ON THE INEQUALITIES CONNECTING THE DOUBLE AND REPEATED UPPER AND LOWER INTEGRALS OF A FUNCTION OF TWO VARIABLES

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1. When $f(x, y)$ is a bounded function of two variables, the inequalities

$$\overline{\iint} f(x,y)\,dx\,dy \geqslant \overline{\int} dy \overline{\int} f(x,y)\,dx \geqslant \underline{\int} dy \underline{\int} f(x,y)\,dx \geqslant \underline{\iint} f(x,y)\,dx\,dy,$$

or, using obvious abbreviations,

upper double $\geqslant$ upper-upper $\geqslant$ lower-lower $\geqslant$ lower double,

between the upper and lower double (proper) integrals and the two extreme repeated (proper) integrals are well known.*

That these inequalities do not necessarily hold when the integrand is an unbounded function, in which case the integrals concerned are improper integrals, may be inferred from the study of stray examples given incidentally by previous writers.

Thus, in the example given by Dr. Hobson,† $f(x, y)$ has an improper double integral, whose value is zero, while its integral with respect to y is infinite for a set of values of x everywhere dense, and is elsewhere zero. Integrating, first with respect to y and then with respect to x, we have, therefore,

double integral $=$ lower integral of integral $= 0$,

while

upper integral of integral $= \infty$,

so that the inequality at the head of this article is violated.

It might be contended that the idiosyncrasies of this function are of an extreme character. Here $f(x, y)$ is a discontinuous function as well as an unbounded one; moreover, its integral with respect to y is infinite at an everywhere dense set of points on the y-axis.

* Prof. Pierpont has given conditions under which these inequalities still hold for integration with respect to a set of points. "On Improper Multiple Integrals," 1906, *Trans. of the American Math. Soc.*, Vol. VII., pp. 155–174.

† "On Absolutely Convergent Improper Integrals," 1906, *Proc. London Math. Soc*, Ser. 2, Vol. 4, p. 156.

Now, in the first place, the admission of infinite values for the integral constantly produces abnormalities. For example, as I have lately pointed out elsewhere,* the theorem that the integral of an unbounded function of a single variable is a continuous function of its upper limit is no longer always true if $+\infty$ or $-\infty$ be allowed as values which the integral may assume, and this is the case even when the integrand is continuous in the extended sense.

In the second place, the doubt might arise whether, if we took as integrand a *continuous* unbounded function $f(x, y)$, the usual inequalities might not inevitably hold good.

I have therefore been at pains to construct an example of an unbounded continuous function such that (1) its double integral is finite, (2) its integral with respect to x is a bounded non-integrable function of y, (3) this function of y has for *lower* integral the double integral.

This example (§ 4), beside setting at rest the doubts in question, is found to throw considerable further light on the various possibilities which may arise with regard to the inequalities which form the main subject of the paper. No systematic investigation of these possibilities appears to exist, and the theorems given in the paper are, I believe, stated for the first time. I shew that (I.) *for functions with a finite upper bound,*

upper double $\geqslant$ *upper-upper ;*

(II.) *for functions with a finite lower bound,*

lower-lower $\geqslant$ *lower double ;*

(III.) *for any functions whatever,*

upper double $\geqslant$ *lower-upper,*

and (IV.) *upper-lower* $\geqslant$ *lower double.*

In case (I.) the sign of equality holds when the integrand is upper semi-continuous, and it holds in case (II.) when the integrand is lower semi-continuous.

The example already referred to (§ 4) shews that the inequality (I.) may be violated if the restriction as to the finitude of the upper bound is removed. Similarly, of course, (II.) may be violated when the corresponding restriction is removed.

It is then shown by means of two examples (§ 8) that a connexion between the inequalities (III.) and (IV.) cannot in general be established ; even in the case of bounded functions, the lower-upper may be either greater or less than the upper-lower.

* "On a Test for Continuity," 1908, *Proc. of the Royal Society of Edinburgh*, Vol. XXVIII., § 8, pp. 254, 255.

From all this it follows, in particular, that the double integral of an integrable function may be found by successive upper and lower integration whenever one bound of the function is finite. If, however, the bounds are both infinite, the double integral will not, in general, be capable of calculation by this method.

