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Notes on the Theory of Logarithms

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N O T E.

It may be well to explain at the outset that this Paper is not intended as a first introduction to the subject, but that a knowledge of the nature and use of Tables of Logarithms is presupposed, also a general knowledge of the Theory, such as may be obtained from the ordinary Algebra text-books, though a good deal of what may be thence derived is dealt with in Part III.

EDINBURGH, *April 22*, 1885.

DIAGRAM

Notes on the Theory of Logarithms.

PART I.

THE Theory of Logarithms was first made known to the world by John Nepair, Baron of Marchiston (now Merchiston), in a volume published at Edinburgh in 1614, under the title, "*Mirifici Logarithmorum Canonis descriptio Ejusque usus in utraq[ue] Trigonometriâ, ut etiam in omni Logistica Mathematicâ, Amplissimi, Facillimi, et expeditissimi, explicatio.*" This work was translated into English by Edward Wright, and published at London in 1618, with the table reduced by one place.

The following is a copy of part of the first page of the tables published in the above work:—

Gr. 0 Min.	0 Sinus.	Logarithmi.	+ - Differentiæ.	Logarithmi.	Sinus.	
0	0	Infinitum.	Infinitum.	0	10 000 000	60
1	2 909	81 425 681	81 425 680	1	10 000 000	59
2	5 818	74 494 213	74 494 211	2	9 999 998	58
3	8 727	70 439 564	70 439 560	4	9 999 996	57
4	11 636	67 562 746	67 562 739	7	9 999 993	56
5	14 544	65 331 315	65 331 304	11	9 999 989	55
						Min. 89 Gr. 89

Each page of these tables contains seven columns.

In the first and seventh columns are given the angles for every minute in the quadrant. In the first column the angle starts as 0° 0', and ends as 44° 60'. In the seventh column the angle starts as 89° 60', and ends as 45° 0'. So that the sum of the angles in the two columns is always equal to 90°.

In the second and sixth columns are given the numerical values of the sines of the angles in columns one and seven, the sines being referred to a radius of 10 000 000.

In the third and fifth columns are given the logarithms of these sines.

In the fourth column is given the difference of the logarithms in columns three and five.

NEPAIR'S ORIGINAL EXPLANATION OF THE NATURE OF LOGARITHMS.

Following on the nature of his tables, Nepair's original explanation of the theory of logarithms referred specifically to sines and their logarithms, but his explanation is of general application.

He first assumes the line AZ (see diagram) to represent the radius. Along this line he assumes a point to travel, starting from A , and moving with continually decreasing velocity, so that it marks off, in equal moments of time, continually decreasing lengths of line. Thus in the first moment of time it reaches the point B , in the second the point C , and so on. The decreasing velocity of the moving point being such that AB (the part of AZ travelled over in the first moment) is to BC (the part travelled over in the second moment) in the same ratio as ZA (the whole line or radius) is to ZB (the remainder)—namely, that $AB:BC::ZA:ZB$.

Thus the moving point, in successive equal moments of time, marks off on AZ lengths of line which are in continued decreasing proportion, so that $AB:BC::BC:CD::CD:DE::$ etc.

The respective remainders being consequently also in continued decreasing proportion, so that $ZA:ZB::ZB:ZC::ZC:ZD::ZD:ZE::$ etc.

The numerical equivalents of these lines will thus form a decreasing geometrical series. And the sines of all angles in the quadrant being less than radius, by taking these remainders near enough together one of them will always be found approximating very closely to the sine of each angle given in the table.

He next assumes a line commencing at a and of indefinite length towards z , along which a point travels with uniform velocity, marking off in equal moments of time the spaces ab, bc, cd , etc., which are all equal. So that ab, ac, ad , etc., will form a series of lines increasing by equal lengths. And their numerical equivalents will consequently form an increasing arithmetical series.

Having got these two series, he then explains that the numerical equivalents of the distances marked off on az will be the logarithms of the numerical equivalents of the remainders marked off on ZA .

So that $ab = \log ZB$, $ac = \log ZC$, etc. The line az having no length at the point a , therefore $0 = \log ZA$.

HOW NEPAIR DEFINES HIS PARTICULAR SYSTEM.

But the above explanation is still defective, as the determination of two independent relations is necessary to define a particular system of logarithms. So having assumed the relation $\log ZA = 0$ as the first, Nepair proceeds completely to define his system by assuming as the second that the velocities of the moving points at the instant of starting are equal.

That is, supposing AB to be an extremely small fraction of AZ , the limit to which AB approaches as it is indefinitely diminished is such that virtually $AB = ab$.

ALGEBRAICAL DEFINITION OF NEPAIR'S ORIGINAL SYSTEM.

Assume that AZ or radius is denoted by R , and that it is divided into n equal parts so that $AB = \frac{R}{n}$, and ab, bc, cd , etc., also each $= \frac{R}{n}$. Then Nepair's definition algebraically expressed is

$$\log \left(R - \frac{R}{n} \right) = \frac{R}{n} \text{ or } \log \left\{ R \times \left(1 - \frac{1}{n} \right) \right\} = R \times \frac{1}{n}.$$

When R is assumed equal to unity this is an entirely satisfactory definition, and this being done the expression becomes $\log \left(1 - \frac{1}{n} \right) = \frac{1}{n}$, where n is indefinitely great or $= \infty$, and consequently $\frac{1}{n}$ is indefinitely small.

ALGEBRAICAL EXPRESSION AND RESULTING SERIES FOR THE BASE OF NEPAIR'S ORIGINAL SYSTEM.

By the theory of indices we know that $\left(1 - \frac{1}{n} \right)^0 = 1$.

We have then the two following series, the terms of the geometrical representing sines or numbers and the corresponding terms of the arithmetical representing their logarithms.

Geometrical series or numbers—

$$\left(1 - \frac{1}{n}\right)^0 \text{ or } 1, \left(1 - \frac{1}{n}\right)^1, \left(1 - \frac{1}{n}\right)^2, \left(1 - \frac{1}{n}\right)^3, \dots \left(1 - \frac{1}{n}\right)^n, \text{ etc.}$$

Arithmetical series or logarithms—

$$0, \quad \frac{1}{n}, \quad \frac{2}{n}, \quad \frac{3}{n}, \quad \dots \quad \frac{n}{n} \text{ or } 1, \text{ etc.}$$

In a system of logarithms, 0 being taken as the logarithm of unity, it follows that the number of which unity is the logarithm is the base of that system.

From this we see that $\log\left(1 - \frac{1}{n}\right)^n$ being $\frac{n}{n} = 1$, therefore the limit of $\left(1 - \frac{1}{n}\right)^n$ when $n = \infty$ is really the base of this system.

To find this limit expand $\left(1 - \frac{1}{n}\right)^n$ by the binomial theorem, then we have—

$$\begin{aligned} \left(1 - \frac{1}{n}\right)^n &= 1 - \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{1.2} \cdot \frac{1}{n^2} - \frac{n(n-1)(n-2)}{1.2.3} \cdot \frac{1}{n^3} \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} \cdot \frac{1}{n^4} - \text{etc.} \\ &= 1 - \frac{n}{n} + \frac{n^2 - n}{2n^2} - \frac{n^3 - 3n^2 + 2n}{6n^3} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{24n^4} - \text{etc.} \\ &= 1 - 1 + \left(\frac{1}{2} - \frac{1}{2n}\right) - \left(\frac{1}{6} - \frac{1}{2n} + \frac{1}{3n^2}\right) + \left(\frac{1}{24} - \frac{1}{4n} + \frac{11}{24n^2} - \frac{1}{4n^3}\right) \\ &\quad - \text{etc.} \end{aligned}$$

Now the parts of the above terms which contain n , viz. $-\left(\frac{1}{2n}\right)$, $+\left(\frac{1}{2n} - \frac{1}{3n^2}\right)$, $-\left(\frac{1}{4n} - \frac{11}{24n^2} + \frac{1}{4n^3}\right)$, etc., are none of them greater than $\frac{1}{2n}$, but when n becomes indefinitely increased these fractions become indefinitely small, and they may be made less than any assigned magnitude by correspondingly increasing n , so that as n approaches the limit of ∞ they will vanish altogether, and we thus find the limit of the expansion to be—

$$\begin{aligned} &= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \text{etc.} \\ &= 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \text{etc.} \end{aligned}$$

NUMERICAL VALUE OF THE SERIES FOR THE BASE OF
NEPAIR'S ORIGINAL SYSTEM.

To obtain the numerical value of this series, each term must be converted into decimals; the sum of the positive and also of the negative terms so converted being then obtained, their difference will be the required value. The following is the actual working, carrying out the decimals to 25 places.

+ Term ₁ = + 1·	= + 1·00000 00000 00000 00000 00000
„ 3 = + $\frac{1}{1.2}$	= + ·50000 00000 00000 00000 00000
„ 5 = + $\frac{1}{1.2.3.4}$	= + ·04166 66666 66666 66666 66667
„ 7 = + $\frac{1}{1.2.3.4.5.6}$	= + ·00138 88888 88888 88888 88889
„ 9 = + $\frac{1}{1.2.3.4.5.6.7.8}$	= + ·00002 48015 87301 58730 15873
„ 11 = etc.	= + ·00000 02755 73192 23985 89065
„ 13 =	= + ·00000 00020 87675 69878 68099
„ 15 =	= + ·00000 00000 11470 74559 77297
„ 17 =	= + ·00000 00000 00047 79477 33239
„ 19 =	= + ·00000 00000 00000 15619 20697
„ 21 =	= + ·00000 00000 00000 00041 10318
„ 23 =	= + ·00000 00000 00000 00000 08897
„ 25 =	= + ·00000 00000 00000 00000 00016
<hr/>	
+ 1·54308 06348 15243 77847 79057	
- Term ₂ = - $\frac{1}{1}$	= - 1·00000 00000 00000 00000 00000
„ 4 = - $\frac{1}{1.2.3}$	= - ·16666 66666 66666 66666 66667
„ 6 = - $\frac{1}{1.2.3.4.5}$	= - ·00833 33333 33333 33333 33333
„ 8 = - $\frac{1}{1.2.3.4.5.6.7}$	= - ·00019 84126 98412 69841 26984
„ 10 = etc.	= - ·00000 27557 31922 39858 90653
„ 12 =	= - ·00000 00250 52108 38544 17188
„ 14 =	= - ·00000 00001 60590 43836 82161
„ 16 =	= - ·00000 00000 00764 71637 31820
„ 18 =	= - ·00000 00000 00002 81145 72543
„ 20 =	= - ·00000 00000 00000 00822 06352
„ 22 =	= - ·00000 00000 00000 00001 95729
„ 24 =	= - ·00000 00000 00000 00000 00387
„ 26 =	= - ·00000 00000 00000 00000 00001
<hr/>	
- 1·17520 11936 43801 45688 23818	

Sum of terms with positive sign	}	=	+	1·54308 06348 15243 77847 79057
Sum of terms with negative sign				
Numerical value of the series to 25 decimal places	}	=	+	·36787 94411 71442 32159 55239

The above quantity ·36787 etc., is therefore theoretically the base of this system.

THE BASE ACTUALLY USED BY NEPAIR IN CONSTRUCTING HIS TABLES.

The base of this system we have seen was theoretically $\left(1 + \frac{1}{n}\right)^n$, where $n = \infty$. But to suit his method of calculation, Nepair assumed, as a sufficiently close approximation for his purpose, $n = 10\,000\,000$, instead of $n = \infty$. He also introduced other modifications, which will be referred to afterwards. But at present we will consider his original definition as thus far modified only.

So that now we may consider $\left(1 - \frac{1}{10\,000\,000}\right)^{10\,000\,000}$ as the actual base of the system.

DIFFERENCE BETWEEN THE NUMERICAL VALUE OF THE BASE $\left(1 - \frac{1}{n}\right)^n$, WHEN $n = \infty$ AND WHEN $n = 10\,000\,000$.

