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On an Extension of the term "Area" to any closed Circuit in Space.

By R. B. HAYWARD, M.A.

[Read March 13th, 1873.]

De Morgan has shown (*Cam. and Dub. Math. Journ.*, vol. v.) that the term "Area" may be applied in a perfectly definite sense to any closed circuit in one plane, however *autotomic* its character, and that it may be regarded as having an algebraical sign determined by the direction in which the circuit is supposed to be described.

I propose to show that there is, for any closed circuit in space, a plane area determined by the circuit itself without any extrinsic elements, which is definite as well in the *aspect* of its plane as in magnitude, and which, since it reduces to the area of the circuit in De Morgan's sense when the circuit is plane, may properly be termed the area of the circuit in the general case.

In this sense, area is no longer a mere magnitude, or a magnitude affected only with the positive or negative sign, but a magnitude affected with direction ; in other words, it is a *vector*, not simply a *scalar*.

Conceive a point to describe continuously a given closed circuit in space of three dimensions. Then its projection on any given plane will describe a closed circuit in that plane, the area of which will have a definite sign depending on the convention which is made as to the positive direction of revolution on the positive aspect of the plane. Let the positive aspects of the rectangular coordinate planes of yz , zx , xy be those which are turned towards the positive sides of the axes of x , y , z respectively, then the projections of the given circuit on the coordinate planes will have areas determinate in sign as well as in magnitude. Let them be denoted by A_1 , A_2 , A_3 respectively. Then, if the direction, cosines of a given plane, that is, the cosines of the angles between its

positive aspect and the positive aspects of the coordinate planes respectively, be denoted by l, m, n , the area of the projection of the given circuit on that plane will be given, both in sign and magnitude, by the expression

$$lA_x + mA_y + nA_z.$$

This expression is well known for a plane circuit, and it will be easily seen to be true for any circuit whatever. For, if PQ be an element of the circuit, and O any fixed point, the expression will be true for the projections of the plane element POQ; and therefore, for the algebraical sums of the projections of all such elements, and these determine the areas of the projections of the complete circuit.

Now, if $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$, the expression $lA_x + mA_y + nA_z$ is a maximum, and is equal to A , when $l = \frac{A_x}{A}$, $m = \frac{A_y}{A}$, $n = \frac{A_z}{A}$.

Hence, on a plane, whose aspect is determined by these values of l, m, n , the area of the projection (A) of the given circuit will be a maximum; and if θ be the angle between the positive aspect of a given plane and that of the plane of A , it will be readily seen that $A \cos \theta$ is the area of the projection of the circuit on that plane. Thus the projection of the area A^* on any plane has an area equal to that of the projection of the circuit itself. Farther, it is obvious that, if the circuit be plane, A is the area of the circuit itself in the ordinary sense. It is therefore a natural extension of the term to speak of this maximum area A as *the* area of the circuit, in the general case of any circuit whatever in space.

Adopting this more general sense of the term area, it is plain that areas are vector magnitudes subject to the laws of vector addition and resolution, like finite lines, forces, couples, velocities, &c.

It is worth noting that on any plane perpendicular to that of the area of a circuit, the area of its projection will be zero, so that on all such planes the projection of the circuit must be either a finite arc of a curve, or an autotomic circuit, in which the positive and negative "primary parts" of the area are equal. Conversely, if *two* planes can be found on which the areas of the projections of a closed circuit are zero, on *every* plane parallel to their common section, the area of its projection will also be zero, and the plane perpendicular to both of them will be the plane of the area of the circuit.

Let me conclude with a few illustrations of the use of this extension of the term area.

It is well known that if a set of forces, acting along the sides of a plane polygon, taken in order, be represented in magnitude by those

* It is to be observed that the projections of the area A and of the circuit are not in general identical or identically equal, but only equal in area.

sides, their resultant will be a couple represented in sign and magnitude by double the area of the polygon. With the extended sense of the term area, it is easily seen that the same proposition is true for any polygon in space, the aspect of the area defining that of the couple.

Again, if a homogeneous incompressible fluid move in a tubular channel of small section, whose axis is a closed curve, and which it completely fills, it is plain that, as the centre of inertia of the fluid is at rest, the total momentum of the fluid is reducible to an angular momentum, which may easily be shown to be in the plane of "the area of the curve," and proportional to that area in intensity.

Lastly, to take an example from pure geometry, the area of a skew quadrilateral is the area of its projection on a plane parallel to both its diagonals. For, on any plane perpendicular to this, its diagonals will be projected into two parallel lines as AC, BD, and the quadrilateral into the plane autotomic quadrilateral ABCD whose area is evidently zero, so that on a plane parallel to both diagonals the projection is a maximum. Hence the area of a skew quadrilateral is the same as that of a plane quadrilateral with diagonals equal to those of the former, and inclined to one another at the same angle; and we may state generally that the area of *any* quadrilateral, skew or plane, is half that of the parallelogram contained by its diagonals. The same result might have been obtained by the vector-addition of the areas of the two triangles, into which the quadrilateral is divided by one of its diagonals.

On the Evaluation of a Class of Definite Integrals involving Circular Functions in the Numerator, and Powers of the Variable only in the Denominator. By J. W. L. GLAISHER, B.A., F.R.A.S., Fellow of Trinity College, Cambridge.

[Read March 13th, 1873.]

The main purpose of the present paper will be best understood by commencing with an example. Suppose it were required to find the value of $\int_0^\infty \frac{\sin ax \sin bx}{x^{\frac{1}{2}}} dx$, which is clearly finite, then the most obvious mode of procedure would be to replace $\sin ax \sin bx$ by

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