One additional result may be noticed. It is known that, if $f(x, y)$ is a bounded continuous function of the ensemble (x, y), its integral with respect to x between fixed limits, say $\int_a^b f(x, y)\,dx$, is a continuous function of y. It is shewn in §§ 1–3 that, when f, remaining continuous, is unbounded even at one point, the integral is in general an upper or lower semi-continuous function of y according as f has a finite upper or lower bound in the interval considered. We may, of course, divide the segment of the axis of x under consideration into a finite number of segments in each of which one of the bounds is finite, since f is continuous.

It should be added, in conclusion, that the paper has been so worded that the definition employed for improper double integrals may be taken to be that of de la Vallée-Poussin. The definition given by myself in my paper quoted below, presented to the Cambridge Philosophical Society, leads more naturally to the results obtained, but I have not explicitly employed it, with the object of rendering the paper more readily comprehensible to those acquainted with the existing literature of the subject.

The range of integration I have always taken to be a finite rectangle, and I have not thought it necessary to enter into the obvious generalisations which arise when the number of independent variables is more than two.

1. The following preliminary theorem is fundamental:—

THEOREM 1.—*If $f(x, y)$ is an upper (lower) semi-continuous function of the ensemble (x, y) having a finite upper (lower) bound,* (1) *its upper (lower) integral with respect to one variable x is an upper (lower) semi-continuous function of the other variable y, and* (2) *its upper (lower) double integral is its upper-upper (lower-lower) integral.*

It will be sufficient to prove these theorems when $f(x, y)$ is *upper* semi-continuous.

Since $f(x, y)$ has a finite upper bound, it may* be expressed as the limit of a monotone descending sequence of bounded continuous

* "On Monotone Sequences of Continuous Functions," § 5, Cor., *Proc. Camb. Phil. Soc.*, Lent Term, 1908.

functions $$f_1(x, y) \geqslant f_2(x, y) \geqslant f_3(x, y) \geqslant \ldots.$$

Keeping y constant and integrating from a to b with respect to x, and denoting the integral of f_n by $F_n(y)$, the functions

$$F_1(y) \geqslant F_2(y) \geqslant F_3(y) \geqslant \ldots$$

form a monotone descending sequence of continuous bounded functions.

The limit of this sequence is therefore an upper semi-continuous function of y. By a known theorem,* however, this limit is the upper integral with respect to x of the upper semi-continuous function $f(x, y)$. This proves the statement (1).

Again, the double integrals of $f_1(x, y)$, $f_2(x, y)$, ... also form a monotone descending sequence whose limit is again, by a known theorem, the upper double integral of $f(x, y)$.

Since $f_n(x, y)$ is bounded and continuous,

$$\iint f_n(x, y)\, dx\, dy = \int dy \int f(x, y)\, dx,$$

so that $$\overline{\iint} f(x, y)\, dx\, dy = \operatorname*{Lt}_{n=\infty} \int dy \int f_n(x, y)\, dx = \operatorname*{Lt}_{n=\infty} \int F_n(y)\, dy.$$

But, since it has been shewn that $F_1(y)$, $F_2(y)$, ... form a monotone decreasing sequence of continuous bounded functions of y, whose limit $F(y)$ is therefore an upper semi-continuous function of y, the last-mentioned limit is the upper integral of $F(y)$, that is, by what has been proved

$$\overline{\iint} f(x, y)\, dx\, dy = \overline{\int} dy\, \overline{\int} f(x, y)\, dx.$$

This proves the statement (2).

2. On account of the fundamental character of the above theorem, we now give an instructive alternative proof of the first result.

We require the following lemma :—

LEMMA.—*If $f(x, y)$ is an upper (lower) semi-continuous function of the ensemble (x, y), and U_y* and *L_y are the upper and lower bounds of $f(x, y)$ on any particular parallel to the axis of x between fixed limits for x, then U_y and L_y are both upper (lower) semi-continuous functions of y.*

We give the proof for an upper semi-continuous function.

* For the upper integral of an upper semi-continuous function is its generalised or Lebesgue integral, and generalised integration, term by term, is allowable in the case of a monotone sequence. It is easy to give an independent proof of the case of this theorem used more than once in our investigation. For the general theorem, *cp*. Beppo Levi, *Atti di Torino*, 1907.