That the numerical value of the base as thus modified is a very close approximation to the value of the base as in the original definition, the following calculation will show :—

Base = $\left(1 - \frac{1}{10\,000\,000}\right)^{10\,000\,000} = (.9999999)^{10\,000\,000}$ and
log base = $10\,000\,000 \times \log .9999999$.

We have, therefore, first to find $\log .9999999$.

From Hutton's tables of common logarithms to 64 places we get the following, shortening the logarithms to 20 places :—

u_x	$\Delta^1 u_x$
$\log_{10} \cdot 999999 = 9 \cdot 99999 \ 95657 \ 05300 \ 94936$	+ 04342 94699 05064
$\log_{10} 1 \cdot 000000 = 0 \cdot 00000 \ 00000 \ 00000 \ 00000$	+ 04342 94264 75616
$\log_{10} 1 \cdot 000001 = 0 \cdot 00000 \ 04342 \ 94264 \ 75616$	+ 04342 93830 46254
$\log_{10} 1 \cdot 000002 = 0 \cdot 00000 \ 08685 \ 88095 \ 21870$	
<hr/>	
$\Delta^2 u_x$	$\Delta^3 u_x$
- 434 29448	+ 87
- 434 29361	

As the numbers given in the table are less by one place than the number required, we must find it by interpolation.

Applying the well-known formula—

$$u_{x+n} = u_x + \frac{n}{1} \cdot \Delta^1 u_x + \frac{n(n-1)}{1 \cdot 2} \cdot \Delta^2 u_x, \text{ etc., we find—}$$

$$u_x + \frac{a}{n} = u_x + \frac{a}{n} \cdot \Delta^1 u_x + \frac{\frac{a}{n} \left(\frac{a}{n} - 1 \right)}{1 \cdot 2} \cdot \Delta^2 u_x + \frac{\frac{a}{n} \left(\frac{a}{n} - 1 \right) \left(\frac{a}{n} - 2 \right)}{1 \cdot 2 \cdot 3} \cdot \Delta^3 u_x + \text{etc.,}$$

$$u_x + \frac{9}{10} = u_x + \frac{9}{10} \cdot \Delta^1 u_x + \frac{\frac{9}{10} \left(-\frac{1}{10} \right)}{1 \cdot 2} \cdot \Delta^2 u_x + \frac{\frac{9}{10} \left(-\frac{1}{10} \right) \left(-\frac{11}{10} \right)}{1 \cdot 2 \cdot 3} \cdot \Delta^3 u_x + \text{etc.,}$$

$$= u_x + \frac{9}{10} \cdot \Delta^1 u_x - \frac{9}{200} \cdot \Delta^2 u_x + \frac{33}{2000} \cdot \Delta^3 u_x - \text{etc.}$$

Interpolating by this formula we have—

$$\begin{aligned} u_x &= + 9 \cdot 99999 \ 95657 \ 05300 \ 94936 \\ + \frac{9}{10} \cdot \Delta^1 u_x &= + \quad \quad \quad 3908 \ 65229 \ 14558 \\ - \frac{9}{200} \cdot \Delta^2 u_x &= + \quad \quad \quad 19 \ 54325 \\ + \frac{33}{2000} \cdot \Delta^3 u_x &= + \quad \quad \quad 0 \end{aligned}$$

$$u_x + \frac{9}{10} = \underline{\underline{9 \cdot 99999 \ 99565 \ 70549 \ 63819}} = \log_{10} \cdot 9999999$$

Then $\log_{10}(\cdot 9999999)^{10\,000\,000}$ will be 10 000 000 times the logarithm found above, or 9·56570 54963 819.

By Hutton's seven-place table the number corresponding to this is ·3678794, and by Vlacq's table it is ·36787 94228.

By the kindness of Mr. Sang, I am, however, able to give the logarithm and its corresponding number with still greater exactitude.

From his most valuable and extensive Tables of Logarithms to 28 places, which are still unpublished, $\log_{10}(\cdot 999\,9999)^{10\,000\,000} = 9\cdot56570\,54963\,82022\,62954$ etc., and the number corresponding is ·36787 94227 7772 etc.

We found the base when $n = \infty$ to be ·36787 94411 7144 etc. (*See page 364.*)

So that the assumption of 10 000 000 instead of ∞ as the value of n in the expression for the base, namely $\left(1 - \frac{1}{n}\right)$, decreases its numerical value by ·00000 00183 9372, or by less than 2 in the eighth decimal place.

It may be interesting to see how it will affect the base if n be taken equal to one million instead of ten millions. In that case base $= \left(1 - \frac{1}{1\,000\,000}\right)^{1\,000\,000} = (\cdot 999999)^{1\,000\,000}$ and $\log \text{ base} = 1\,000\,000 \times \log \cdot 999999$.

As given above, $\log_{10} \cdot 999999 = 9\cdot99999\,95657\,05301$, and this multiplied by one million gives 9·56570 5301, the number corresponding to which by Hutton's seven-place table is ·3678793, which only differs by one in the seventh place from the value for the base found when $n = \infty$.

RELATIONS OF NUMBERS TO THEIR LOGARITHMS AS EXHIBITED IN THE DIAGRAM.

Having found the base of the system, we may now more clearly see the relations of the base and other numbers to their logarithms, as illustrated in the diagram. We have seen that Nepair takes for his base $\left(1 - \frac{1}{10\,000\,000}\right)^{10\,000\,000}$ as being for his purpose a sufficiently near approximation to $\left(1 - \frac{1}{\infty}\right)^{\infty}$.

For graphic illustration, let us take $\left(1 - \frac{1}{10}\right)^{10}$ as the base: this is a very rough approximation, but it is on precisely the same

lines as Nepair's, and is as close an approximation as can conveniently be shown in the diagram.

By continual subtraction of $\frac{1}{10}$ th we get the successive powers of $\left(1 - \frac{1}{10}\right)$ as shown below, and this method is made use of here as it was the one by which Nepair calculated the powers of $\left(1 - \frac{1}{10\,000\,000}\right)$, and so formed his table.

Since $\left(1 - \frac{1}{10}\right)^{10}$ is the base, the logarithm of which is 1, $\left(1 - \frac{1}{10}\right)$ is the 10th root of the base, and its logarithm is $\frac{1}{10}$ or $\cdot 1$. The decimals are not carried beyond 5 places.

	Numbers.	Logarithms.	Numbers— <i>continued.</i>	Logarithms— <i>contd.</i>
	1·00000	$\frac{0}{10}$	·34868	$\frac{10}{10}$
Less $\frac{1}{10}$ th	·10000		·03487	
	·90000	$\frac{1}{10}$	·31381	$\frac{11}{10}$
	·09000		·03138	
	·81000	$\frac{2}{10}$	·28243	$\frac{12}{10}$
	·08100		·02824	
	·72900	$\frac{3}{10}$	·25419	$\frac{13}{10}$
	·07290		·02542	
	·65610	$\frac{4}{10}$	·22877	$\frac{14}{10}$
	·06561		·02288	
	·59049	$\frac{5}{10}$	·20589	$\frac{15}{10}$
	·05905		·02059	
	·53144	$\frac{6}{10}$	·18530	$\frac{16}{10}$
	·05314		·01853	
	·47830	$\frac{7}{10}$	·16677	$\frac{17}{10}$
	·04783		·01668	
	·43047	$\frac{8}{10}$	·15009	$\frac{18}{10}$
	·04305		·01501	
	·38742	$\frac{9}{10}$	·13508	$\frac{19}{10}$
	·03874		·01351	
	·34868	$\frac{10}{10}$	·12157	$\frac{20}{10}$

We thus find $\cdot34868$ etc., to be the value of the base when we use $\left(1-\frac{1}{10}\right)^{10}$ instead of $\left(1-\frac{1}{\infty}\right)^{\infty}$, the value of which, as found on page 364, is $\cdot36787$ etc. So that even this very rough approximation gives a value for the base not altogether unlike the exact value.

The lengths of the lines representing numbers and logarithms, with their approximate values to three decimal places, should thus be as follows, and in the diagram you will see they are represented with tolerable exactitude.

Lengths of line representing numbers, Numerical values,	{	<i>ZA</i>	<i>ZB</i>	<i>ZC</i>	<i>ZD</i>	<i>ZE</i>	<i>ZF</i>	<i>ZG</i>	<i>ZH</i>	<i>ZI</i>	<i>ZJ</i>	<i>ZK</i>
	{	1·000	·900	·810	·729	·656	·590	·531	·478	·430	·387	·349
Lengths of line representing logarithms, Numerical values,	{	at point <i>a</i>	<i>ab</i>	<i>ac</i>	<i>ad</i>	<i>ae</i>	<i>af</i>	<i>ag</i>	<i>ah</i>	<i>ai</i>	<i>aj</i>	<i>ak</i>
	{	0	·100	·200	·300	·400	·500	·600	·700	·800	·900	1·000
Do. Continued.	{	<i>ZL</i>	<i>ZM</i>	<i>ZN</i>	<i>ZO</i>	<i>ZP</i>	<i>ZQ</i>	<i>ZR</i>	<i>ZS</i>	<i>ZT</i>	<i>ZU</i>	etc.
	{	·314	·282	·254	·229	·206	·185	·167	·150	·135	·122	
Do. Continued.	{	<i>al</i>	<i>am</i>	<i>an</i>	<i>ao</i>	<i>ap</i>	<i>aq</i>	<i>ar</i>	<i>as</i>	<i>at</i>	<i>au</i>	etc.
	{	1·100	1·200	1·300	1·400	1·500	1·600	1·700	1·800	1·900	2·000	

The length of *AZ* is assumed to be unity, and it is divided into equal spaces of $\frac{1}{10}$ each, which, commencing at *Z* or 0, are numbered $\cdot100$, $\cdot200$, $\cdot300$, etc., up to 1·000.

The length of *az* is indefinite towards *z*, and commencing at *a* it is divided into lengths, each equal to *AZ* or 1, and these are each again subdivided into ten equal parts, so that beginning at *a* or 0 they are numbered $\cdot100$, $\cdot200$, $\cdot300$, etc., indefinitely onwards.

Before either of the points travelling along the lines *AZ* and *az* has moved, the length of the line *AZ* is 1, and that of *az* at the point *a* is 0; so that 0 is the logarithm of 1.

When the point moving along *AZ* has got to *B*, leaving the remainder *ZB*=·900, the point moving along *az* has got to *b* marking off *ab*=·100, so that ·100 is the logarithm of ·900.

When the first-mentioned point has got to *C*, leaving the remainder *ZC*=·810, the second has got to *c*, marking off *ac*=·200, so that ·200 is the logarithm of ·810.

The point moving along *AZ* having successively marked off 10 decreasing parts, reaches *K*, leaving the remainder *ZK*=·349, and the point moving along *az* having marked off 10 parts, each equal

to $\cdot 100$, reaches k , marking off $ak=1\cdot 000$, so that $1\cdot 000$ is the logarithm of $\cdot 349$, and consequently ZK represents the base of the system.

In the same way, after the point moving along AZ has marked off other 10 decreasing parts, it will have reached U , leaving the remainder $ZU=\cdot 122$, and the point moving along az will have marked off 10 more parts, each equal to $\cdot 100$, reaching u , and marking off $au=2\cdot 000$, so that $2\cdot 000$ is the logarithm of $\cdot 122$, or the base raised to the 2d power.

NEPAIR'S LOGARITHMS AS PUBLISHED.

The tables published by Nepair being tables of sines and their logarithms, in order to make the sines whole numbers he made radius= $10\ 000\ 000$, and then assumed the sine of any angle to be the whole number nearest its exact value, disregarding the decimals as unimportant.

