The following presents were received :—

- "Proceedings of the Royal Society," Vol. XLVII., No. 287.
- "Proceedings of the Cambridge Philosophical Society," Vol. VII., Part I.
- "Educational Times," for April.
- "Bulletin des Sciences Mathématiques," Tome XIV. ; March, 1890.
- "Beiblätter zu den Annalen der Physik und Chemie," Band XIV., Stück 3.
- "Memorias de la Sociedad Científica—'Antonio Alzate,'" Tomo III., No. 3 ; Mexico, 1889.
- "Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 101.
- "Jornal de Sciencias Mathematicas e Astronomicas," Vol. IX., No. 4 ; Coimbra, 1889.
- "Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VI., Fasc. 1, 2, 3.
- "Annali di Matematica pura ed applicata," Tome XVII., Fasc. 4.
- "Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," XXXIX.—LIII.
- "Bulletin International de l'Académie des Sciences de Cracovie—Comptes Rendus des Séances de l'Année 1890," Février, 1890.

*On the Properties of some Circles connected with a Triangle formed
by Circular Arcs. By R. LACHLAN, M.A.*

[Read April 3rd, 1890.]

Introduction, §§ 1, 2.

1. Let ABC be a triangle formed by three given circular arcs, and let the complete circles be drawn intersecting again in A', B', C' ; we obtain three triangles $A'BC$, $AB'C$, and ABC' which may be called the associated triangles of ABC , and four triangles $A'B'C'$, $AB'C$, $A'BC'$, $A'B'C$ which are the inverse triangles with respect to the circle, which cuts the given circles orthogonally, of ABC and its associated triangles respectively.

Each of these triangles has a circum-circle, an inscribed circle, and a Hart-circle, which last corresponds to the nine-point circle of an ordinary triangle. In the present paper I propose to exhibit the properties of these groups of circles in as complete a form as possible, but as some of these properties have already been investigated in a paper published in the *Quarterly Journal*, Vol. XXI., pp. 1–59, and in my Memoir on "Circles and Spheres," in the *Phil. Trans.*, Vol. CLXXVII., Part II., pp. 481–625, I shall merely quote results to be found in those papers.*

* References to these papers will be indicated by the letters $Q.$, $T.$, respectively.

Descriptive properties, viz., those involving only the angles at which the circles intersect, must obviously be true, whether the fundamental triangle is formed by three plane circular arcs or three spherical circular arcs. Also, since formulæ involving the radii of coplanar circles can always be deduced from the general formula (*cf.* *T.*, § 6 and § 129),

$$\pi \left(\begin{matrix} 1, 2, 3, 4, 5 \\ 6, 7, 8, 9, 10 \end{matrix} \right) = 0,$$

connecting the powers of two systems of circles, which is the same for spherical circles, it follows that any formula involving the radii of plane circles can be transformed into a formula involving radii of spherical circles by substituting the tangents of the spherical radii for the radii of the plane circles. It will be sufficient, then, to confine our attention to coplanar circles.

2. From the figure we see that, if α, β, γ are the angles of the fundamental triangle ABC , the angles of the associated triangle $A'BC$ are $\alpha, \pi - \beta, \pi - \gamma$; and that the angles of any pair of inverse triangles are the same. As pointed out in *Q.*, § 41, from any formula for the triangle ABC , corresponding formulæ may be found for the associated triangles by changing α, β, γ according to the scheme—

ABC	α	β	γ
$A'BC$	α	$\pi - \beta$	$\pi - \gamma$
$AB'C$	$\pi - \alpha$	β	$\pi - \gamma$
ABC'	$\pi - \alpha$	$\pi - \beta$	γ

But to obtain from a formula for any triangle ABC the corresponding formula for the inverse triangle $A'B'C'$, we must take into account the position of the orthogonal circle; in fact, if O be the centre of the orthogonal circle, we have (see *Q.*, § 5) the theorem—

If the centre of inversion O lie within or without each of two given circles, the angle at which they intersect is equal to the angle at which the inverse circles intersect; but if O lie within one circle and without the other, the angle at which they intersect is the supplement of the angle at which the inverse circles intersect.

In the sequel, unless otherwise stated, we shall suppose the radical centre of the given circles to lie within the triangle ABC , and the orthogonal circle of the given circles to be imaginary.

The Group of In-Circles, §§ 3, 4.

3. It has been proved in *T.*, § 45, that the in-circles of the eight triangles are touched by eight other circles; the theorem may be stated thus:—

The in-circles of any triangle and the in-circles of the three associated triangles are touched by another circle, which touches the first internally and the rest externally.