For, if $y_1, y_2, \ldots$ is a sequence of values of y having y_0 as limit, there will be a point P_n on the line $y = y_n$ where $f(x, y_n)$ *assumes* its upper bound U_{y_n}. These points P_n for all values of n have one or more limiting points lying on $y = y_0$, and at such a limiting point, $f(x, y)$ being upper semi-continuous with respect to the ensemble (x, y), the value of $f(x, y_0)$ is not less than any limit approached by the quantities U_{y_n}; *a fortiori*, the same is true of U_{y_0}, which shews that U_y is an upper semi-continuous function of y.

Again, on the line $y = y_0$ there will be a point P_0 where $f(x, y)$ has a value less than L, where L is any quantity greater than L_y. Since $f(x, y)$ is upper semi-continuous with respect to y, its values on all neighbouring lines $y = y_n$ on the ordinate of P_0 are also less than L, and therefore the same is true of the corresponding lower bounds L_{y_n}. That is to say, L_y is an upper semi-continuous function of y.

COR. 1.—*If $f(x, y)$ is a continuous function of the ensemble (x, y), its upper and lower bounds U_y and L_y on any parallel to the axis of x between fixed limits for x, are continuous functions of y.*

COR. 2.—*If $f(x, y)$ be upper (lower) semi-continuous with respect to y only, then the lower (upper) bound only is an upper (lower) semi-continuous function of y.*

Alternative proof of (1) in Theorem 1 :—

Since $f(x, y)$ has a finite upper bound, it may be shewn* that its upper integral with respect to x is the lower limit of its upper summations.

Now, taking any fixed division of the segment (a, b) into a finite number of segments, and for fixed y taking the upper limit of $f(x, y)$ in each segment and summing, so as to form one of these upper summations, we get, by the preceding theorem, an upper semi-continuous function of y.

Taking any monotone sequence of such upper summations, descending to the lower limit, it follows that that limit is an upper semi-continuous function of y, which proves the required result, viz., that the upper integral of the upper semi-continuous function $f(x, y)$ with respect to x is an upper semi-continuous function of y when $f(x, y)$ has a finite upper bound.

3. As a special case of Theorem 1, we note the following :—

THEOREM 2.—*If $f(x, y)$ is a continuous function of the ensemble (x, y) with a finite upper (lower) bound, its double integral is the upper (lower) integral with respect to y of the integral with respect to x.*

* This theorem, which is easily proved, is probably contained in Severini's *Thèse*, which, however, I have not been able to procure.

Further, the integral with respect to x of such a function is an upper (lower) semi-continuous function of y.

The following example shews that the result just stated is all that can, in general, be predicated:—

Construction of a Continuous Unbounded Positive Function $f(x, y)$, whose Integral

$$F(y) = \int_0^1 f(x, y)\, dx,$$

with respect to x, is a Bounded Lower Semi-Continuous Function of y, not Continuous.

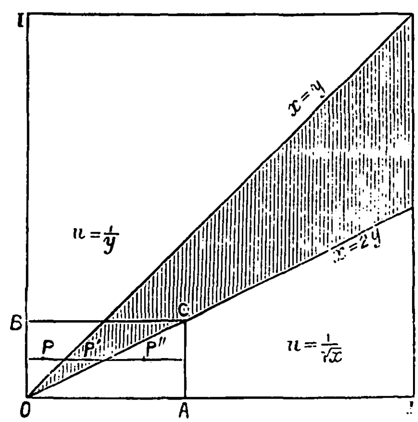

FIG. 1.

Take the unit square (Fig. 1) and divide it by the lines

$$x = y, \qquad x = 2y,$$

into three triangles.

In the isosceles right-angled triangle put

$$u = f(x, y) = \frac{1}{y}. \tag{1}$$

In the other right-angled triangle put

$$u = f(x, y) = x^{-\frac{1}{2}}. \tag{2}$$

In the remaining triangle, shaded in the figure, put

$$u = f(x, y) = \frac{2y - x}{y^2} + \frac{x - y}{\sqrt{2}\, y^{\frac{3}{2}}} = \frac{2}{y} - \frac{1}{\sqrt{2y}} - x\left(\frac{1}{y^2} - \frac{1}{\sqrt{2}\, y^{\frac{3}{2}}}\right), \tag{3}$$

so that in this triangle, for any constant value of y,

$$\frac{1}{y} \geqslant f(x, y) \geqslant x^{-\frac{1}{2}}, \tag{4}$$

the sign of equality holding only at the respective extreme points, and $f(x, y)$ decreasing in a monotone manner from the former to the latter value. Thus, for constant y, $f(x, y)$ is a monotone decreasing continuous function of x; it is always positive, and, except on the axis of y, always finite. On the axis of x it is $x^{-\frac{1}{2}}$.