Having for this reason made radius= $10\ 000\ 000$, he still retained 0 as its logarithm, so that in his tables 0 stands as the logarithm of $10\ 000\ 000$ instead of as the logarithm of 1.

Had he stopped here we would simply have required, when using his tables, to keep in view that the numbers in column "Sinus" were all referred to a radius of $10\ 000\ 000$, and that consequently they were not really *numbers* but ratios; thus any number, say x , in that column would represent not the *number* x , but the ratio $x : 10\ 000\ 000 = \text{the ratio } \frac{x}{10\ 000\ 000} : 1 = \text{the real}$

number $\frac{x}{10\ 000\ 000}$. So that his table might have been used as a table of logarithms of ordinary *numbers*, by advancing the decimal point of the numbers in column "sinus" seven places. As an example, the sine of 0° or radius, which is entered as $10\ 000\ 000$, would then become $1\cdot 0000000$, and its logarithm 0 would then appear correctly as that of 1.

But further, owing to using the formula $\log \left\{ R \times \left(1 - \frac{1}{n} \right) \right\} = R \times \frac{1}{n}$ instead of the more simple and satisfactory one, $\log \left(1 - \frac{1}{n} \right) = \frac{1}{n}$, the assumption of $R=10\ 000\ 000$ led Nepair at the same time to make

$\log \left\{ 10\,000\,000 \times \left(1 - \frac{1}{10\,000\,000} \right) \right\} = 10\,000\,000 \times \frac{1}{10\,000\,000}$
 or 1, so that he multiplied his logarithms by 10 000 000 as well as his sines. Now, bearing in mind that the numbers in column “sinus” are really antecedents of ratios referred to the common consequent 10 000 000, in doing this he altered the base of his system from $\left(1 - \frac{1}{10\,000\,000} \right)^{10\,000\,000}$ to $\left(1 - \frac{1}{10\,000\,000} \right)$, thereby disguising the fact that the former is essentially the base of the system.

The part of his original explanation which ultimately defines the system is that the decrement of the line representing radius in the first moment of time, is equal to the increment of the line representing logarithms in that first moment. But by this alteration the latter is made 10 000 000 times greater than the former. This makes his published logarithms 10 000 000 times greater than the logarithms of the corresponding numbers in the system which has for its base $\left(1 - \frac{1}{10\,000\,000} \right)^{10\,000\,000}$. The Modulus of the system as published compared with the latter system is therefore 10 000 000.

It will however be simpler, and more in keeping with Napier’s reason for introducing these modifications, to consider that he arbitrarily multiplied both his numbers and his logarithms by 10 000 000 so as to avoid the use of decimals.

Therefore, to bring his table into harmony with the system having base $= \left(1 - \frac{1}{10\,000\,000} \right)^{10\,000\,000}$, the numbers in both his columns “Sinus” and “Logarithmi” must be divided by 10 000 000.

For further reference to this subject, see page 373.

THE SYSTEM OF LOGARITHMS, NOW CALLED “NAPIERIAN” OR “HYPERBOLIC,” OR OTHERWISE THE “NATURAL SYSTEM OF LOGARITHMS.”

The base of the system of logarithms, now called Napierian, is $\left(1 + \frac{1}{n} \right)^n$ where $n = \infty$. To find the limit of this expression, and also its numerical value, expand $\left(1 + \frac{1}{n} \right)^n$ by the binomial theorem.

$$\begin{aligned}
 \left(1 + \frac{1}{n}\right)^n &= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{1.2} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3} \cdot \frac{1}{n^3} \\
 &\quad + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} \cdot \frac{1}{n^4} + \text{etc.} \\
 &= 1 + \frac{1}{1} \cdot \left(\frac{n}{n}\right) + \frac{1}{1.2} \cdot \left(\frac{n^2}{n^2} - \frac{n}{n^2}\right) + \frac{1}{1.2.3} \cdot \left(\frac{n^3}{n^3} - \frac{3n^2}{n^3} + \frac{2n}{n^3}\right) \\
 &\quad + \frac{1}{1.2.3.4} \cdot \left(\frac{n^4}{n^4} - \frac{6n^3}{n^4} + \frac{11n^2}{n^4} - \frac{6n}{n^4}\right) + \text{etc.} \\
 &= 1 + \frac{1}{1} \cdot (1) + \frac{1}{1.2} \cdot \left(1 - \frac{1}{n}\right) + \frac{1}{1.2.3} \cdot \left(1 - \frac{3}{n} + \frac{2}{n^2}\right) \\
 &\quad + \frac{1}{1.2.3.4} \cdot \left(1 - \frac{6}{n} + \frac{11}{n^2} - \frac{6}{n^3}\right) + \text{etc.}
 \end{aligned}$$

and the parts of these terms containing n , viz.—

$$-\left(\frac{1}{2n}\right), -\left(\frac{3}{6n} - \frac{2}{6n^2}\right), -\left(\frac{6}{24n} - \frac{11}{24n^2} + \frac{6}{24n^3}\right), \text{etc.},$$

as explained before, will vanish as n approaches ∞ , so that the limit of the expansion will be—

$$= 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{etc.}$$

Having converted the individual terms of this series into decimals, their sum is the numerical value of the Base, and this, as shown below, is 2.71828 18284 59045 23536 02875 etc., which value is usually symbolised by e .

On page 363 are given the values of the individual terms of the series expressed in decimals. There the terms are alternately positive and negative, here they are all positive, so that instead of taking the difference of the sums of the alternate terms as we did then, we must now take their sum, which is—

$$\begin{array}{r}
 + 1\text{st, 3rd, 5th, etc. terms} = + 1.54308\ 06348\ 15243\ 77847\ 79057 \\
 + 2\text{nd, 4th, 6th, etc. terms} = + 1.17520\ 11936\ 43801\ 45688\ 23818 \\
 \hline
 + 2.71828\ 18284\ 59045\ 23536\ 02875
 \end{array}$$

RELATION BETWEEN THE BASE OF THE HYPERBOLIC SYSTEM AND THE BASE OF THE SYSTEM AS ORIGINALLY DEFINED BY NEPAIR.

It has been shown on pages 361 to 364 that the limit of $\left(1 - \frac{1}{n}\right)^n$ was

$$1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \text{etc.}, \text{ and that its numerical value was}$$

obtained by taking the difference of the sums of the alternate terms of the series as below—

$$\begin{array}{r} + \text{1st, 3rd, 5th, etc. terms} = + 1\cdot54308\ 06348\ 15243\ 77847\ 79057 \\ - \text{2nd, 4th, 6th, etc. terms} = - 1\cdot17520\ 11936\ 43801\ 45688\ 23818 \\ \hline + \quad \cdot36787\ 94411\ 71442\ 32159\ 55239 \end{array}$$

Thus the numerical values of the limits of the two expressions are—

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 2\cdot71828\ 18284 \text{ etc.} \\ \left(1 - \frac{1}{n}\right)^n &= \cdot36787\ 94411 \text{ etc.} \end{aligned}$$

If we divide unity by the value of the latter expression, we get the value of the first, thus—

$$\begin{array}{r} \cdot36787\ 94411 \text{ etc.})\ 1\cdot0000000000\ (2\cdot71828\ 1828 \text{ etc.} \\ \quad \cdot7357588823 \\ \hline \quad 2642411177 \\ \quad 2575156088 \\ \hline \quad \quad 67255089 \\ \quad \quad 36787944 \\ \hline \quad \quad 30467145 \\ \quad \quad 29430355 \\ \hline \quad \quad \quad 1036790 \\ \quad \quad \quad 735759 \\ \hline \quad \quad \quad 301031 \\ \quad \quad \quad 294304 \\ \hline \quad \quad \quad \quad 6727 \\ \quad \quad \quad \quad 3679 \\ \hline \quad \quad \quad \quad 3048 \\ \quad \quad \quad \quad 2943 \\ \hline \quad \quad \quad \quad \quad 105 \\ \quad \quad \quad \quad \quad 73 \\ \hline \quad \quad \quad \quad \quad 32 \\ \quad \quad \quad \quad \quad 30 \\ \hline \quad \quad \quad \quad \quad \quad 2 \end{array}$$

This leads to the conclusion that the limit of $\frac{1}{\left(1 - \frac{1}{n}\right)^n}$ must be the same as the limit of $\left(1 + \frac{1}{n}\right)^n$.

That this is so, is proved as follows, expanding by the binomial theorem—

$$\begin{aligned}\frac{1}{\left(1 - \frac{1}{n}\right)^n} &= \left(1 - \frac{1}{n}\right)^{-n} \\ &= + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n+1)}{1.2} \cdot \frac{1}{n^2} + \frac{n(n+1)(n+2)}{1.2.3} \cdot \frac{1}{n^3} + \text{etc.} \\ &= 1 + \frac{1}{1} \cdot (1) + \frac{1}{1.2} \cdot \left(1 + \frac{1}{n}\right) + \frac{1}{1.2.3} \cdot \left(1 + \frac{3}{n} + \frac{2}{n^2}\right) + \text{etc.}\end{aligned}$$

and in a similar way to that formerly shown $\left(\frac{1}{n}\right)$, $\left(\frac{3}{n} + \frac{2}{n^2}\right)$, etc., vanish as n approaches ∞ , so that the limit of the expansion is—

$$= 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \text{etc.}$$

being the same as we obtained for $\left(1 + \frac{1}{n}\right)^n$.

Thus we see that the base of the natural system of logarithms being denoted by e , the base of the system as originally defined by Nepair is $\frac{1}{e}$ or e^{-1} .

COMPARISON OF THE LOGARITHMS PUBLISHED BY NEPAIR, WITH HYPERBOLIC LOGARITHMS.

We are now in a position to examine the relation which the logarithms in Nepair's published tables bear to those in our tables of Hyperbolic logarithms.

As before explained, the numbers and logarithms in Nepair's tables are each multiplied by 10 000 000. So that we have first to eliminate this constant multiplier by dividing all his numbers and logarithms by 10 000 000.

Having done so, then any logarithm of the same numerical value in the two systems shall have for its number in the one system the reciprocal of that which it has in the other. For instance, if $l = \log N$ in Nepair's system, then shall $l = \log \frac{1}{N}$ in the Hyperbolic system.

And conversely the logarithm of any given number shall in the one system have the opposite sign from that which it has in the other. For instance, if $\log N = l$ in Nepair's system, then shall $\log N = -l$ in the Hyperbolic system.

Underneath is given a table in *Part 1* of which is extracted as it stands a portion of Neper's original table.

In *Part 2*, omitting the angles, and now calling "Sinus" Number and "Logarithmi" Logarithm, are given the same figures as in *Part 1*, but each divided by 10 000 000, the constant by which Neper multiplied both his numbers and logarithms.

In *Part 3*, the logarithms remain the same as in *Part 2*, but the numbers are given which correspond to these logarithms in the Hyperbolic system. They are left in the form of vulgar fractions for the sake of clearness, but they may be reduced to the form of decimals by division when required in that form for comparison.

In *Part 4*, the numbers remain the same as in *Part 2*, but the logarithms are converted into the Hyperbolic logarithms of the numbers. As explained above, this is done by simply prefixing the negative sign to the logarithms in *Part 2*.

In *Part 5*, the numbers in *Part 4* have been multiplied by 10 000, and the logarithms in it have been correspondingly increased. That is, they being hyperbolic logarithms, have each been increased by the hyperbolic logarithm of 10 000 = $4 \times \log_e 10 = 4 \times 2.3025851 = 9.2103404$. So that the numbers and logarithms in *Parts 4* and *5* are exactly similar, the change in *Part 5* being made to enable us to compare them with Hutton's table of Hyperbolic logarithms, which does not contain logarithms of numbers less than unity, which those in *Part 4* are.

TABLE.

