ &\quad \left. \left. - \frac{\sin \omega}{k} \right] \cos \sqrt{\frac{k}{x}} \varphi + \frac{\cos \omega}{\sqrt{xk}} \sin \sqrt{\frac{k}{x}} \varphi \right\} \\ e_1''' &= E \cos(\varphi - \omega) - E \varepsilon^{-(r/2x)\phi} \left\{ \cos \omega \cos \sqrt{\frac{k}{x}} \varphi \right. \\ &\quad \left. - \frac{\sqrt{xk} \cos(\omega + j)}{z} \sin \sqrt{\frac{k}{x}} \varphi \right\} \end{aligned} \right\} \quad (37)$$

or approximately, since all terms are negligible compared with ψ''' and e_1''' :

$$\left. \begin{aligned} i &= \frac{E}{z} \varepsilon^{-(r/2x)\phi} \cos(\omega + j) \cos \sqrt{\frac{k}{x}} \varphi \\ e_1 &= \frac{E \sqrt{xk}}{z} \varepsilon^{-(r/2x)\phi} \cos(\omega + j) \sin \sqrt{\frac{k}{x}} \varphi \end{aligned} \right\} \quad (38)$$

These values are a maximum, if the circuit is opened at the moment: $\alpha = -j$, that is at the maximum value of the short-circuit current, and are then:

$$\left. \begin{aligned} i &= \frac{E}{z} \varepsilon^{-(r/2x)\phi} \cos \sqrt{\frac{k}{x}} \varphi \\ e_1 &= \frac{\sqrt{xk}}{z} E \varepsilon^{-(r/2x)\phi} \sin \sqrt{\frac{k}{x}} \varphi \end{aligned} \right\} \quad (39)$$

The amplitude of oscillation of condenser potential difference is:

$$\frac{\sqrt{xk}}{z} E$$

or, neglecting the line resistance, as rough approximation:
 $\omega = z$:

$$\sqrt{\frac{k}{x}} E$$

That is, the potential difference at the condenser is increased above the impressed E. M. F. in the proportion of the square root of the ratio of capacity reactance to inductive reactance, or inverse proportional to the square root of inductance voltage times capacity current, as fraction of impressed voltage and full load current.

Thus in this case, the rise of voltage is excessive.

The minimum intensity of the oscillation due to rupturing short-circuit occurs if the circuit is broken at the moment: $\alpha = 90^\circ - j$, that is, at the zero value of short-circuit current. It is then:

$$\left. \begin{aligned} i &= \frac{E}{\varphi} \cos(\varphi + j) - E \varepsilon^{-(r/2x)\varphi} \left\{ \frac{\cos j}{\varphi} \cos \sqrt{\frac{\varphi}{x}} \varphi \right. \\ &\quad \left. - \frac{\sin j}{\sqrt{x}\varphi} \sin \sqrt{\frac{\varphi}{x}} \varphi \right\} \\ e_1 &= E \sin(\varphi + j) - E \varepsilon^{-(r/2x)\varphi} \sin j \cos \sqrt{\frac{k}{x}} \varphi \end{aligned} \right\} \quad (40)$$

That is, the potential difference at the condenser is less than twice the impressed E. M. F., thus moderate.

Hence, a short-circuit can be opened safely only at or near the zero value of the short-circuit current.

The phenomenon ceases to be oscillating, and becomes an ordinary logarithmic discharge, if $\sqrt{r_2^2 - 4xk}$ is real, or:

$$r < 2\sqrt{xk}.$$

Some instances may illustrate the phenomena discussed in the preceding:

I.

Let in a transmission line carrying 100 amperes at full load, under an impressed E. M. F. of 20,000 volts, the resistance drop equal 8%, the inductance voltage 15% of the impressed voltage, and the charging current 8% of full load current. Assuming as resistance drop in the step-up transformers, 1%, and as reactance voltage $2\frac{1}{2}\%$.

The resistance drop between constant potential generator terminals and middle of transmission line is then 5%, or $r = 10$ ohms. The inductance voltage is 10%, or $x = 20$ ohms. The

charging current of the line is 8 amperes, thus the capacity reactance $k = 2500$ ohms.

It is then, assuming a sine wave of impressed E. M. F.:

$$E = 20000 \sqrt{2} = 28280 \text{ volts.}$$

$$i' = -11.3 \sin(\varphi - \omega)$$

$$e_1' = 28280 \cos(\varphi - \omega)$$

$$i'' = -11.3 \epsilon^{-.25\phi} [\sin \omega \cos 11.2 \varphi - 11.2 \cos \omega \sin 11.2 \varphi]$$

$$e_1'' = -28280 \epsilon^{-.25\phi} [\cos \omega \cos 11.2 \varphi - .089 \sin \omega \sin 11.2 \varphi]$$

FIG. 1.