The formulæ (1), (2) and (3) being continuous functions, it follows that, at any point of the unit square not on one of the dividing lines

$$x = y, \qquad x = 2y,$$

$f(x, y)$ is a continuous function of the ensemble (x, y). The same is true on the dividing lines, with the possible exception of the origin, since the expressions for $f(x)$ in the two triangles having that line as boundary agree on that line. The only doubt remains at the origin, where $f(x, y)$ is infinite. But, if we draw any rectangle, as $OACB$ in the figure, having the corner opposite the origin on the line

$$x = 2y,$$

and in this rectangle draw any line parallel to the x-axis, since $f(x, y)$ decreases monotonely along this line, its value at any point, such as P, or P' or P'' in the figure, is greater than that on the bounding ordinate AC, that is, greater than at A, where it is $x^{-\frac{1}{2}}$. Thus, by taking A sufficiently near to O, so that

$$OA < \frac{1}{k^2}\,.$$

all the values of $f(x, y)$ in the rectangle are greater than k, so that f is continuous at its infinity, the origin.

Integrating, we have

$$F(0) = \int_0^1 x^{-\frac{1}{2}}\,dx = 2, \tag{5}$$

and when y is not zero,

$$F(y) = \int_0^y f(x, y)\,dx + \int_y^{2y} f(x, y)\,dx + \int_{2y}^1 f(x, y)\,dx,$$

or, using the formulæ (1), (2) and (3), and integrating

$$F(y) = 1 + 2 - \sqrt{\frac{y}{2}} - \frac{3}{2}\left(1 - \sqrt{\frac{y}{2}}\right) + 2(1 - \sqrt{2y}) = \tfrac{7}{2}\left(1 - \sqrt{\frac{y}{2}}\right). \tag{6}$$

From (5) and (6), we see that

$$F(0) = 2 < \tfrac{7}{2} < \operatorname*{Lt}_{y=0} F(y),$$

so that $F(y)$, though elsewhere continuous, is only lower semi-continuous at the origin.

4. *Construction of a Continuous Unbounded Positive Function* $f(x, y)$, *whose Integral*

$$F(y) = \int_0^1 f(x, y)\, dx,$$

with respect to x *is a Bounded Non-Integrable Lower Semi-Continuous Function of* y.

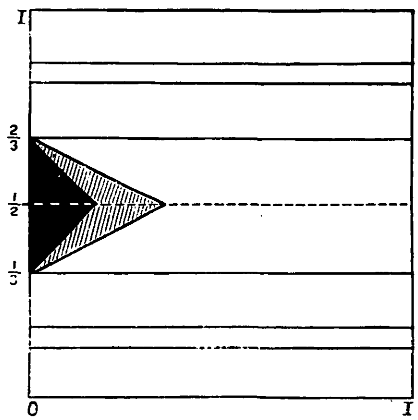

FIG. 2.

Ex. 2.—We define $f(x, y)$ in the unit square as follows :—

On the y-axis take the typical ternary perfect set of positive content* (lying between $\frac{1}{3}$ and $\frac{2}{3}$), say G. On each black interval of G erect an isosceles right-angled triangle (black in the figure), and an isosceles triangle whose altitude is equal to its base (shaded outside the black triangle).

Let a_i be the lower, and b_i the upper end of the i-th black interval, and m_i its middle point. Then the line

$$y = m_i$$

divides the strip bounded by the parallels through a_i and b_i symmetrically.

In each half-strip we have one black and one shaded triangle and a white part. We define $f(x, y)$ in the lower half-strip as follows, and then change $(y-a_i)$ into (b_i-y), so as to get the corresponding formulæ in the upper half-strip.