PART 1.			PART 2.		PART 3.	
Gr. Min.	Sinus.	Logarithmi.	Numbers.	Logarithms.	Numbers.	Logarithms.
5° 45'	1 001 881	23 007 056	·100 1881	2·300 7056	$\frac{1}{\cdot 100\ 1881}$	2·300 7056
46	1 004 775	22 978 212	·100 4775	2·297 8212	$\frac{1}{\cdot 100\ 4775}$	2·297 8212
47	1 007 669	22 949 449	·100 7669	2·294 9449	$\frac{1}{\cdot 100\ 7669}$	2·294 9449
48	1 010 563	22 920 769	·101 0563	2·292 0769	$\frac{1}{\cdot 101\ 0563}$	2·292 0769
49	1 013 457	22 892 172	·101 3457	2·289 2172	$\frac{1}{\cdot 101\ 3457}$	2·289 2172

PART 4.		PART 5.	
Numbers.	Logarithms.	Numbers.	Logarithms.
·100 1881	— 2·300 7056	1001·881	6·909 6348
·100 4775	— 2·297 8212	1004·775	6·912 5192
·100 7669	— 2·294 9449	1007·669	6·915 3955
·101 0563	— 2·292 0769	1010·563	6·918 2635
·101 3457	— 2·289 2172	1013·457	6·921 1232

Let us now take as an example from *Part 5* of the above table, $\log 1\,010\cdot563 = 6\cdot918\,2635$, and compare it with the logarithm of the same number as derived from Hutton's table of Hyperbolic logarithms.

From Hutton's table :—

	Diff. of log.	Diff. of number.	Correction.
$\log_e 1\,011 = 6\cdot918\,6952$	9896	$\times \quad \cdot563$	$= \quad 5571$
$\log_e 1\,010 = 6\cdot917\,7056$			
correction =	5571		

$\log_e 1\,010\cdot563 = 6\cdot918\,2627$, which is less than the logarithm given in *Part 5* by 8 in the seventh decimal place.

Taking the corresponding example from *Part 3*, we find by dividing out that $\frac{1}{\cdot101\,0563} = 9\cdot895\,474$, and its logarithm is given there as 2·292 0769.

In Hutton's table we do not find $\log_e 9\cdot895\,474$, but $\log_e 989$ is given, so that we may get a comparison by multiplying the number by 100, and correspondingly increasing its logarithm thus—

$\log \quad 9\cdot895\,474 = 2\cdot292\,0769$
$\log 100 \cdot \quad \quad = 4\cdot605\,1702$
therefore $\log 989\cdot547\,400 = 6\cdot897\,2471$

From Hutton's table we have—

	Diff. of log.	Diff. of number.	Correction.
$\log_e 990 = 6\cdot897\,7049$	10106	$\times \quad \cdot5474$	$= \quad 5532$
$\log_e 989 = 6\cdot896\,6943$			
correction =	5532		

$\log_e 989\cdot547 = 6\cdot897\,2475$, which is greater than the logarithm found above by 4 in the seventh decimal place.

SUMMARY OF THE NATURE OF THE DIFFERENT SYSTEMS OF
LOGARITHMS AS EXEMPLIFIED BY THE RELATION SUBSISTING
BETWEEN THE TERMS OF A GEOMETRICAL SERIES AND THOSE
OF AN ARITHMETICAL SERIES.

So far as we have gone, this paper has been based mainly on that view of the subject which explains the relation subsisting between numbers and their logarithms, by that subsisting between the terms of a geometrical series and those of an arithmetical series, the former representing numbers, and the latter logarithms.

In this view the result of our examination of different systems of logarithms may now be briefly summarised.

Assuming r as the common ratio of the geometrical series, and 1 as the common difference of the arithmetical series, then a pure system of logarithms such as the Common system or the Hyperbolic system may be represented as follows:—

Geometrical Series or Numbers, $r^0, \quad r^1, \quad r^2, \quad r^3, \quad r^4, \quad \text{etc.}$

Arithmetical Series or Logarithms, 0, 1, 2, 3, 4, etc.

The intermediate terms of the series may be found by interpolating geometrical means between the terms of the former, and corresponding arithmetical means between the terms of the latter.

This pure system may be complicated by multiplying the terms of either or both of these series by some constant quantities, as a and b , so that the series become—

Geometrical Series or Numbers,

$a \times r^0, \quad a \times r^1, \quad a \times r^2, \quad a \times r^3, \quad a \times r^4, \quad \text{etc.}$

Arithmetical Series or Logarithms,

$b \times 0, \quad b \times 1, \quad b \times 2, \quad b \times 3, \quad b \times 4, \quad \text{etc.}$

Let us take as examples the systems which have actually been in use, and taking the series last given as the most general.

I. *The Common System.*—In it $r=10$, and $a=b=1$, so that the series become—

Geometrical Series or Numbers, $10^0, \quad 10^1, \quad 10^2, \quad 10^3, \quad 10^4, \quad \text{etc.}$

Arithmetical Series or Logarithms, 0, 1, 2, 3, 4, etc.

II. *The Hyperbolic System.*—In it $r=e$, and $a=b=1$, so that the series become—

Geometrical Series or Numbers, $e^0, \quad e^1, \quad e^2, \quad e^3, \quad e^4, \quad \text{etc.}$

Arithmetical Series or Logarithms, 0, 1, 2, 3, 4, etc.

III. *The System originally published by Neper.*—In it $r = \frac{1}{e}$ approximately, and $a = b = 10\,000\,000$, so that the series become—
Geometrical Series or Numbers,

$$10\,000\,000 \times \frac{1}{e^0}, \quad 10\,000\,000 \times \frac{1}{e^1}, \quad 10\,000\,000 \times \frac{1}{e^2}, \quad \text{etc.}$$

Arithmetical Series or Logarithms,

$$10\,000\,000 \times 0, \quad 10\,000\,000 \times 1, \quad 10\,000\,000 \times 2, \quad \text{etc.}$$

THE MODULUS.

In two different systems, the logarithms of the same numbers bear a uniform ratio to each other. And it is to this uniform ratio, or constant multiplier, by which the logarithms in one system may be converted into the logarithms of the same numbers in the other system, that the term *Modulus* is applied.

To prove that this constant relation exists, and to find what it is, let us assume N to denote any given number; a to denote the base of the one system, b that of the other; x the logarithm of N in the first system, and y its logarithm in the second. Then will $N = a^x = b^y$, and $\log_a N = x \times \log_a a = y \times \log_a b$; but the logarithm of the base in its own system is 1, therefore $\log_a a = 1$, and consequently $x = y \times \log_a b$ and $y = x \times \frac{1}{\log_a b}$, substituting the equivalent

symbols for x and y , $\log_b N = \log_a N \times \frac{1}{\log_a b}$. Now $\frac{1}{\log_a b}$ is independent of N , and is therefore a constant quantity as regards the two systems, depending solely upon the values of a and b . Thus it has been shown that the logarithm in system b , of any number N , may be found from the logarithm in system a of the same number, by multiplying the latter by the constant quantity $\frac{1}{\log_a b}$.

The numerical value of $\frac{1}{\log_a b}$ is therefore the *Modulus* of system b in relation to system a , or $M_{b:a} = \frac{1}{\log_a b}$.

We may thus express the manner in which the logarithms in one system may be deduced from those of another system when the *Modulus* of the second in reference to the first is known by the formula $\log_b N = M_{b:a} \times \log_a N$.

Conversely, if the logarithms of the same number in two systems

are known, the Modulus of the second in reference to the first may be found from the formula $\frac{\log_b N}{\log_a N} = M_{b:a}$.

To test these formulæ—(1.) Let it be required to find the logarithm of 1133 in the system which has 10 for its base from the logarithm of 1133 in the system which has e for its base.

Substituting e for a and 10 for b in the formula $M_{b:a} = \frac{1}{\log_a b}$, we have $M_{10:e} = \frac{1}{\log_e 10}$.

From Hutton's tables we find $\log_e 10 = 2.3025851$, therefore $M_{10:e} = \frac{1}{2.3025851} = .43429448$ etc., as shown below.

$$\begin{array}{r}
 2.3025851 \ 1.00000000 \ (.43429448 \\
 \underline{92103404} \\
 7896596 \\
 \underline{6907755} \\
 988841 \\
 \underline{921034} \\
 67807 \\
 \underline{46052} \\
 21755 \\
 \underline{20723} \\
 1032 \\
 \underline{921} \\
 111 \\
 \underline{92} \\
 19 \\
 \underline{18} \\
 1
 \end{array}$$

Again substituting in formula $\log_b N = M_{b:a} \times \log_a N$, we have $\log_{10} N = M_{10:e} \times \log_e N$.

From Hutton's tables we find $\log_e 1133 = 7.0326243$, so that $\log_{10} 1133$ should be equal to $M \times \log_e 1133 = .43429448 \times 7.0326243$ and this, as shown below, $= 3.0542299$.

$$\begin{array}{r}
 .43429448 \\
 7.0326243 \\
 \underline{3.0400614} \\
 130288 \\
 8686 \\
 2606 \\
 87 \\
 17 \\
 \underline{1} \\
 3.0542299
 \end{array}$$

Referring to Hutton's tables again, we find $\log_{10} 1133 = 3.054\ 2299$ which agrees exactly with the value found above.

(2.) Let it be required to find $M_{10:e}$ when $\log_{10} 779 = 2.891\ 5375$ and $\log_e 779 = 6.658\ 0110$.

Substituting these values in the second formula given, we have
 $M_{10:e} = \frac{\log_{10} 779}{\log_e 779} = \frac{2.891\ 5375}{6.658\ 0110}$ and this, as shown below, $= .434\ 29449$,
 which agrees very nearly with the value of the Modulus as already found.

$$\begin{array}{r}
 6.6580110) \ 2.89153750 \ (.43429449 \\
 \underline{2.66320440} \\
 22833310 \\
 \underline{19974033} \\
 2859277 \\
 \underline{2663204} \\
 196073 \\
 \underline{133160} \\
 62913 \\
 \underline{59921} \\
 2992 \\
 \underline{2663} \\
 329 \\
 \underline{266} \\
 63 \\
 \underline{60}
 \end{array}$$

As the values of $\log_e 10$ and of $M_{10:e}$ are of importance, I give them to 30 decimal places—

$$\begin{array}{l}
 \log_e 10 = 2.30258\ 50929\ 94045\ 68401\ 79914\ 54684 \\
 M_{10:e} = .43429\ 44819\ 03251\ 82765\ 11289\ 18916
 \end{array}$$

ESSENTIAL NATURE OF THE MODULUS.

Referring to Nepair's method of explaining the nature of logarithms, the Modulus is the ratio that the velocities which the moving points, travelling along the lines representing numbers and logarithms, bear to each other at the moment of starting.

We saw in the Hyperbolic system that these velocities are the same at the moment of starting; their ratio is therefore one of equality, which is denoted by unity, and consequently the Modulus

of this system is 1, and this is the reason for calling it the Natural system.

In Nepair's system, as originally defined, the ratio of the initial velocities of the moving points was also one of equality; but whereas in the Hyperbolic system the point moving along the line representing numbers has an increasing or positive motion, in Nepair's system it has a decreasing or negative motion, and the Modulus of his system is consequently -1 . The numbers in Nepair's system decrease as the logarithms increase, and *vice versa*; thus the two velocities are seen to have a contrary character, so that when the one is increasing the other is decreasing—that is, when the one is positive the other is negative, the consequence being that the logarithms of all numbers greater than unity are negative. Now the numbers being the principals are naturally considered as positive, and therefore the logarithms, being related and subordinate to them, must in this system have the negative sign.

