TWO ILLUSTRATIONS OF ELIMINATION.

We can, therefore, so arrange them that not only is $G$ the centroid but also $\Sigma xx = q \Sigma x$ as well. The foregoing proof is therefore perfectly general.

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NOTE ON THE SOLUTION OF CUBIC AND Biquadratic EQUATIONS.

1. Let $x^3 + px^2 + qx + r = 0$ be a cubic equation, whose roots are $a, \beta, \gamma$. Then, defining the quantity $\lambda$ by means of the equation $\lambda^3 = 1$, we have

$$(a + \lambda \beta + \lambda^2 \gamma)^3 = a^3 + \beta^3 + \gamma^3 + 6a\beta\gamma + 3\lambda(\beta^2 \gamma + \gamma^2 a + a^2 \beta) + 3\lambda^2(\beta^2 \gamma + \gamma^2 a + a^2 \beta)$$

$$= A + \lambda B + \lambda^2 C.$$  

Since this is true for all values of $\lambda$ that satisfy the equation $\lambda^3 = 1$; we have, $\omega$ being one of the imaginary cube roots of unity, the three equations

$$a + \beta + \gamma = (A + B + C)^{1/3},$$

$$a + \omega \beta + \omega^2 \gamma = (A + \omega B + \omega^2 C)^{1/3},$$

$$a + \omega^2 \beta + \omega \gamma = (A + \omega^2 B + \omega C)^{1/3}.$$ 

From these equations it follows that

$$3a = (A + B + C)^{1/3} + (A + \omega B + \omega^2 C)^{1/3} + (A + \omega^2 B + \omega C)^{1/3},$$

$$3\beta = (A + B + C)^{1/3} + \omega^2(A + \omega B + \omega^2 C)^{1/3} + \omega(A + \omega^2 B + \omega C)^{1/3},$$

$$3\gamma = (A + B + C)^{1/3} + \omega(A + \omega B + \omega^2 C)^{1/3} + \omega^2(A + \omega^2 B + \omega C)^{1/3},$$

Thus, if $A, B, C$ can be expressed in terms of the coefficients of the given cubic equation, we have a solution of the said equation. Now $A$ is a symmetrical function of the roots of the said equation, and may therefore be so expressed. In fact we have

$$A = -p^3 + 3pq - qr.$$ 

Writing

$$U = \beta^2 \gamma + \gamma^2 \alpha + \alpha^2 \beta,$$

$$V = \beta^2 \gamma + \gamma^2 \alpha + \alpha^2 \beta,$$

we see that $U + V$ is a symmetrical function of the roots. In fact we have

$$U + V = -pq + 3r. \quad (1)$$

Also

$$U - V = \beta \gamma (\beta - \gamma) + \gamma \alpha (\gamma - \alpha) + \alpha \beta (\alpha - \beta)$$

$$= (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta).$$

This shows that $(U - V)^2$ is a symmetrical function of the roots; and, as $U + V$ is of the same character, it follows that $UV$ must be so also. In fact we have

$$UV = \beta^2 \gamma \alpha^2 + \gamma^2 \alpha^2 + \alpha^2 \beta^2 + 3\alpha^2 \beta^2 \gamma^2 + a\beta \gamma (\alpha^3 + \beta^3 + \gamma^3)$$

$$= p^2 r + q^2 r + 6pqr + qr^2. \quad (2)$$

From equations (1) and (2) the values of $U$ and $V$ can be readily obtained by a known method. Thus the values of $B$ and $C$, in terms of the coefficients of the equation, can be found, and the solution completed.

2. Next suppose that we have a biquadratic equation

$$x^4 + px^2 + qx + r = 0,$$

whose roots are $a, \beta, \gamma, \delta$. This time we will employ two quantities, $\lambda$ and $\mu$, defined by the equations

$$\lambda^2 = 1, \quad \mu^2 = 1.$$
We have \((a + \lambda \beta + \mu \gamma + \lambda \mu \delta)^2 = a^2 + \beta^2 + \gamma^2 + \delta^2 + 2\lambda(a\beta + \gamma\delta) + 2\mu(a\gamma + \beta\delta) + 2\lambda\mu(a\delta + \beta\gamma) = A + \lambda\beta + \mu\gamma + \lambda\mu \delta\). 

Now this equation is satisfied by any pair of possible values of \(\lambda\) and \(\mu\).

We have, therefore, four cases, viz.,

\[
\begin{align*}
\lambda &= 1, & \mu &= 1, \\
\lambda &= -1, & \mu &= 1, \\
\lambda &= 1, & \mu &= -1, \\
\lambda &= -1, & \mu &= -1.
\end{align*}
\]

Thus we obtain

\[
\begin{align*}
a + \beta + \gamma + \delta &= (A + B + C + D)^\frac{1}{3}, \\
a - \beta + \gamma - \delta &= (A - B + C - D)^\frac{1}{3}, \\
a + \beta - \gamma - \delta &= (A + B - C - D)^\frac{1}{3}, \\
a - \beta - \gamma + \delta &= (A - B - C + D)^\frac{1}{3}.
\end{align*}
\]

Thus, if \(A, B, C, D\) can be expressed in terms of the coefficients of the equation, we readily obtain the solution in the following form:

\[
\begin{align*}
4a &= (A + B + C + D)^\frac{1}{3} + (A - B + C - D)^\frac{1}{3} + (A + B - C - D)^\frac{1}{3} + (A - B - C + D)^\frac{1}{3}, \\
4\beta &= (A + B + C + D)^\frac{1}{3} - (A - B + C - D)^\frac{1}{3} + (A + B - C - D)^\frac{1}{3} - (A - B - C + D)^\frac{1}{3}, \\
4\gamma &= (A + B + C + D)^\frac{1}{3} + (A - B + C - D)^\frac{1}{3} - (A + B - C - D)^\frac{1}{3} - (A - B - C + D)^\frac{1}{3}, \\
4\delta &= (A + B + C + D)^\frac{1}{3} - (A - B + C - D)^\frac{1}{3} - (A + B - C - D)^\frac{1}{3} + (A - B - C + D)^\frac{1}{3}.
\end{align*}
\]

Now \(A\) is a symmetrical function of the roots, and we have

\(A = p^2 - 2q\).

The solution of the equation, therefore, depends upon the expression of the three quantities

\(a\beta + \gamma\delta,\ a\gamma + \beta\delta,\ a\delta + \beta\gamma\)

in terms of the coefficients of the equation. Now these three quantities clearly form a group, i.e. they are such that any interchanges effected among the quantities \(a, \beta, \gamma, \delta\) leave us with the same three quantities. Thus it is clear that if we form a cubic equation with these three quantities for roots, its coefficients will be expressible in terms of the coefficients of the given equation. In fact, writing \(U, V, W\) for the three quantities, we have

\[
\begin{align*}
U + V + W &= \Sigma a\beta = q, \\
VW + WU + UV &= \Sigma a^2\beta\gamma = pq - 4s, \\
UVW &= \Sigma a^3\beta\gamma\delta + \Sigma a^3\beta^2\gamma^2 \\
&= p^2q - 4qs + 2.
\end{align*}
\]

Thus the auxiliary cubic, whose roots are \(U, V, W\), is

\[x^3 - qx^2 + (pq - 4s)x - p^2q + 4qs - r^2 = 0.\]

The solution of this equation will give \(U, V, W\), and therefore \(B, C, D\) in terms of the coefficients of the biquadratic. J. Brill.