In the black triangle $\quad f(x, y) = \dfrac{1}{y-a_i};$ (1)

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* Young's *Theory of Sets of Points*, Ex. 1, p. 78. The construction is by means of division into 3, 3^2, 3^3, ..., 3^n, ... equal parts. The complementary set of "black intervals" consists of the central third of the segment (0, 1), the central ninth of (0, $\frac{1}{3}$) and ($\frac{2}{3}$, 1), and so on.

in the white part $$f(x, y) = x^{-\frac{1}{2}}; \tag{2}$$

in the shaded triangle $$f(x, y) = \frac{2y'-x}{y'^2} + \frac{x-y'}{y'\sqrt{2y'}}, \tag{3}$$

where, for shortness, we write $y' = y - a_i$. (3a)

This function is, by the preceding example, continuous throughout the half strip, and therefore throughout the whole strip, since the values of the given expressions in the two half strips agree on the median line. On the two extreme lines

$$y = a_i, \qquad y = b_i,$$

we have $$f(x, y) = x^{-\frac{1}{2}}, \tag{4}$$

which also expresses the function on any parallel through a point of the perfect set G. The function $f(x, y)$, so defined for the whole unit square is then clearly a positive continuous function of the ensemble (x, y), and is finite everywhere except at the points of G, where it is infinite.

Integrating, we have

$$F(y) = 2 \text{ at all the points of the perfect set } G,$$

while elsewhere

$$F(y) = \tfrac{7}{2}[1-(y-a_i)^{\frac{1}{2}}] \quad \text{or} \quad \tfrac{7}{2}[1-(b_i-y)^{\frac{1}{2}}],$$

according as y lies in the lower or the upper half of the black interval (a_i, b_i) or d_i. Thus $F(y)$ is lower semi-continuous at every point of the perfect set G and is elsewhere continuous; it is therefore a non-integrable function of y.

Since $F(y)$ is lower semi-continuous, its lower integral is its generalized integral, that is

$$2I + \Sigma 2\int_{a_i}^{m_i} F(y)\,dy = 2I + 7\Sigma[\tfrac{1}{2}d_i - \tfrac{2}{3}(\tfrac{1}{2}d_i)^{\frac{3}{2}}] = \tfrac{7}{2} - \tfrac{3}{2}I - \frac{7\sqrt{2}}{6}\Sigma d_i^{\frac{3}{2}}$$

(where I is the content of G), which is the value of the double integral of $f(x, y)$ over the unit square.

It may be noticed that the upper integral of $F(y)$ only differs from the above by taking $\frac{7}{2}$ instead of 2 at all the points of G, so that

$$\overline{\int_0^1} F(y)\,dy = \frac{7}{2} - \frac{7\sqrt{2}}{6}\Sigma d_i^{\frac{3}{2}}.$$

Thus the upper integral of the integral is greater than the double integral, contrary to what can happen when the integrand is bounded, in which case the double integral is always greater than or equal to the upper-upper integral, whether the integrand is continuous or not.

If we integrate $f(x, y)$ with respect to y from 0 to 1, we get a function of x which is clearly finite and continuous for all values of x other than zero, and is always greater than $Ix^{-\frac{1}{2}}$, so that it has the limit $+\infty$ at the origin. The value at the origin is also $+\infty$. Thus the function is a continuous unbounded function, having a single infinity at the origin. The integral of this function is, of course, the double integral of $f(x, y)$ over the unit square, and has therefore the value already found.

THEOREM 3.—*Given any function $f(x, y)$ whatever with a finite upper bound, its upper-upper integral is less than or equal to its upper double integral, that is*

$$\overline{\int} dy \overline{\int} f(x, y)\, dx \leqslant \overline{\iint} f(x, y)\, dx\, dy.$$

Let $\phi(x, y)$ be the associated upper limiting function of $f(x, y)$, that is the function got by taking at each point (x, y) the highest value which can be approached as limit by $f(x, y)$ in the neighbourhood of the point in question. Then ϕ is an upper semi-continuous function of the ensemble (x, y), and it has the same finite upper bound as f itself. Hence, by Theorem 1, its upper double integral is its upper-upper integral. Since, however, as is easily seen, f and ϕ have the same upper double integral, this proves that

$$\overline{\int} dy \overline{\int} \phi(x, y)\, dx = \overline{\iint} f(x, y)\, dx\, dy. \tag{1}$$

Now the associated upper limiting function of $f(x, y)$ when y is constant is evidently less than or equal to $\phi(x, y)$ at each point (x, y). Hence the upper integral of $f(x, y)$ with respect to x, y being constant, which is the same as the upper integral of that associated upper limiting function, is less than or equal to the upper integral of ϕ with respect to x, that is

$$\overline{\int} f(x, y)\, dx \leqslant \overline{\int} \phi(x, y)\, dx. \tag{2}$$