Thus, the oscillation produced in starting the transmission line is:

$$i = -11.3 [\sin(\varphi - \omega) + \epsilon^{-.25\phi} (\sin \omega \cos 11.2 \varphi - 11.2 \cos \omega \sin 11.2 \varphi)]$$

$$e_1 = 28280 [\cos(\varphi - \omega) - \epsilon^{-.25\phi} (\cos \omega \cos 11.2 \varphi - .089 \sin \omega \sin 11.2 \varphi)]$$

Hence the maximum value, for: $\omega = 0$:

$$i = -11.3 (\sin \varphi - 11.2 \epsilon^{-.25\phi} \sin 11.2 \varphi)$$

$$e_1 = 28280 (\cos \varphi - \epsilon^{-.25\phi} \cos 11.2 \varphi)$$

and the minimum value, for: $\omega = 90^\circ$:

$$i = 11.3 (\cos \varphi - \varepsilon^{-.25 \phi} \cos 11.2 \varphi)$$

$$e_1 = 28280 (\sin \varphi + .089 \varepsilon^{-.25 \phi} \sin 11.2 \varphi)$$

These values are plotted in Figs. 1 and 2, with the current i in dotted and the potential difference e_1 in drawn line. The stationary values are plotted also, in thin lines.

(a.) Opening the circuit under full load, it is:

FIG. 2.

$$i = -11.3 \sin (\varphi - \omega) + \varepsilon^{-.25 \phi} (i_0 - 11.3 \sin \omega) \cos 11.2 \varphi$$

$$e_1 = 28280 \cos (\varphi - \omega) + 224 i_0 \varepsilon^{-.25 \phi} \sin 11.2 \varphi$$

These values are maximum for $\omega = 0$ and non-inductive circuit, or: $i_0 = 141.4$, and are:

$$i = -11.3 \sin \varphi + 141.4 \varepsilon^{-.25 \phi} \cos 11.2 \varphi$$

$$e_1 = 28280 \cos \varphi + 31600 \varepsilon^{-.25 \phi} \sin 11.2 \varphi$$

These values are plotted, in Fig. 3, in the same manner as Figs. 1 and 2:

(b.) Rupturing the line under short-circuit, it is:

$$z = 22.4$$

$$i_o = 1265 \cos (\omega + j)$$

thus:

$$i = -11.3 \sin (\varphi - \omega) + 1265 \varepsilon^{-.25 \phi} \{ [\cos (\omega + j) - .0089 \sin \omega] \cos 11.2 \varphi + .1 \cos \omega \sin 11.2 \varphi \}$$

$$e_1 = 28280 \{ \cos (\varphi + \omega) - \varepsilon^{-.25 \phi} [\cos \omega \cos 11.2 \varphi - 10 \cos (\omega + j) \sin 11.2 \varphi] \}$$

FIG. 3.

These values are a maximum for: $\omega = -j = -63^\circ$:

$$i = -11.3 \sin (\varphi + 63^\circ) + 1260 \varepsilon^{-.25 \phi} (\cos 11.2 \varphi + .044 \sin 11.2 \varphi)$$

$$e_1 = 28280 \cos (\varphi + 63^\circ) - 282,800 \varepsilon^{-.25 \phi} (.044 \cos 11.2 \varphi - \sin 11.2 \varphi)$$

That is, the potential difference rises about tenfold, to 282,800 volts.

These values are plotted in Fig. 4.

II.

In the instance referred to in the beginning of the paper, where I first observed the destructive voltage induced by a short-circuiting arc, on an experimental 10,000-volt 40-cycle line, during the occurrence of the phenomenon the generator voltage probably fell to about 5,000, due to limited capacity. The resistance of the system was very low, about $r = 1$ ohm, while the inductive reactance may be estimated as $x = 10$ ohms, and the capacity reactance as $k = 20,000$ ohms.

It is thus: $\tan j = 10$, or approximately: $j = 90^\circ$.

FIG. 4.

Herefrom it follows:

$$i = 707 \varepsilon^{-.05 \phi} \cos 44.7 \phi$$

$$e_1 = 316000 \varepsilon^{-.05 \phi} \sin 44.7 \phi$$

That is, the oscillation has a frequency of about 1,800 cycles, and a maximum E. M. F. of nearly one-third million volts, which fully accounts for its disruptive effects.

As conclusion, it follows herefrom:

1. The most important source of destructive high voltage phenomena in high potential circuits containing inductance and capacity are not resonance phenomena with the wave of im-

pressed E. M. F. or its higher harmonics, in case of the deviation of the wave from sine shape, but are the electric oscillations produced by a change of circuit conditions, as starting, opening circuit, etc.

2. These phenomena are essentially independent of frequency and wave shape of impressed E. M. F., but depend upon the conditions under which the circuit is changed, as the manner of change and the point of the impressed E. M. F. and current wave at which the change occurs.

3. The electric oscillations occurring in connecting a transmission line to the generator are not of dangerous potential, but the oscillations produced by opening the transmission circuit under load may reach destructive voltages, and the oscillations caused by interrupting a short-circuit are liable to reach voltages far beyond the strength of any insulation. Thus special precautions should be taken in opening a high potential circuit under load. But the most dangerous phenomenon is a low resistance short-circuit in open space.

4. The voltages produced by the oscillations in open-circuiting a transmission line under load or under short-circuit are moderate, if the opening of the circuit occurs at a certain point of the E. M. F. wave. This point approximately coincides with the moment of zero current.

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