We can, therefore, so arrange them that not only is G the centroid but also $\Sigma xa = g\Sigma x$ as well. The foregoing proof is therefore perfectly general.

R. F. DAVIS.

NOTE ON THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

1. Let
$$x^3+px^2+qx+r=0$$
be a cubic equation, whose roots are α, β, γ. Then, defining the quantity λ by means of the equation $\lambda^3=1$, we have
$$\begin{aligned}(\alpha+\lambda\beta+\lambda^2\gamma)^3 &= \alpha^3+\beta^3+\gamma^3+6\alpha\beta\gamma+3\lambda(\beta^2\gamma+\gamma^2\alpha+\alpha^2\beta)\\ &\qquad +3\lambda^2(\beta\gamma^2+\gamma\alpha^2+\alpha\beta^2)\\ &= A+\lambda B+\lambda^2 C.\end{aligned}$$

Since this is true for all values of λ that satisfy the equation $\lambda^3=1$; we have, ω being one of the imaginary cube roots of unity, the three equations
$$\alpha+\beta+\gamma=(A+B+C)^{\frac{1}{3}},$$
$$\alpha+\omega\beta+\omega^2\gamma=(A+\omega B+\omega^2 C)^{\frac{1}{3}},$$
$$\alpha+\omega^2\beta+\omega\gamma=(A+\omega^2 B+\omega C)^{\frac{1}{3}}.$$

From these equations it follows that
$$3\alpha=(A+B+C)^{\frac{1}{3}}+(A+\omega B+\omega^2 C)^{\frac{1}{3}}+(A+\omega^2 B+\omega C)^{\frac{1}{3}},$$
$$3\beta=(A+B+C)^{\frac{1}{3}}+\omega^2(A+\omega B+\omega^2 C)^{\frac{1}{3}}+\omega(A+\omega^2 B+\omega C)^{\frac{1}{3}},$$
$$3\gamma=(A+B+C)^{\frac{1}{3}}+\omega(A+\omega B+\omega^2 C)^{\frac{1}{3}}+\omega^2(A+\omega^2 B+\omega C)^{\frac{1}{3}},$$

Thus, if A, B, C can be expressed in terms of the coefficients of the given cubic equation, we have a solution of the said equation. Now A is a symmetrical function of the roots of the said equation, and may therefore be so expressed. In fact we have
$$A=-p^3+3pq-qr.$$
Writing
$$U=\beta^2\gamma+\gamma^2\alpha+\alpha^2\beta,$$
and
$$V=\beta\gamma^2+\gamma\alpha^2+\alpha\beta^2,$$
we see that $U+V$ is a symmetrical function of the roots. In fact we have
$$U+V=-pq+3r. \qquad (1)$$
Also
$$\begin{aligned}U-V &= \beta\gamma(\beta-\gamma)+\gamma\alpha(\gamma-\alpha)+\alpha\beta(\alpha-\beta)\\ &= -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).\end{aligned}$$

This shows that $(U-V)^2$ is a symmetrical function of the roots; and, as $U+V$ is of the same character, it follows that UV must be so also. In fact we have
$$\begin{aligned}UV &= \beta^3\gamma^3+\gamma^3\alpha^3+\alpha^3\beta^3+3\alpha^2\beta^2\gamma^2+\alpha\beta\gamma(\alpha^3+\beta^3+\gamma^3)\\ &= p^3r+q^3-6pqr+qr^2. \qquad (2)\end{aligned}$$

From equations (1) and (2) the values of U and V can be readily obtained by a known method. Thus the values of B and C, in terms of the coefficients of the equation, can be found, and the solution completed.

2. Next suppose that we have a biquadratic equation
$$x^4+px^3+qx^2+rx+s=0,$$
whose roots are α, β, γ, δ. This time we will employ two quantities, λ and μ, defined by the equations
$$\lambda^2=1, \quad \mu^2=1.$$

We have $$(\alpha+\lambda\beta+\mu\gamma+\lambda\mu\delta)^2=\alpha^2+\beta^2+\gamma^2+\delta^2+2\lambda(\alpha\beta+\gamma\delta)$$
$$+2\mu(\alpha\gamma+\beta\delta)+2\lambda\mu(\alpha\delta+\beta\gamma)$$
$$=A+\lambda B+\mu C+\lambda\mu D.$$

Now this equation is satisfied by any pair of possible values of λ and μ. We have, therefore, four cases, viz.,

$$\lambda=1, \quad \mu=1,$$
$$\lambda=-1, \quad \mu=1,$$
$$\lambda=1, \quad \mu=-1,$$
$$\lambda=-1, \quad \mu=-1.$$

Thus we obtain
$$\alpha+\beta+\gamma+\delta=(A+B+C+D)^{\frac{1}{2}},$$
$$\alpha-\beta+\gamma-\delta=(A-B+C-D)^{\frac{1}{2}},$$
$$\alpha+\beta-\gamma-\delta=(A+B-C-D)^{\frac{1}{2}},$$
$$\alpha-\beta-\gamma+\delta=(A-B-C+D)^{\frac{1}{2}}.$$

Thus, if A, B, C, D can be expressed in terms of the coefficients of the equation, we readily obtain the solution in the following form :

$$4\alpha=(A+B+C+D)^{\frac{1}{2}}+(A-B+C-D)^{\frac{1}{2}}+(A+B-C-D)^{\frac{1}{2}},$$
$$+(A-B-C+D)^{\frac{1}{2}},$$
$$4\beta=(A+B+C+D)^{\frac{1}{2}}-(A-B+C-D)^{\frac{1}{2}}+(A+B-C-D)^{\frac{1}{2}},$$
$$-(A-B-C+D)^{\frac{1}{2}},$$
$$4\gamma=(A+B+C+D)^{\frac{1}{2}}+(A-B+C-D)^{\frac{1}{2}}-(A+B-C-D)^{\frac{1}{2}},$$
$$-(A-B-C+D)^{\frac{1}{2}},$$
$$4\delta=(A+B+C+D)^{\frac{1}{2}}-(A-B+C-D)^{\frac{1}{2}}-(A+B-C-D)^{\frac{1}{2}},$$
$$+(A-B-C+D)^{\frac{1}{2}}.$$

Now A is a symmetrical function of the roots, and we have
$$A=p^2-2q.$$

The solution of the equation, therefore, depends upon the expression of the three quantities
$$\alpha\beta+\gamma\delta, \quad \alpha\gamma+\beta\delta, \quad \alpha\delta+\beta\gamma$$
in terms of the coefficients of the equation. Now these three quantities clearly form a group, *i.e.* they are such that any interchanges effected among the quantities α, β, γ, δ leave us with the same three quantities. Thus it is clear that if we form a cubic equation with these three quantities for roots, its coefficients will be expressible in terms of the coefficients of the given equation. In fact, writing U, V, W for the three quantities, we have

$$U+V+W=\Sigma\alpha\beta=q,$$
$$VW+WU+UV=\Sigma\alpha^2\beta\gamma=pr-4s,$$
$$UVW=\Sigma\alpha^3\beta\gamma\delta+\Sigma\alpha^2\beta^2\gamma^2$$
$$=p^2s-4qs+r^2.$$

Thus the auxiliary cubic, whose roots are U, V, W, is
$$x^3-qx^2+(pr-4s)x-p^2s+4qs-r^2=0.$$

The solution of this equation will give U, V, W, and therefore B, C, D in terms of the coefficients of the biquadratic. J. BRILL.