In this view, let us now consider the Modulus of the Common system in relation to the Hyperbolic system. When the points travelling along the lines representing logarithms in the two systems have each got to 1, the point travelling along the line representing numbers has in the Common system got to 10, and in the Hyperbolic system to 2.71828; now conversely, when each of the points travelling along the lines representing numbers have got to 10, the points travelling along the lines representing logarithms will in the Common system have got to $\log_{10} 10$, or 1, and in the Hyperbolic system to $\log_e 10$, so that the ratio of their velocities is $1 : \log_e 10$ or $\frac{1}{\log_e 10} : 1$, and the velocities of the points travelling along the lines representing logarithms being uniform, this will be the ratio of their initial velocities, and $\frac{1}{\log_e 10}$ will therefore be the Modulus of the Common system in relation to the Hyperbolic. This law holds universally: thus, let b represent the base of any system, then $M_{b:e} = \frac{1}{\log_e b}$.

In dealing with a particular system of logarithms when the Modulus is spoken of, it is understood to be the Modulus of that system in relation to the Hyperbolic or Natural system which is referred to, and the symbol M , without any suffixes, is used as sufficient to denote it.

PART II.

After the publication of Nepair's book, mathematicians began to consider whether some clearer explanation of the relation between logarithms and their numbers could not be given than that by the motion of points. The result was that, about 1624, Kepler published his explanation showing that the relation between numbers and their logarithms was the same as that between ratios and their measures.

Properly to understand this explanation, however, it will be desirable to define some of the terms to be employed.

DEFINITION OF TERMS.

1. *Ratio*.—This term expresses the relation which one quantity bears to another in respect of magnitude. Thus the number of days in a year are to the number of days in a week in the ratio of 365 to 7, and this relation is expressed symbolically thus, 365 : 7.

This expression consists of two parts connected by the sign : . The first part, 365, is called the antecedent, and the latter part, 7, is called the consequent. Using the initial letters for these words, the general expression for a ratio may thus be written $a : c$. Taking the above example, we see that the ratio of the number of days in 6 years to the number of days in 6 weeks is the same as the ratio of the number of days in 1 year to the number of days in 1 week. From this observation we deduce the law that a ratio is not altered when both antecedent and consequent are multiplied by the same quantity, and similarly of division, so that

$$\text{ratio } a : c = \text{ratio } na : nc = \text{ratio } \frac{a}{n} : \frac{c}{n}.$$

From this it follows that any ratio $a : c$ may be transformed into an equivalent ratio, having for its consequent unity or any other quantity p , thus $\text{ratio } a : c = \text{ratio } \frac{a}{c} : 1 = \text{ratio } \frac{ap}{c} : p$.

So we may define ratios as equal when the quotients of the several antecedents by their respective consequents are equal.

2. *Proportion*.—This term expresses the equality of ratios, and

is written $a:b::c:d$, thus stating the proposition that ratio $a:b$ = ratio $c:d$ or $\frac{a}{b} = \frac{c}{d}$. A proportion may consist of any number of such statements as $a:b::c:d::d:e::$ etc.

3. *Continued Proportion.*—When the terms in a proportion, intermediate to the first and last, have the double relation of consequents and antecedents, as $a:b::b:c::c:d::$ etc., this is called a continued proportion. Any geometrical series, as 1, 2, 4, 8, 16, etc., will serve for an example, as it may be put in the form $1:2::2:4::4:8::8:16::$ etc.

4. *Compound Ratios.*—In the above example, the ratio 1:16 is said to be compounded of all the connecting ratios among the interposed quantities, namely, 1:2, 2:4, 4:8, 8:16; and the law is that ratios may be compounded by multiplying together all the antecedents to form the new antecedent and all the consequents to form the new consequent. Thus, multiplying $1 \times 2 \times 4 \times 8 = 64$ for the new antecedent, and $2 \times 4 \times 8 \times 16 = 1024$ for the new consequent, we find the compounded ratio is 64:1024, and eliminating common factors we get ratio 1:16 as above.

5. *Measure of Ratios.*—To measure the relative magnitude of different ratios, we must fix upon a ratio to be the standard of measurement, with which all ratios are to be compared, and whose measure shall be denoted by unity. Let the ratio $r:p$ be assumed as the standard of measurement, or shortly, the ‘Standard Ratio,’ then the measure of a given ratio is the number of times the Standard Ratio has to be compounded with itself in order to become the given ratio. This measure may be either integral or fractional.

As an example, if the Standard Ratio be 4:1, the measure of the ratio 32:1 is 2½, because 4:1 must be compounded 2½ times to become 32:1; that is, the ratio 32:1 equals the ratios 4:1, 4:1, and $\sqrt{4}:\sqrt{1}$ compounded, equals $4 \times 4 \times 2:1 \times 1 \times 1$ equals 32:1.

RATIOS AND THEIR MEASURES.

Let us now compare certain simple ratios with our Standard Ratio, and see what their measures are.

In doing this all ratios must be referred to the same consequent as the Standard Ratio.

Thus when compared with the Standard Ratio $r:p$, the following ratios have for their measures the values given in the table:—

Ratio.	Ratio referred to consequent p .	Measure of Ratio.	Notes.
$p:p$	$p:p$	0	This ratio expressing equality between antecedent and consequent, its measure is 0.
$r:p$	$r:p$	1	This being the Standard Ratio when referred to itself, its measure is 1.
$r^2:p^2$	$\frac{r^2}{p}:p$	2	This ratio is said to be duplicate of ratio $r:p$, being compounded of two ratios $r:p$ for $\frac{r^2}{p}:r::r:p$.
$r^3:p^3$	$\frac{r^3}{p^2}:p$	3	This ratio is triplicate of ratio $r:p$ for $\frac{r^3}{p^2}:\frac{r^2}{p}::\frac{r^2}{p}:r::r:p$.
$r^4:p^4$	$\frac{r^4}{p^3}:p$	4	This ratio is quadruplicate of ratio $r:p$ for $\frac{r^4}{p^3}:\frac{r^3}{p^2}::\frac{r^3}{p^2}:\frac{r^2}{p}::\frac{r^2}{p}:r::r:p$.
$r^5:p^5$	$\frac{r^5}{p^4}:p$	5	
$r^6:p^6$	$\frac{r^6}{p^5}:p$	6	
$r^n:p^n$	$\frac{r^n}{p^{n-1}}:p$	n	
$r^{\frac{1}{2}}:p^{\frac{1}{2}}$	$r^{\frac{1}{2}}p^{\frac{1}{2}}:p$	$\frac{1}{2}$	This ratio is said to be subduplicate of ratio $r:p$ as two of the ratios $r^{\frac{1}{2}}p^{\frac{1}{2}}:p$ have to be compounded to form ratio $r:p$ for $r:r^{\frac{1}{2}}p^{\frac{1}{2}}::r^{\frac{1}{2}}p^{\frac{1}{2}}:p$.
$r^{\frac{1}{3}}:p^{\frac{1}{3}}$	$r^{\frac{1}{3}}p^{\frac{2}{3}}:p$	$\frac{1}{3}$	This ratio is subtriplicate of ratio $r:p$ for $r:r^{\frac{2}{3}}p^{\frac{1}{3}}::r^{\frac{2}{3}}p^{\frac{1}{3}}:r^{\frac{1}{3}}p^{\frac{2}{3}}::r^{\frac{1}{3}}p^{\frac{2}{3}}:p$.
$\frac{s}{r^t}:\frac{s}{p^t}$	$\frac{s}{r^t}p^{\frac{t-s}{t}}:p$	$\frac{s}{t}$	

RELATION BETWEEN RATIOS AND THEIR MEASURES.

When one ratio is compounded with another, the measure of the compound ratio is equal to the sum of the measures of the compound ratios. Thus if ratio $\frac{r^2}{p}:p$ becompounded with $\frac{r^3}{p^2}:p$ the

resulting ratio is $\frac{r^5}{p^3} : p^2$ or $\frac{r^5}{p^4} : p$; and as the measures of the component ratios are 2 and 3 respectively, therefore the measure of the compound ratio, according to the law, should be $2 + 3 = 5$, and referring to the above table we see that this is so.

When one ratio is divided by another, the measure of the resulting ratio is equal to the measure of the first, less the measure of the second. For example, if ratio $\frac{r^6}{p^5} : p$ be divided by ratio $\frac{r^3}{p^2} : p$ the resulting ratio is $\frac{r^3}{p^3} : \frac{p}{p}$ or $\frac{r^3}{p^2} : p$, and its measure should be $6 - 3 = 3$, which we find is the case by referring to the table.

When a ratio is compounded with itself, the measure of the resulting ratio is the measure of the original ratio multiplied by the number of times it is compounded. Thus if the ratio $\frac{r^2}{p} : p$ be compounded three times with itself, the result is $\frac{r^6}{p^3} : p^3$ or $\frac{r^6}{p^5} : p$, and its measure is $2 \times 3 = 6$.

When the root of a ratio is extracted, the measure of the resulting ratio is the measure of the original ratio divided by the index of the root extracted. Thus the square root of the ratio $\frac{r^4}{p^3} : p$ is $\sqrt{\frac{r^4}{p^3}} : \sqrt{p} = \frac{r^2}{p^{1\frac{1}{2}}} : p^{\frac{1}{2}} = \frac{r^2}{p} : p$, and its measure is $4 \div 2 = 2$.

The above laws hold good for any number of operations.

When the consequents of the ratios we have to deal with are all unity, they may be neglected and the antecedents then become ordinary numbers, in which case the expressions "to divide" and "to extract the root" are strictly applicable; and in the above laws the expressions for the operations on the ratios become identical with the expressions to multiply, to divide, to raise to a power, and to extract a root as applied to numbers.

APPLICATION TO THE THEORY OF LOGARITHMS.

We have seen that when the consequents of all the ratios we have to deal with are made unity the antecedents become iden-

tical with ordinary numbers, and the measures of the ratios are identical with the logarithms of the numbers. In fact, the term “measures of ratio” is simply a translation of the word logarithm. The “theory of logarithms” is therefore really the theory of ratios and their logarithms, and it is only when we restrict the ratios to a particular class, namely—those having unity for consequent—that it becomes the theory of numbers and their logarithms. Practically it comes to much the same thing, as all ratios may be converted into equivalent ones having unity for consequent, and the treatment of the subject in this restricted sense is more direct and simple.

But in this part we must adhere to the more general view of the theory as that of ratios and their logarithms.

DEFINITION OF A LOGARITHM.

The Standard Ratio being denoted by $r : p$, the measure of which is 1, then the term logarithm is applied to the measure of the ratio of a given number N to p (the consequent of the Standard Ratio), as compared with $r : p$, the Standard Ratio; thus to find the logarithm is to find how many times, integrally or fractionally, the ratio $r : p$ has to be compounded with itself in order to give the ratio $N : p$.

SYSTEM OF LOGARITHMS FOUNDED ON RATIO 2 : 3.

A system of logarithms may be formed by assuming any particular ratio as its standard of measurement.

Let us therefore give a particular numerical value to the Standard Ratio, say 2 : 3, and sketch out a table of logarithms.

Proceeding in a similar way to that which we adopted in forming the table of ratios and their measures, we have, log of ratio 3 : 3 = 0, log of ratio 2 : 3 = 1, log of ratio 4 : 9 or $\frac{4}{3} : 3 = 2$, log of ratio 8 : 27 or $\frac{8}{9} : 3 = 3$, etc.; then, converting the antecedents

into decimals, we obtain the results given in the following table:—

Antecedents referred to Common Consequent 3. Corresponding to Numbers	Measure of Ratio or Logarithm.
3·000 000	0·000 000
2·000 000	1·000 000
1·333 333	2·000 000
·888 889	3·000 000
·592 593	4·000 000
·395 062	5·000 000
·263 374	6·000 000
·175 583	7·000 000
·117 055	8·000 000
etc.	etc.

With the table we have made, let us now perform one of the ordinary calculations which are facilitated by the use of a table of logarithms.

Let it be required to multiply ·395 062 by ·888 889.