Thus each triangle has a circle corresponding to the nine-points circle of a plane rectilinear triangle,—it is called the Hart-circle.

It is easily proved (*T.*, § 46) that the Hart-circle of any triangle, whose angles are α, β, γ , cuts the sides at angles $\beta \sim \gamma, \gamma \sim \alpha, \alpha \sim \beta$.

4. If the inscribed circle of the triangle ABC cut the orthogonal circle at the angle ω , we have (*T.*, § 43),

$$K \cos^3 \omega = 2(1 + \cos \alpha)(1 + \cos \beta)(1 + \cos \gamma),$$

where

$$K = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma - 1.$$

Also, if r_1, r_2, r_3 are the radii of the given circles, and r the radius of the orthogonal circle, we have (*T.*, § 44) for the radius of the inscribed circle,

$$\begin{vmatrix} \frac{1}{\rho} + \frac{\cos \omega}{r}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3} \\ -1, & -1, & \cos \gamma, & \cos \beta \\ -1, & \cos \gamma, & -1, & \cos \alpha \\ -1, & \cos \beta, & \cos \alpha, & -1 \end{vmatrix} = 0.$$

Denoting by Δ the area of the triangle formed by the centres of the given circular arcs, we have (T., § 18)

$$4r^3\Delta^2 = r_1^2 r_2^2 r_3^2 K,$$

whence we deduce, for the radius of the inscribed circle,

$$\begin{aligned} \frac{K}{o} &= \frac{8\Delta}{r_1 r_2 r_3} \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \\ &\quad + (1 + \cos \alpha)(1 - \cos \alpha + \cos \beta + \cos \gamma) \frac{1}{r_1} \\ &\quad + (1 + \cos \beta)(1 + \cos \alpha - \cos \beta + \cos \gamma) \frac{1}{r_2} \\ &\quad + (1 + \cos \gamma)(1 + \cos \alpha + \cos \beta - \cos \gamma) \frac{1}{r_3}. \end{aligned}$$

The Group of Circum-circles, §§ 5-7.

5. From the figure we see that the circum-circle of the triangle ABC cuts the sides at angles

$$\frac{\pi}{2} + (s - \alpha), \quad \frac{\pi}{2} + (s - \beta), \quad \frac{\pi}{2} + (s - \gamma).$$

The radius of the circle is given by (T., § 39)

$$-\frac{2 \cos s}{\rho} = \frac{2\Delta}{r_1 r_2 r_3} + \frac{\sin \alpha}{r_1} + \frac{\sin \beta}{r_2} + \frac{\sin \gamma}{r_3},$$

and the angle at which it cuts the orthogonal circle is given by

$$4 \cos^2 \omega = K \sec^2 s.$$

6. The angles at which the circum-circles of triangle ABC and its associated triangles intersect are easily found from the figure; they are tabulated below:—

	ABC	$A'B'C'$	$A'BC$	$AB'C'$	$AB'C$	$A'BC'$	ABC'	$AB'C$
ABC	π		α	$\beta \sim \gamma$	β	$\gamma \sim \alpha$	γ	$\alpha \sim$
$A'B'C'$		π	$\pi - (\beta \sim \gamma)$	$\pi - \alpha$	$\pi - (\gamma \sim \alpha)$	$\pi - \beta$	$\pi - (\alpha \sim \beta)$	$\pi - \gamma$
$A'BC$	α	$\pi - (\beta \sim \gamma)$	π		$\alpha + \beta - \pi$	$\pi - \gamma$	$\alpha + \gamma - \pi$	$\pi - \beta$
$AB'C'$	$\beta \sim \gamma$	$\pi - \alpha$		π	$\pi - \gamma$	$\alpha + \beta - \pi$	$\pi - \beta$	$\alpha + \gamma - \pi$
$AB'C$	β	$\pi - (\gamma \sim \alpha)$	$\alpha + \beta - \pi$	$\pi - \gamma$	π		$\beta + \gamma - \pi$	$\pi - \alpha$
$A'BC'$	$\gamma \sim \alpha$	$\pi - \beta$	$\pi - \gamma$	$\alpha + \beta - \pi$		π	$\pi - \alpha$	$\beta + \gamma - \pi$
ABC'	γ	$\pi - (\alpha \sim \beta)$	$\alpha + \gamma - \pi$	$\pi - \beta$	$\beta + \gamma - \pi$	$\pi - \alpha$	π	
$A'B'C$	$\alpha \sim \beta$	$\pi - \gamma$	$\pi - \beta$	$\alpha + \gamma - \pi$	$\pi - \alpha$	$\beta + \gamma - \pi$		π