Hence, by (1) and (2),

$$\overline{\int} dy \overline{\int} f(x, y)\, dx \leqslant \overline{\iint} f(x, y)\, dx\, dy.$$

A similar argument proves the alternative theorem :—

Given any function $f(x, y)$ whatever with a finite lower bound, the lower-lower integral is greater than or equal to the lower double integral, that is

$$\underline{\iint} f(x, y)\, dx\, dy \leqslant \underline{\int} dy \underline{\int} f(x, y)\, dx.$$

COR. 1.—*If $f(x, y)$ is any bounded function of the ensemble (x, y),*

$$\underline{\iint} f(x, y)\,dx\,dy \leqslant \underline{\int} dy\, \underline{\int} f(x, y)\,dx \leqslant \overline{\int} dy\, \overline{\int} f(x, y)\,dx \leqslant \overline{\iint} f(x, y)\,dx\,dy.$$

COR. 2.—*If $f(x, y)$ is an integrable bounded function, then in the preceding inequality the sign of equality must be taken throughout.*

6. THEOREM 4.—*If $f(x, y)$ be any function of x and y, its lower-upper integral is less than or equal to its upper double integral, that is*

$$\underline{\int} dy\, \overline{\int} f(x, y)\,dx \leqslant \overline{\iint} f(x, y)\,dx\,dy.$$

Let ϕ be the upper limiting function of f. Then ϕ is an upper semi-continuous function of the ensemble (x, y), and therefore* can be expressed as the limit of a monotone descending sequence of continuous functions each having a finite lower bound,

$$\phi_1(x, y) \geqslant \phi_2(x, y) \geqslant \dots,$$

and the upper double integral of ϕ is the limit of the double integral of ϕ_n. Thus, if the upper double integral of ϕ is finite, we can find n so that

$$\iint \phi_n(x, y)\,dx\,dy - \overline{\iint} \phi(x, y)\,dx\,dy < e, \tag{1}$$

e being any positive quantity previously chosen at will.

But, by Theorem 1, since ϕ_n has a finite lower bound, and is continuous,

$$\iint \phi_n(x, y)\,dx\,dy = \underline{\int} dy \int \phi_n(x, y)\,dx\,dy$$

$$= \underline{\int} dy\, \overline{\int} \phi_n(x, y)\,dx\,dy \geqslant \underline{\int} dy\, \overline{\int} f(x, y)\,dx, \tag{2}$$

since at every point $\phi_n \geqslant \phi \geqslant f$.

Since f and ϕ have the same upper double integral, it now follows from (1) and (2) that

$$\overline{\iint} f(x, y)\,dx\,dy + e \geqslant \underline{\int} dy\, \overline{\int} f(x, y)\,dx,$$

which, e being at our disposal, proves the theorem.

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* Theorem of § 5, Case 3, of my paper, "Note on Monotone Sequences of Continuous Functions."

7. Summing up our results so far, we have shewn that for functions with a finite upper bound,

upper double $\geqslant$ upper-upper;

for functions with a finite lower bound,

lower-lower $\geqslant$ lower double;

for any functions,

upper double $\geqslant$ lower-upper,

upper-lower $\geqslant$ lower double.

The example just given shews that it *may* happen for functions with an infinite upper bound, that

upper double $>$ upper-lower;

and for functions with an infinite lower bound, that

lower-upper $>$ lower double.

Thus for a bounded function,

upper double $\geqslant$ upper-upper $\geqslant$ lower-upper,

upper-lower $\geqslant$ lower-lower $\geqslant$ lower double,

while for an unbounded function the upper double may be displaced so as to occupy the second position in its own sequence, but not the third, while in the other sequence it may be displaced so as to occupy the second place, between upper-lower and lower-lower, but no further. Similar remarks apply, of course, to the lower double. Thus in the case of a function which has infinite upper and lower bounds, the following is a possible inequality:—

upper-lower $>$ upper double $>$ lower double $>$ lower-upper,

or, more fully,

upper-upper $>$ upper-lower $>$ upper double $>$ lower double

$>$ lower-upper $>$ lower-lower.

One point only rests in doubt, namely, as to the possible relative positions of the lower-upper and the upper-lower. It is clear that in the case of an unbounded function they may change positions, that is, we cannot say, *a priori*, which of the two is greater. It remains, however, still to discuss their relative position when the function is bounded. In this case also either position is possible, as is shewn by Examples 3 and 4.