In the ordinary way of using a table of logarithms we would proceed thus: looking up the tables we find $\log \cdot 395\ 062 = 5\cdot000\ 000$ and $\log \cdot 888\ 889 = 3\cdot000\ 000$; adding these logarithms we get $8\cdot000\ 000$ as the logarithm of the product of the numbers, and again looking up the table we find $8\cdot000\ 000 = \log \cdot 117\ 055$.

Let us see if this is correct. Multiplying out—

$$\begin{array}{r}
 \cdot 395\ 062 \\
 \cdot 888\ 889 \\
 \hline
 \cdot 316\ 050 \\
 31\ 605 \\
 3\ 160 \\
 316 \\
 31 \\
 3 \\
 \hline
 \cdot 351\ 165
 \end{array}$$

This is not the result we got from the table, but is 3 times greater than it, the reason being that, in using the table, we have omitted to take into account that we were not multiplying together the numbers $\cdot 395\ 062$ and $\cdot 888\ 889$ but were compounding the ratios $\cdot 395\ 062 : 3$ and $\cdot 888\ 889 : 3$, the result being $\cdot 351\ 165 : 9$. But the ratios in the table are all referred to the consequent 3, so that to obtain the correct result, the value found from the table must, for each multiplication, be multiplied by the consequent of the standard ratio.

METHOD 'OF USING SUCH A TABLE.

From the foregoing example you see that in using a table such as this for the multiplication of *numbers* it must be kept in view that the *numbers* in the table are the antecedents of ratios, and that in compounding ratios we compound the consequents as well as the antecedents, so that the product of two *numbers* will have $3^2=9$ for its consequent. But the *numbers* in the table being all referred to the consequent 3, therefore the *number* corresponding to the sum of the logarithms will only be $\frac{3}{9}$, or one-third of the product of the *numbers*. For example, if a and b denote the *numbers* to be multiplied, in using the table, we are really compounding the ratios $a : 3$ and $b : 3$, the result being $ab : 9$; this referred to consequent 3 becomes $\frac{ab}{3} : 3$, and the value in the table opposite the sum of the logarithms is consequently not ab but $\frac{ab}{3}$; therefore for each operation in multiplication the value found from the table must be multiplied once by the common consequent 3. Thus, if three *numbers* have to be multiplied together there are two operations, namely, multiplying the first by the second and their product by the third, so that the value found in the table opposite the sum of their logarithms must be multiplied by 9.

Instead of making the corrections on the final result obtained, another way of using the table is to commence by converting the numbers to be multiplied into ratios referred to the common consequent. Thus, to multiply a by b we might first convert a or $a : 1$ into its equivalent $3a : 3$, and b or $b : 1$ into $3b : 3$, and looking up

the table for the logarithms of $3a$ and $3b$, their sum would be in the table the logarithm of $3ab : 3$, for ratio $3a : 3$ compounded with ratio $3b : 3 = \text{ratio } 9ab : 9 = 3ab : 3$, and the value found in the table being now divided by 3 would give the true product of the numbers. This method has the advantage that however many numbers are multiplied or divided, the result finally obtained from the table has invariably to be divided simply by the common consequent 3, and thus it is unnecessary to keep note of the different operations performed. For example, to find the result of

$$a \times b \times c \times d \times e \div f$$

we have, converting each into ratio referred to 3 and finding resulting ratio—

$$\text{Antecedent} = 3a \times 3b \times 3c \times 3d \times 3e \div 3f = \frac{3^5}{3} \times \frac{abcde}{f}$$

$$\text{Consequent} = 3 \times 3 \times 3 \times 3 \times 3 \div 3 = \frac{3^5}{3}$$

$$\text{Final ratio} = \frac{3^5}{3} \times \frac{abcde}{f} : \frac{3^5}{3} = 3 \times \frac{abcde}{f} : 3.$$

As a numerical example, let it be required to multiply $\cdot 296\ 296$ by $\cdot 131\ 687$. This is the same as compounding the ratios $\cdot 296\ 296 : 1$ and $\cdot 131\ 687 : 1$; now, converting these into equivalent ratios, with 3 for consequent, namely, $\cdot 888\ 889 : 3$ and $\cdot 395\ 062 : 3$, we find from the table—

$$\log \cdot 888\ 889 = 3\cdot 000\ 000$$

$$\log \cdot 395\ 062 = 5\cdot 000\ 000$$

$$\underline{8\cdot 000\ 000} = \log (\cdot 117\ 055 : 3) = \log (\cdot 039\ 018 : 1)$$

Multiplying the numbers together—

$$\begin{array}{r} \cdot 296\ 296 \\ \cdot 131\ 687 \\ \hline \cdot 029\ 630 \\ 8\ 889 \\ 296 \\ 178 \\ 23 \\ 2 \\ \hline \cdot 039\ 018 \end{array}$$

the result obtained above.

It is evident that a table of logarithms founded on a ratio such as the above would be extremely troublesome to work with, and having been accustomed to our present tables, we might think no one would calculate such a table ; but, as before explained, the table published by Nepair was precisely similar, the ratio he used being $9\,999\,999 : 10\,000\,000$ (or the ten-millionth power of it, namely, $3\,678\,794 : 10\,000\,000$, *see* Part I.). The numbers in his table are thus all referred to the common consequent $10\,000\,000$, so that when $\log 10\,000\,000$ is entered as 0, the logarithm refers not to the number $10\,000\,000$, but to the ratio $(10\,000\,000 : 10\,000\,000) = \text{ratio } (1 : 1)$; or again, when $\log 1\,000\,000$ is entered as $23\,025\,851$, the logarithm meant is that of ratio $(1\,000\,000 : 10\,000\,000) = \text{ratio } \left(\frac{1}{10} : 1\right)$.

LIMITATIONS DESIRABLE IN SELECTING A "STANDARD RATIO."

Having found the inconvenience of requiring always to refer to the consequent, we will now inquire how this may be obviated. The answer is *by making Unity the consequent of the Standard Ratio*. Then any number of ratios may be compounded without any alteration taking place in the consequent, for unity does not alter by being multiplied by itself. For example, compounding the ratios $a : p$, $b : p$, $c : p$, we have $abc : p^3 = \frac{abc}{p^3} : p$; but when $p = 1$, then $abc : p^3 = abc : p = abc : 1$.

And as unity does not alter by being multiplied by itself, neither does it alter by being divided by itself, nor by being raised to any power, nor by having any root extracted.

The advantages of adopting this limitation are then :—*First*, That the antecedent of the Standard Ratio becomes the base of the system : and *Second*, That the logarithms may be treated as logarithms of numbers, instead of logarithms of ratios (for strictly the logarithm is the number of ratios of the base to unity which the ratio of a given number to unity contains), and therefore in using a table calculated in this way, we may with safety treat the numbers of which the logarithms are given as simple *numbers*.

Again, referring to our table, we see that the numbers decrease as the logarithms increase. The reason is that the antecedent of

the Standard Ratio is less than the consequent. To obviate this *the antecedent of the Standard Ratio must be greater than the consequent*, in which case the logarithms will vary directly as the numbers. In Napier's table the Standard Ratio being 9 999 999 : 10 000 000, the numbers, as in our example, decrease as the logarithms increase.

A necessary limitation naturally comes before the two above referred to, namely, that *the consequent of the Standard Ratio should be positive*. In all ratios the antecedent must be of the same affection as the consequent, that is, either positive or negative. Therefore if the consequent be negative, the antecedents will all be negative, and there will thus be logarithms of negative numbers only; conversely the consequent being positive, the antecedents will all be positive, and there will be logarithms of positive numbers only.

The last limitation to which I shall refer is that *the antecedent of the Standard Ratio be the same as the radix of our system of numeration, that is, 10*. When this is so, the mantissa or decimal part of the logarithm of any series of significant digits remains the same by whatever integral power of 10 the number may be multiplied or divided. Besides other advantages, this enables the operator with a table of logarithms of a given size to have command of a vastly greater range of numbers with which to work.

We have thus been led to the ratio 10 : 1 as that most suitable to be adopted as the Standard Ratio for a table of logarithms for ordinary use. But for other purposes other ratios may be more suitable; thus, for analytic purposes, the ratio 2·71828 : 1 is the most advantageous, being that employed in the system known as the Hyperbolic or ordinary Napierian. Again, Napier's original ratio 9 999 999 : 10 000 000 suited his purpose, as he used it for calculating the logarithms of sines referred to a radius assumed to be divided into 10 000 000 parts.

To sum up the conclusions arrived at, we have found that a System of Logarithms may be formed by taking any ratio as its standard of measurement; it being, however, essential in a ratio that the antecedent and consequent be of the same affection. But while this is true, we have also found that in practice it is more desirable to adopt certain classes of ratios than others. The limitations we have thus found to be desirable in fixing on the ratio to be used as the standard of measurement are :—

1. *That its consequent be positive.* Otherwise we should only have the logarithms of negative numbers.

2. *That the antecedent be greater than the consequent.* Otherwise as the logarithms increase the numbers decrease.

3. *That the consequent be unity*—in which case the antecedent becomes what is known as the Base of the system. Otherwise in every operation reference must be made to the consequent.

4. *That the antecedent be the same as the radix of our scale of numeration, namely 10.* Otherwise an alteration of the decimal point in any series of significant digits entirely alters the corresponding logarithm; whereas the base being 10, the mantissa remains the same whatever alteration be made in the place of the decimal point of the number.

PART III.

In Part I. the relation subsisting between numbers and their logarithms was explained by the relation subsisting between the terms of a geometrical series and the corresponding terms of an arithmetical series.

In Part II. that relation was explained by the relation subsisting between ratios and their measures.

And now, restricting our inquiry to the logarithms of ratios having unity for their consequent, we may explain that relation by the relation subsisting between powers of the base and the exponents of such powers.

DEFINITION OF LOGARITHMS AS EXPONENTS OF POWERS OF THE BASE.

In this view the logarithm of a number is the exponent or index of the power to which the base of the system must be raised in order to equal the number of which the logarithm is required. And the relation subsisting between numbers and their logarithms is therefore defined by the equation $N=B^l$, where N denotes any given number, B the base of the system, and l the logarithm of N in that system.

LOGARITHMS OF THE BASE AND OF UNITY.

In all systems, restricted as above mentioned, and to which our inquiries are now limited, it follows from the equation of defini-

tion $N=B^l$ or $B^l=N$, that, as $B^1=B$, therefore 1 is the logarithm of the base in its own system, and as $B^0=1$, therefore 0 is the logarithm of 1 in all such systems.

USE OF A TABLE OF LOGARITHMS.

By means of a table of logarithms, the product of numbers may be found by the addition of their logarithms.

For example if $\log_B A=a$ then $A=B^a$, and if $\log_B C=c$, then $C=B^c$, but $A \times C=B^a \times B^c$ or $AC=B^{a+c}$, consequently $\log AC=a+c$, and the product AC is the number in the table corresponding to the logarithm $a+c$.

If one number be divided by another, the result may be found from the tables by subtracting the logarithm of the divisor from the logarithm of the dividend. Thus $\frac{A}{C}=\frac{B^a}{B^c}=B^a \times B^{-c}=B^{a-c}$, and the quotient is therefore the number in the table corresponding to the logarithm $a-c$.

Similarly powers or roots of numbers may be found by multiplying or dividing their logarithms. For if $A=B^a$, then $A^n=B^{na}$ and $A^{\frac{1}{n}}=B^{\frac{a}{n}}$, so that the n th power of a number is the number in the table corresponding to n times the logarithm of the original number, and the n th root of a number is the number in the table corresponding to $\frac{1}{n}$ -th part of the logarithm of the original number.

CONDITIONS NECESSARY TO DEFINE A PARTICULAR SYSTEM OF LOGARITHMS.