We notice that the circles $AB'C'$, $A'BO'$, $A'B'C$ intersect at angles

$$\beta + \gamma - \pi, \quad \gamma + \alpha - \pi, \quad \alpha + \beta - \pi,$$

and that the circle ABC cuts them at angles

$$\beta \sim \gamma, \quad \gamma \sim \alpha, \quad \alpha \sim \beta,$$

respectively; hence it follows that the circle ABC is the Hart-circle of the triangle $A'B'O'$ formed by the circular arcs $AB'C'$, $A'BO'$, $A'B'O$.

Similarly we see that the circle $A'B'C'$ is the Hart-circle of the triangle ABC formed by the circular arcs $A'BO$, $AB'O$, ABO' .

7. This theorem may be otherwise stated, for we know that the Hart-circle of a triangle touches the inscribed and escribed circles of the triangle; hence—

The circum-circle of any triangle and the circum-circles of the triangles inverse to its associated triangles are touched by four circles, each of which touches one of the former internally and the rest externally.

We thus obtain a group of eight circles, and if we consider that one which touches internally the circum-circle of any triangle, such as ABC , we may say that there is one circle belonging to each of the eight triangles of the system. Thus that circle of the group which corresponds to the triangle ABC will be the circle which touches the circles $AB'C'$, $A'BO'$, $A'B'O$ externally.

In the sequel these circles will be called the *T*-circles.

The theorem of this article was communicated to me by Mr. A. Larmor, of Clare College, Cambridge, who derived it by considering the dual nature of theorems respecting curves described on a sphere.

The T-circles, §§ 8, 9.

8. To find the angle at which the *T*-circle corresponding to the triangle ABC cuts the orthogonal circle of the triangle, let 1, 2, 3 denote the given circles, 4 the orthogonal circle, and let 5, 6, 7, 8 denote, respectively, the circles ABC , $AB'O$, $A'BO'$, $A'B'C'$; then, *T* denoting the circle which touches ABC internally and the others externally, and ω the angle at which the circle *T* cuts the circle 4, we have, from the formula,

$$\pi \begin{pmatrix} 1, 2, 3, 4 \\ 4, 5, 6, 7, 8 \end{pmatrix} = 0,$$

$$\begin{vmatrix} \cos \omega, & 0, & 0, & 0, & -1 \\ -1, & -\sin (s-a), & -\sin (s-\beta), & -\sin (s-\gamma), & \cos \omega_1 \\ 1, & -\sin s, & \sin (s-\gamma), & \sin (s-\beta), & \cos \omega_2 \\ 1, & \sin (s-\gamma), & -\sin s, & \sin (s-a), & \cos \omega_3 \\ 1, & \sin (s-\beta), & \sin (s-a), & -\sin s, & \cos \omega_4 \end{vmatrix} = 0,$$

where

$$\begin{aligned} -2 \cos \omega_1 \cos s &= 2 \cos \omega_2 \cos (s-a) \\ &= 2 \cos \omega_3 \cos (s-\beta) = 2 \cos \omega_4 \cos (s-\gamma) = K^4. \end{aligned}$$

Hence, we find,

$$K^4 \cos \omega = -2 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}.$$

Thus we see that, if the inscribed circle of the triangle ABC cuts the orthogonal circle at the angle Ω ,

$$\cos \Omega = 2 \cos \omega.$$

Hence, if $\bar{\rho}, \bar{\rho}'$ denote the radii of the T -circles corresponding to the triangles $ABC, A'B'C'$, and ρ, ρ' the radii of the inscribed circles of these triangles, we shall have [see *T.*, § 35 (47)]

$$\frac{1}{\rho} - \frac{1}{\rho'} = 2 \left(\frac{1}{\bar{\rho}} + \frac{1}{\bar{\rho}'} \right).$$

These two formulæ may be compared with the formulæ obtained in *T.*, § 46, viz.: if the circum-circle and Hart-circle of the triangle ABC cut the orthogonal circle at angles ϕ, Φ , respectively, then

$$\cos \Phi = 2 \cos \phi;$$

and if R, R' be the radii of the circum-circles of the triangles $ABC, A'B'C'$, and r, r' the radii of the Hart-circles of these triangles,

$$\frac{1}{r} - \frac{1}{r'} = 2 \left(\frac{1}{R} - \frac{1}{R'} \right).$$

9. The radius of the T -circle corresponding to the triangle ABC may be most easily found from the formula

$$\pi \left(\begin{matrix} T, 1, 2, 3, 4 \\ \theta, 5, 6, 7, 8 \end{matrix} \right) = 0,$$

where θ denotes the line at infinity.