8. The following simple example shews that in the case of a bounded function the following relative position is possible,

$$\text{upper-lower} > \text{lower-upper}.$$

Ex.—Take in the segment (0, 1) of the y-axis a perfect set nowhere dense of positive content I. For every value of y belonging to this perfect set, let

$$f(x, y) = 1,$$

and elsewhere $$f(x, y) = 0,$$

the region of integration being the unit square.

Then $f(x, y)$ is, for every value of y, a continuous function of x and therefore integrable, so that

$$\underline{\int} f(x, y)\, dx = \overline{\int} f(x, y)\, dx,$$

and has the value 1 or 0 according as y does or does not belong to the perfect set. The function of y so defined is therefore non-integrable, having the upper integral $= I$, and the lower integral $= 0$, so that

$$I = \text{upper-lower} > \text{lower-upper} = 0.$$

The following example, on the other hand, shews that in the case of a bounded function the relative position may be reversed, so that we may have

$$\text{lower-upper} > \text{upper-lower}.$$

Ex.—Take in the segment (0, 1) of the x-axis a perfect set nowhere dense and of positive content I, and on each of the ordinates through its points place a similar set. We thus get a plane perfect set nowhere dense of content I^2, such as is given in Ex. 5 and Fig. 24 of my *Theory of Sets of Points*, pp. 173, 174.

At every point of this plane perfect set let

$$f(x, y) = 0,$$

and elsewhere have the value 1.

Then for every value of y belonging to the perfect set of content I on the y-axis, we have

$$\overline{\int} f(x, y)\, dx = 1,$$

but $$\underline{\int} f(x, y)\, dx = 1 - I,$$

while for other values of y both the upper and the lower integrals have the value 1. Thus the upper integral with respect to x is an integrable function of y, while the lower integral with respect to x is a non-integrable function of y, hence

$$\text{lower-upper} = \text{upper-upper} > \text{upper-lower}.$$

In fact
$$1 = \underline{\int} dy \overline{\int} f(x, y)\, dx > \overline{\int} dy \underline{\int} f(x, y)\, dx = 1 - I.$$

9. From the above results we can at once deduce the following :—

(1) *If an integrable function have a finite upper bound,*

upper double = upper-upper = upper-lower = lower double.

(2) *If an integrable function have a finite lower bound,*

upper double = lower-upper = lower-lower = lower double.

(3) *In the case of an integrable function having an infinite upper and an infinite lower bound, the method of repeated upper and lower integration will totally fail in general to give the value of the double integral.*

[*Added March* 14*th*, 1908.—In the above paper I have deliberately avoided the use of the concept of Lebesgue integration. I hoped in this way to appeal to a larger public. I should like to point out, however, what is indeed obvious to any one acquainted with the Lebesgue theory, that the reasoning by which the inequalities (III.) and (IV.) were obtained, really gives us a slightly more extended result, viz.,

(V.) upper double $\geqslant$ middle-upper,

(VI.) middle-lower $\geqslant$ lower double,

where the word " middle " is used to denote the generalised, or Lebesgue, integral, which, as is well known, lies in general between the upper and lower integrals, and may be equal to either or both. When the function is lower semi-continuous the Lebesgue integral is equal to the lower integral, which I write, symbolically,

$$\overline{\int} f(x)\, dx = \mathop{-\!\!\!\!\!\int} f(x)\, dx,$$

or, perhaps,

$$\overline{\int} f(x)\, dx = \dot{\int} f(x)\, dx.$$

Hence, in § 6, equation (2) may be written, since $\int \phi_n(x, y)\,dx$ is a lower semi-continuous function of y,

$$\iint \phi_n(x, y)\,dx\,dy = \underline{\int} dy \int \phi_n(x, y)\,dx$$

$$= \dot{\int} dy \int \phi_n(x, y)\,dx$$

$$= \dot{\int} dy \overline{\int} \phi_n(x, y)\,dx$$

$$\geqslant \dot{\int} dy \overline{\int} f(x, y)\,dx.$$

Hence, for a function unbounded above and below, the most general inequality will be

upper-upper > upper-lower > upper double > middle-upper

> middle-lower > lower double > lower-upper > lower-lower.]