Two independent conditions are necessary to limit a system of logarithms. The conditions employed in defining the Hyperbolic system are that 0 be the logarithm of 1, and that the initial velocities of the moving points be the same, which latter condition is equivalent to determining the modulus.

The conditions employed in defining the common system are, that 0 be the logarithm of 1, and that 1 be the logarithm of 10, which latter condition is equivalent to determining the base of the system.

In the systems restricted as above mentioned, and with which we are alone dealing in this part, the first of those assumptions is

tacitly made in the equation of definition $N=B^l$, for if $B^l=N$, of necessity, by the law of indices, $B^0=1$; and this is the reason the above equation is not applicable to all possible systems.

LIMITATIONS IN SELECTING THE BASE OF A SYSTEM.

Since all positive numbers may be regarded as powers of any given positive number, unity excepted, an indefinite number of systems are possible, as any one of these numbers may be selected for the base of a system.

The following quantities, however, are excluded from our choice :—

1. *Unity.* Since all powers of unity are unity.
2. *Infinite quantities.* Since all powers of an infinite quantity are infinite.
3. *Negative quantities.* Since the powers of a negative quantity may be either positive, negative, or imaginary, and are consequently not subject to the law of continuity ; in other words, since no value can be found which will generally satisfy the equation $-N=B^l$.

It is also desirable to limit our choice to numbers greater than unity, otherwise as the logarithms increase the numbers decrease, and consequently the logarithms of all numbers greater than unity will be negative ; but this part of the subject was more particularly dealt with in Part II.

EXISTING SYSTEMS OF LOGARITHMS.

There are only two systems of logarithms in general use—the Common System, which has 10 for its base, and the Hyperbolic, which has e for its base.

INVESTIGATION OF FORMULÆ BEARING ON THE CALCULATION OF LOGARITHMS.

We may now investigate the formulæ on which our methods for calculating the logarithm of any specified number are based.

These are all derived from the equation of definition $N=B^l$.

To simplify the working out of these we may put $N=1+n$, and $B=1+b$.

(1.) To expand $(1+b)^l$ in a series ascending by powers of l , that is to expand a number in a series ascending by powers of its logarithm to a given base.

Expanding by the binomial theorem—

$$(1+b)^l = 1 + \frac{l}{1} \cdot b + \frac{l(l-1)}{1.2} \cdot b^2 + \frac{l(l-1)(l-2)}{1.2.3} \cdot b^3 + \frac{l(l-1)(l-2)(l-3)}{1.2.3.4} \cdot b^4 + \text{etc.}$$

Arranging this according to powers of l —

$$(1+b)^l = 1 + l(b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \frac{1}{4}b^4 + \text{etc.}) + l^2(\quad) + l^3(\quad) + l^4(\quad) + \text{etc.}$$

The co-efficients of l , l^2 , l^3 , l^4 , etc., contain powers of b only, and are therefore dependent for their value on that of b alone.

Let us denote those coefficients by c , c_2 , c_3 , c_4 , etc.; where $c = (b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \frac{1}{4}b^4 + \text{etc.})$, and the values of the other co-efficients are to be afterwards ascertained, then—

$$(1+b)^l = 1 + c \cdot l + c_2 \cdot l^2 + c_3 \cdot l^3 + c_4 \cdot l^4 + \text{etc.}$$

Substituting x for l —

$$(1+b)^x = 1 + c \cdot x + c_2 \cdot x^2 + c_3 \cdot x^3 + c_4 \cdot x^4 + \text{etc.}$$

Substituting $l+x$ for l —

$$(1+b)^{l+x} = 1 + c \cdot (l+x) + c_2(l+x)^2 + c_3(l+x)^3 + c_4(l+x)^4 + \text{etc.}$$

Multiplying the first two expressions together—

$$\begin{array}{rcl} (1+b)^l & = & 1 + c \cdot l + c_2 \cdot l^2 + c_3 \cdot l^3 + c_4 \cdot l^4 + \text{etc.} \\ (1+b)^x & = & 1 + c \cdot x + c_2 \cdot x^2 + c_3 \cdot x^3 + c_4 \cdot x^4 + \text{etc.} \\ \hline (1+b)^{l+x} & = & \left\{ \begin{array}{l} 1 + c \cdot l + c_2 \cdot l^2 + c_3 \cdot l^3 + c_4 \cdot l^4 + \text{etc.} \\ + c \cdot x + c_4 \cdot lx + cc_2 \cdot l^2x + cc_3 \cdot l^3x + \text{etc.} \\ + c_2 \cdot x^2 + cc_2 \cdot lx^2 + c_2^2 \cdot l^2x^2 + \text{etc.} \\ + c_3 \cdot x^3 + cc_3 \cdot lx^3 + \text{etc.} \\ + c_4 \cdot x^4 + \text{etc.} \\ + \text{etc.} \end{array} \right. \end{array}$$

Expanding perpendicularly each term of the third expression—

$$(1+b)^{l+x} = \left\{ \begin{array}{l} 1 + c \cdot l + c_2 \cdot l^2 + c_3 \cdot l^3 + c_4 \cdot l^4 + \text{etc.} \\ + c \cdot x + 2c_2 \cdot lx + 3c_3 \cdot l^2x + 4c_4 \cdot l^3x + \text{etc.} \\ + c_2 \cdot x^2 + 3c_3 \cdot lx^2 + 6c_4 \cdot l^2x^2 + \text{etc.} \\ + c_3 \cdot x^3 + 4c_4 \cdot lx^3 + \text{etc.} \\ + c_4 \cdot x^4 + \text{etc.} \\ + \text{etc.} \end{array} \right.$$

Since the two expressions for $(1+b)^{l+x}$ are identical, the coefficients of the corresponding powers of lx are equal, and therefore we may find the values of c_2, c_3, c_4 , etc., in terms of c , by equating like powers of lx , thus:—

$$\begin{aligned} 2c_2 \cdot lx &= c^2 \cdot lx, \therefore c_2 = \frac{1}{2} c^2 &= \frac{1}{2} c^2 \text{ or } \frac{1}{1.2} c^2 \\ 3c_3 \cdot l^2x &= cc_2 \cdot l^2x, \therefore c_3 = \frac{1}{3} cc_2 = \frac{1}{3} c \cdot \frac{1}{2} c^2 = \frac{1}{6} c^3 \text{ or } \frac{1}{1.2.3} c^3 \\ 4c_4 \cdot l^3x &= cc_3 \cdot l^3x, \therefore c_4 = \frac{1}{4} cc_3 = \frac{1}{4} c \cdot \frac{1}{6} c^3 = \frac{1}{24} c^4 \text{ or } \frac{1}{1.2.3.4} c^4 \text{ etc.} \end{aligned}$$

Substituting these values—

$$(1+b)^l = 1 + c \cdot l + \frac{1}{1.2} c^2 \cdot l^2 + \frac{1}{1.2.3} c^3 \cdot l^3 + \frac{1}{1.2.3.4} c^4 \cdot l^4 + \text{etc.}$$

when l becomes 1—

$$(1+b)^1 = 1 + \frac{1}{1} c + \frac{1}{1.2} c^2 + \frac{1}{1.2.3} c^3 + \frac{1}{1.2.3.4} c^4 + \text{etc.}'$$

Now $(1+b)^1 = B$, or the base of the system, and, referring to page 371, we there found the base of the Hyperbolic system, or

$$e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{etc.}$$

therefore when the base is e , then c must equal 1. But the expression for which c was substituted is

$$b - \frac{1}{2} b^2 + \frac{1}{3} b^3 - \frac{1}{4} b^4 + \text{etc.}$$

and in the Hyperbolic system $e = 1 + b$; $\therefore b = e - 1$.

Substituting this in the expression we have—

$$c = (e - 1) - \frac{1}{2} (e - 1)^2 + \frac{1}{3} (e - 1)^3 - \frac{1}{4} (e - 1)^4 + \text{etc.} = 1.$$

(2.) *To expand 1 in a series ascending by powers of n, that is, to expand a logarithm to a given base in a series ascending by powers of its number.*

Let the equation $N = B^l$ be converted as before explained into $(1+n) = (1+b)^l$.

Raising both sides to the x th power, in which x may have any value, $(1+n)^x = (1+b)^{lx}$.

expanding both sides—

$$1 + \frac{x}{1} n + \frac{x(x-1)}{1.2} n^2 + \frac{x(x-1)(x-2)}{1.2.3} n^3 + \frac{x(x-1)(x-2)(x-3)}{1.2.3.4} n^4 + \text{etc.}$$

$$= 1 + \frac{lx}{1} b + \frac{lx(lx-1)}{1.2} b^2 + \frac{lx(lx-1)(lx-2)}{1.2.3} b^3 + \frac{lx(lx-1)(lx-2)(lx-3)}{1.2.3.4} b^4 + \text{etc.}$$

subtracting 1 from each side, and dividing the remainders by x

$$\left\{ n + \frac{(x-1)}{1.2} n^2 + \frac{(x-1)(x-2)}{1.2.3} n^3 + \frac{(x-1)(x-2)(x-3)}{1.2.3.4} n^4 + \text{etc.} \right\}$$

$$= l \left\{ b + \frac{(lx-1)}{1.2} b^2 + \frac{(lx-1)(lx-2)}{1.2.3} b^3 + \frac{(lx-1)(lx-2)(lx-3)}{1.2.3.4} b^4 + \text{etc.} \right\}$$

making $x=0$

$$\left\{ n - \frac{1}{2} n^2 + \frac{1}{3} n^3 - \frac{1}{4} n^4 + \text{etc.} \right\} = l \left\{ b - \frac{1}{2} b^2 + \frac{1}{3} b^3 - \frac{1}{4} b^4 + \text{etc.} \right\}$$

therefore

$$l = \left\{ n - \frac{1}{2} n^2 + \frac{1}{3} n^3 - \frac{1}{4} n^4 + \text{etc.} \right\} \times \left\{ \frac{1}{b - \frac{1}{2} b^2 + \frac{1}{3} b^3 - \frac{1}{4} b^4 + \text{etc.}} \right\}$$

The first part of this expression for the value of l evidently depends solely on the value of the number, and the latter part depends solely on the value of the base. Substituting for l its equivalent $\log_B N$, for n its equivalent $N-1$, and for b its equivalent $B-1$, we have—

$$\log_B N = \left\{ (N-1) - \frac{1}{2} (N-1)^2 + \frac{1}{3} (N-1)^3 - \frac{1}{4} (N-1)^4 + \text{etc.} \right\}$$

$$\times \left\{ \frac{1}{(B-1) - \frac{1}{2} (B-1)^2 + \frac{1}{3} (B-1)^3 + \frac{1}{4} (B-1)^4 + \text{etc.}} \right\}$$

When e , the base of the Hyperbolic system, is substituted for B , the above expression becomes—

$$\log_e N = \left\{ (N-1) - \frac{1}{2} (N-1)^2 + \frac{1}{3} (N-1)^3 - \frac{1}{4} (N-1)^4 + \text{etc.} \right\}$$

$$\times \left\{ \frac{1}{(e-1) - \frac{1}{2} (e-1)^2 + \frac{1}{3} (e-1)^3 - \frac{1}{4} (e-1)^4 + \text{etc.}} \right\}$$

It was shown on page 395 that

$$(e-1) - \frac{1}{2} (e-1)^2 + \frac{1}{3} (e-1)^3 - \frac{1}{4} (e-1)^4 + \text{etc.} = 1,$$

so that—

$$\log_e N = (N-1) - \frac{1}{2} (N-1)^2 + \frac{1}{3} (N-1)^3 - \frac{1}{4} (N-1)^4 + \text{etc.}$$

Substituting B for N , as N may have any value that B may have,

$$\log_e B = (B-1) - \frac{1}{2} (B-1)^2 + \frac{1}{3} (B-1)^3 - \frac{1}{4} (B-1)^4 + \text{etc.};$$

therefore

$$\frac{1}{(B-1) - \frac{1}{2} (B-1)^2 + \frac{1}{3} (B-1)^3 - \frac{1}{4} (B-1)^4 + \text{etc.}} = \frac{1}{\log_e B}.$$

Substituting this symbol in the expression for $\log_B N$ we get—

$$\log_B N = \left\{ (N-1) - \frac{1}{2} (N-1)^2 + \frac{1}{3} (N-1)^3 - \frac{1}{4} (N-1)^4 + \text{etc.} \right\} \times \frac{1}{\log_e B}$$

On page 380 it was shown that $\frac{1}{\log_e B}$ was the modulus of system B in relation to the Hyperbolic or natural system, so writing $M_{B:e}$ instead of $\frac{1}{\log_e B}$

$$\log_B N = \left\{ (N-1) - \frac{1}{2} (N-1)^2 + \frac{1}{3} (N-1)^3 - \frac{1}{4} (N-1)^4 + \text{etc.} \right\} \times M_{B:e}$$

Finally, as a system is determined by its modulus, we may omit the suffixes, and simply write—

$$\log N = M \times \left\{ (N-1) - \frac{1}{2} (N-1)^2 + \frac{1}{3} (N-1)^3 - \frac{1}{4} (N-1)^4 + \text{etc.} \right\}$$

FORMULÆ FOR USE IN CALCULATING LOGARITHMS.