Denoting the radius by ρ , we have at once

$$\begin{vmatrix} \frac{1}{\rho}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4} \\ -1, & -\sin(s-a), & -\sin(s-\beta), & -\sin(s-\gamma), & \cos \omega_1 \\ 1, & -\sin s, & \sin(s-\gamma), & \sin(s-\beta), & \cos \omega_2 \\ 1, & \sin(s-\gamma), & -\sin s, & \sin(s-a), & \cos \omega_3 \\ 1, & \sin(s-\beta), & \sin(s-a), & -\sin s, & \cos \omega_4 \end{vmatrix} = 0.$$

The Hart-circle, §§ 10-13.

10. The Hart-circle of a triangle might be defined as the circle which cuts the sides at angles $\beta \sim \gamma$, $\gamma \sim a$, $a \sim \beta$; α, β, γ being the angles of the triangle.

If ρ be the radius of the circle, ω the angle at which it cuts the orthogonal circle, we have (*Q.*, § 32)

$$\frac{\cos s}{\rho} = \frac{2\Delta}{r_1 r_2 r_3} + \frac{\cos(s-a)}{r_1} + \frac{\cos(s-\beta)}{r_2} + \frac{\cos(s-\gamma)}{r_3},$$

and

$$\cos^2 \omega = K \sec^2 s.$$

Starting with this definition we might prove, as in *Q.*, § 32, that the circle touches the inscribed circle and the three escribed circles of the triangle.

11. Several interesting properties of this circle may be derived from considering that the circle ABO is the Hart-circle of the triangle $A'BC'$ formed by the circular arcs $AB'C'$, $A'BO'$, $A'B'C'$.

Substituting the letters P, Q, R for A, B, C , and A', B', C' for A', B', C' , in §§ 6, 7, we infer that the Hart-circle of the triangle ABC cuts the sides in three points P, Q, R , such that—

- (i.) the straight lines AP, BQ, CR meet in a point L ;
- (ii.) each of the three groups of points (B, C, Q, R) , (C, A, R, P) , (A, B, P, Q) are concyclic;
- (iii.) L is the radical centre of the three circles $BCQR$, $CARP$, $ABPQ$;
- (iv.) the Hart-circle PQR is the inverse of the circum-circle ABC with respect to the circle which cuts the circles $BCQR$, $CARP$, $ABPQ$ orthogonally;

(v.) the inverse circles with respect to this orthogonal circle of the arcs BO , CA , AB are the circles circumscribing the triangles AQR , BRP , CPQ .

12. If the circles $BCQR$, $CARP$, $ABPQ$ intersect at angles θ , ϕ , ψ , then, referring to the table given in § 6, we have $\alpha = \phi + \psi - \pi$, &c. Now the circle $BCQR$ cuts the circle BPC at the angle $\frac{3\pi}{2} - \sigma$, where $2\sigma = \theta + \phi + \psi$; hence the circle $BCQR$ cuts the circle BPC at the angle $\frac{3\pi}{4} - \frac{s}{2}$, where $2s = \alpha + \beta + \gamma$.

Hence, the triangle ABC being given, we have the following simple construction for finding the points P , Q , R ; draw the circles $BCQR$, $CARP$, $ABPQ$ cutting the arcs BC , CA , AB , respectively, at the angle $\frac{3\pi}{4} - \frac{s}{2}$; and then the circle PQR is the Hart-circle of the triangle ABC .

13. The points P , Q , R evidently correspond to the feet of the perpendiculars of a plane rectilinear triangle, and the point L to the orthocentre. The circle which cuts the circles $BCQR$, $CARP$, $ABPQ$ orthogonally, evidently corresponds to the self-polar circle of a plane rectilinear triangle. Calling this circle the *I*-circle of the triangle, we see that there is such a circle corresponding to each of the eight triangles.

The *I*-circle, §§ 14–17.

14. Let θ , ϕ , ψ be the angles of intersection of the circles $BCQR$, $CARP$, $ABPQ$; then, ω_1 , ω_2 , ω_3 being the angles at which the *I*-circle cuts the circular arcs BOP , CAQ , ABR , we have, by § 5,

$$4 \cos^2 \omega_1 = L \sec^2 (\sigma - \theta),$$

and two similar formulæ, where

$$2\sigma = \theta + \phi + \psi,$$

$$\text{and} \quad L = 4 \cos \sigma \cos (\sigma - \theta) \cos (\sigma - \phi) \cos (\sigma - \psi).$$

$$\text{But} \quad \alpha = \phi + \psi - \pi, \quad \beta = \psi + \theta - \pi, \quad \gamma = \theta + \phi - \pi;$$

$$\text{therefore} \quad s = 2\sigma - \frac{3\pi}{2},$$

$$\text{and} \quad L = 4 \cos \left(\frac{s}{2} + \frac{3\pi}{4} \right) \cos \left(\frac{\pi}{4} + \alpha - \frac{s}{2} \right) \\ \cos \left(\frac{\pi}{4} + \beta - \frac{s}{2} \right) \cos \left(\frac{\pi}{4} + \gamma - \frac{s}{2} \right).$$

Hence

$$\frac{\cos^2 \omega_1}{\sec^2 \left(\frac{\pi}{4} + \alpha - \frac{s}{2} \right)} = \frac{\cos^2 \omega_2}{\sec^2 \left(\frac{\pi}{4} + \beta - \frac{s}{2} \right)} = \frac{\cos^2 \omega_3}{\sec^2 \left(\frac{\pi}{4} + \gamma - \frac{s}{2} \right)} = \frac{1}{4} L.$$

15. Again, let S , H denote the circum-circle and Hart-circle of the triangle; then, X denoting any other circle, we have, since S is the inverse of H with respect to I ,

$$\frac{\pi_{X,S}}{\pi_{S,I}} + \frac{\pi_{H,X}}{\pi_{I,H}} = 2 \frac{\pi_{X,I}}{\pi_{I,I}};$$

whence, taking X to be the line at infinity, we have

$$\frac{1}{\pi_{S,I}} + \frac{1}{\pi_{I,H}} = \frac{2}{\pi_{I,I}};$$

or, if R , r are the radii of S and H , and ρ the radius of I , and ω the angle at which S cuts I ,

$$\frac{1}{R} - \frac{1}{r} = \frac{2 \cos \omega}{\rho};$$

and we know, by § 5, that

$$4 \cos^2 \omega = L \sec^2 \sigma,$$

where L , σ have the same values as in § 14.

Hence, by § 5 and § 10,

$$\frac{L^{\frac{1}{2}} \sec \left(\frac{s}{2} + \frac{3\pi}{4} \right)}{\rho} = \frac{1}{R} \sim \frac{1}{\rho},$$

or

$$\frac{4L^{\frac{1}{2}} \cos \left(\frac{s}{2} + \frac{\pi}{4} \right)}{\rho} = \frac{2 \cos s}{R} \sim \frac{2 \cos s}{\rho} \\ = \frac{2\Delta}{r_1 r_2 r_3} + \frac{1}{r_1} [2 \cos (s - \alpha) - \sin \alpha] \\ + \frac{1}{r_2} [2 \cos (s - \beta) - \sin \beta] \\ + \frac{1}{r_3} [2 \cos (s - \gamma) - \sin \gamma].$$

16. To find the angle χ at which the I-circle cuts the orthogonal circle, let ϖ and ϖ' be the angles at which the circles S , H cut the orthogonal circle, and then, taking X as this circle, the formula given in § 15 reduces to

$$\cos \varpi \sim \cos \varpi' = \cos \chi \cos \omega.$$

But $\cos^2 \varpi' = K \sec^2 s$, by § 10,

and $4 \cos^2 \varpi = K \sec^2 s$, by § 5;

hence $K^{\frac{1}{2}} \sec s = 2 \cos \chi L^{\frac{1}{2}} \sec \sigma$,

therefore
$$\begin{aligned} \cos^2 \chi &= \frac{1}{4} \frac{K \cos^2 \sigma}{L \cos^2 s} \\ &= \frac{1}{4} \frac{\cos^2 \left(\frac{3\pi}{4} + \frac{s}{2} \right)}{\cos^2 s} \cdot \frac{K}{L} \\ &= \frac{1}{16} \sec^2 \left(\frac{\pi}{4} + \frac{s}{2} \right) \cdot \frac{K}{L}. \end{aligned}$$

17. We may notice that the I-circle of the triangle ABC is real when L is positive; this can only be the case when one of the angles,

$$\left(\frac{\pi}{4} + \alpha - \frac{s}{2} \right) \text{ say, is } > \frac{\pi}{2};$$

i.e., provided that one of the angles of the triangle ABC , is greater than $\frac{1}{2}(\beta + \gamma + \pi)$.