The expression —

$$\log N = M \times \left\{ (N-1) - \frac{1}{2} (N-1)^2 + \frac{1}{3} (N-1)^3 - \frac{1}{4} (N-1)^4 + \text{etc.} \right\} \quad (1)$$

being divergent when N exceeds 2, is in its present form of little use for the actual calculation of logarithms, and we have now to derive from it formulæ which will be rapidly converging, and which may thus enable us to calculate the logarithms of numbers with the minimum of labour.

Since $\log N = a \times \log N^{\frac{1}{a}}$, we have—

$$\log N = aM \times \left\{ (N^{\frac{1}{a}} - 1) - \frac{1}{2} (N^{\frac{1}{a}} - 1)^2 + \frac{1}{3} (N^{\frac{1}{a}} - 1)^3 - \frac{1}{4} (N^{\frac{1}{a}} - 1)^4 + \text{etc.} \right\} \quad (2)$$

in which if N be greater than 1, it is always possible to make $N^{\frac{1}{a}}$ exceed 1 by as small a quantity as we choose, and thus make the series convergent.

But the most useful series are derived from reverting to the expression in which N is replaced by its equivalent $(1+n)$ namely—

$$\log(1+n) = M \left\{ n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \text{etc.}, \right\} \quad . \quad . \quad . \quad (3)$$

Changing $(+n)$ into $(-n)$ this expression becomes—

$$\log(1-n) = M \left\{ -n - \frac{1}{2}n^2 - \frac{1}{3}n^3 - \frac{1}{4}n^4 - \text{etc.} \right\} \quad . \quad . \quad . \quad (4)$$

Dividing the number in formula (3) by that in (4), and subtracting their logarithms, which are equivalent processes, we have—

$$\log\left(\frac{1+n}{1-n}\right) = 2M \left\{ n + \frac{1}{3}n^3 + \frac{1}{5}n^5 + \frac{1}{7}n^7 + \text{etc.} \right\} \quad . \quad . \quad . \quad (5)$$

In the actual use of formulæ (3), (4), and (5), n is changed into $\frac{1}{n}$, thus becoming—

$$\log\left(1 + \frac{1}{n}\right) = M \left\{ \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \text{etc.} \right\} \quad . \quad . \quad . \quad (3^a)$$

$$\log\left(1 - \frac{1}{n}\right) = M \left\{ -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} - \frac{1}{4n^4} - \text{etc.} \right\} \quad . \quad . \quad . \quad (4^a)$$

$$\log\left(\frac{1 + \frac{1}{n}}{1 - \frac{1}{n}}\right) = 2M \left\{ \frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \frac{1}{7n^7} + \text{etc.} \right\} \quad . \quad . \quad . \quad (5^a)$$

Assume $\frac{1+n}{1-n} = \frac{p}{q}$, then $n = \frac{p-q}{p+q}$, and substituting in (5)—

$$\log\left(\frac{p}{q}\right) = 2M \left\{ \left(\frac{p-q}{p+q}\right) + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \frac{1}{7}\left(\frac{p-q}{p+q}\right)^7 + \text{etc.} \right\} \quad (6)$$

By this formula $\log p$ may be found if $\log q$ is known, since $\log p = \log q + \text{formula (6)}$.

Assume $p = q + z$, and substituting in (6)—

$$\log\left(\frac{q+z}{q}\right) = 2M \left\{ \left(\frac{z}{2q+z}\right) + \frac{1}{3}\left(\frac{z}{2q+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2q+z}\right)^5 + \frac{1}{7}\left(\frac{z}{2q+z}\right)^7 + \text{etc.} \right\} \quad (7)$$

Making $z=1$ in above—

$$\log\left(\frac{q+1}{q}\right) = 2M \left\{ \left(\frac{1}{2q+1}\right) + \frac{1}{3}\left(\frac{1}{2q+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2q+1}\right)^5 + \frac{1}{7}\left(\frac{1}{2q+1}\right)^7 + \text{etc.} \right\} \quad (7^a)$$

from which we may find the log of a number 1 greater than a number the log of which is known, since $\log(q+1) = \log q + \text{formula } (7^a)$.

In (6) substituting x^2 for p and x^2-1 for q , then $\frac{p-q}{p+q} = \frac{1}{2x^2-1}$, and—

$$\log\left(\frac{x^2}{x^2-1}\right) = 2N \left\{ \left(\frac{1}{2x^2-1}\right) + \frac{1}{3}\left(\frac{1}{2x^2-1}\right)^3 + \frac{1}{5}\left(\frac{1}{2x^2-1}\right)^5 + \frac{1}{7}\left(\frac{1}{2x^2-1}\right)^7 + \text{etc.} \right\} \quad (8)$$

from which, if we know the logs of any two of the consecutive numbers $x-1$, x , and $x+1$, the log of the other can be found, as

$$\log \frac{x^2}{x^2-1} = 2 \log x - \log(x+1) - \log(x-1).$$

The above are a few of the principal formulæ which may be used for calculating the logarithms of numbers, but with a little ingenuity others suitable for special purposes may easily be derived from them.

CALCULATION OF LOGARITHMS FOR THE FORMATION OF A TABLE.

We may now exhibit the application of some of the formulæ found above in the actual calculation of logarithms.

In doing so we will calculate the logarithms to base 10, which will show the process fully, as in the calculation of Hyperbolic logarithms the modulus, being 1, does not appear in the calculations.

The system which has 10 for its base has for its modulus .43429 44819 etc. (See page 379.)

In forming a table, it is only necessary to calculate the logarithms of prime numbers, as the logarithms of composite numbers may be formed by the addition of the logarithms of their factors.

(1.) Let us commence by finding $\log 2$, the first prime number.

Since $2 = \frac{4}{2} = \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}$ we have, using formula (5^a)—

$$\log \left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} \right) = 2M \left\{ \frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \text{etc.} \right\}$$

$$2M = \cdot 86858 \ 89638$$

$$\frac{2M}{3^1} = \cdot 28952 \ 96546 \div 1 = \cdot 28952 \ 96546$$

$$\frac{2M}{3^2} = 3216 \ 99616 \div 3 = 1072 \ 33205$$

$$\frac{2M}{3^5} = 357 \ 44402 \div 5 = 71 \ 48881$$

$$\frac{2M}{3^7} = 39 \ 71600 \div 7 = 5 \ 67372$$

$$\frac{2M}{3^9} = 4 \ 41289 \div 9 = 49032$$

$$\frac{2M}{3^{11}} = 49032 \div 11 = 4457$$

$$\frac{2M}{3^{13}} = 5448 \div 13 = 419$$

$$\frac{2M}{3^{15}} = 605 \div 15 = 40$$

$$\frac{2M}{3^{17}} = 67 \div 17 = 4$$

$$\underline{\underline{\cdot 30102 \ 99956}} = \log 2.$$

And since $\log 5 = \log \frac{10}{2} = \log 10 - \log 2$ we easily find $\log 5$ thus—

$$\begin{array}{r} \log 10 = 1 \cdot 00000 \ 00000 \\ \log 2 = \cdot 30102 \ 99956 \\ \hline \log 5 = \underline{\underline{\cdot 69897 \ 00044}} \end{array}$$

(2.) The logarithms of the prime numbers 3 and 11 may

be found by a single calculation. Using formulæ (3^a) and (4^a) we have—

$$\log \left(1 \pm \frac{1}{10} \right) = M \left\{ \pm \frac{1}{10} - \frac{1}{2} \cdot \frac{1}{10^2} \pm \frac{1}{3} \cdot \frac{1}{10^3} - \frac{1}{4} \cdot \frac{1}{10^4} \pm \text{etc.} \right\}$$

$$M = \cdot 43429\ 44819$$

$$\pm \frac{M}{10} = \pm \cdot 04342\ 94482$$

$$- \frac{M}{2 \cdot 10^2} = - \quad \quad \quad - \cdot 00217\ 14724$$

$$\pm \frac{M}{3 \cdot 10^3} = \pm \quad 14\ 47648$$

$$- \frac{M}{4 \cdot 10^4} = \quad \quad \quad - \quad 1\ 08574$$

$$\pm \frac{M}{5 \cdot 10^5} = \pm \quad \quad 8686$$

$$- \frac{M}{6 \cdot 10^6} = \quad \quad \quad - \quad \quad 724$$

$$\pm \frac{M}{7 \cdot 10^7} = \pm \quad \quad 62$$

$$- \frac{M}{8 \cdot 10^8} = \quad \quad \quad - \quad \quad 5$$

$$\begin{array}{r} \pm \cdot 04357\ 50878 \\ - \cdot 00218\ 24027 \\ \hline \end{array}$$

$$\log \frac{11}{10} = + \cdot 04139\ 26851 \text{ as } \log \left(1 + \frac{1}{10} \right) = \log \frac{11}{10}$$

$$\log \frac{9}{10} = - \cdot 04575\ 74905 \text{ as } \log \left(1 - \frac{1}{10} \right) = \log \frac{9}{10}$$

Multiplying the numbers by 10, and adding $\log 10 = 1$ to the logarithms—

$$\log 11 = 1 \cdot 04139\ 26851$$

$$\log 9 = \cdot 95424\ 25095$$

Extracting the square root of 9, and dividing its logarithm by 2—

$$\log 3 = \cdot 47712\ 12547$$

We have thus by two calculations found the logarithms of the prime numbers 2, 3, 5, 11, and the others may be similarly found.

By a judicious selection of the numerical values to be substituted in the formulæ, the labour may be greatly abridged. With this

object we must endeavour to find a multiple of the prime (the logarithm of which is required), which will be 1 greater or 1 less than some large number easy to divide by, such as a power of 10 or simple multiple of it, the logarithms of the other factors of which large numbers have already been found. The finding of such suitable numbers is very much facilitated by the use of a Table of the Factors of Numbers, the best of which is the "*Table des Diviseurs pour tous les nombres, etc.*," by J. Ch. Burckhardt, published at Paris in 1817, with subsequent continuations.

Thus, to calculate $\log 37$, we find that $37 \times 27 = 999$; and by substituting $\frac{1}{1000}$ for $\frac{1}{n}$ in formula (4^a) we get for $\log \left(1 - \frac{1}{1000} \right)$ a very rapidly converging series, from the value of which $\log 37$ may be found if $\log 3$ is known, since $\log \left(1 - \frac{1}{1000} \right) = \log \frac{999}{1000} = \log 37 + \log 27 - \log 1000 = \log 37 + 3 \log 3 - 3$.

FINIS.