Thursday, May 8th, 1890.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

The President announced that a member, Lieut.-Col. J. R. Campbell, had asked to be allowed to give a donation of £500 to the Society, and that the Council had accepted the gift. On the motion of the Treasurer, seconded by Mr. S. Roberts, the following resolution was carried unanimously:—"That the cordial thanks of the London Mathematical Society be given to Lieut.-Col. Campbell for his generous gift of £500 to the general fund of the Society."

The following communications were made:—

On the Function which denotes the Excess of the Divisors of a Number which $\equiv 1, \text{ mod. } 3$, over those of a Number which $\equiv 2, \text{ mod. } 3$: Dr. Glaisher.

A Table of Complex Multiplication Moduli: Prof. Greenhill.

On a Theorem relating to Bicircular Quartics and Twisted Quartics: R. Lachlan.

On the Genesis of Binodal Quartic Curves from Conics: H. M. Jeffery.

On the Arithmetical Theory of the Form $x^3 + ny^3 + n^2z^3 - 3nxyz$: Prof. Mathews.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLVII., No. 288.

"Educational Times," for May.

"Mathematics from the 'Educational Times,'" Vol. LII.

"Transactions of the Royal Irish Academy," Vol. XXIX., Part XIII.

"Annals of Mathematics," Vol. v., No. 2; University of Virginia, Oct., 1889.

"Bulletin des Sciences Mathématiques," Tome XIV., April, 1890.

"Bulletin de la Société Mathématique de France," Tome XVII., No. 6; Tome XVIII., Nos. 2, 3, and 4.

"Atti della Reale Accademia dei Lincei," Vol. VI., Fasc. 4 and 5; Roma, 1890.

"Journal de l'École Polytechnique," 59th Cahier; Paris, 1889.

"Annuario del Circolo Matematico di Palermo—Rules," &c.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 102, 103, and 104.

"Jahrbuch über die Fortschritte der Mathematik," Band XIX., Heft 2; Jahrgang, 1887.

"Journal für die Reine und Angewandte Mathematik," Band CVI., Heft 2.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome IV.; Tome V., 1^{er} Cahier.

"Observations Pluviométriques et Thermométriques faites dans le Département de la Gironde de Juin 1887 à Mai 1888, et de Juin 1888 à Mai 1889; Bordeaux.

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"Esquisse Historique sur la Marche du Développement de la Géométrie du Triangle," by E. Vigarié; 8vo pamphlet (read before the French Association for the Advancement of Science—Congrès de Paris—1889).

"Sur les Axes de Steiner et l'Hyperbole de Kiepert," et "Sur les Foyers de Steiner d'un Triangle," MM. J. Neuberg and A. Gob (read before the French Association for the Advancement of Science—Congrès de Paris—1889).

"Sur les Projections et Contre-Projections d'un Triangle Fixe, et Sur le Système de Trois Figures directement Semblables," M. J. Neuberg. (Offprint from Tome XLIV. of the "Mémoires Couronnés et autres Mémoires," publiés par l'Académie Royale de Belgique, 1890.)

*On a Theorem relating to Bicircular Quartics and Twisted
Quartics. By R. LACHLAN, M.A.*

[Read May 8th, 1890.]

The present paper deals with a theorem analogous to the theorem that: Every plane cubic which passes through eight fixed points must pass through a ninth. It will be proved that a very similar theorem exists for plane bicircular quartics, and for twisted quartic curves, and, as might be expected, the theorem leads to as many, if not more, interesting results than the one for the cubic. It has been thought advisable to state for circular curves the corresponding theorems to those which are given in Salmon ("Higher Plane Curves," §§ 30-34) for ordinary plane curves, although of course the former may be easily deduced from these. And it should be stated that a circular curve of the $2n^{\text{th}}$ order is always understood to mean a curve of order $2n$, having each of the circular points at infinity as multiple points of the n^{th} order.

The paper is divided into three sections: in the first are given the general theory of the intersection of circular curves; in the second section a particular theorem relating to plane bicircular quartics is developed; and in the third section the corresponding theorem for twisted quartics is proved independently, and it is explained how the particular theorems in the second section may be translated so as to apply to twisted quartics.

General Theorems on Circular Curves, §§ 1-4.

1. The general equation of a curve of the $2n^{\text{th}}$ order having multiple points of the n^{th} order at each of the circular points, is of