On the Calculus of Equivalent Statements. (Seventh Paper.)

By HUGH MACCOLL, B.A. Received November 10th, 1897.

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DEF. 1.—Let ϕ denote any implication whose elementary constituents may change their values, being sometimes of the class ϵ , sometimes of the class η , and sometimes of the class θ . Let ϕ^a assert that ϕ is *true* for *all* values of its constituents; let ϕ^s denote ϕ^a and assert that ϕ is *false* for *all* values of its constituents; and let ϕ^v denote $\phi^a \cdot \phi^s$ and assert that ϕ is neither true for all values of its constituents, nor false for all values of its constituents. Thus ϕ^a would apply to the statement $AB : B + C$; ϕ^s would apply to the statement $A + A' : BB'$; and ϕ^v would apply to $A : BC$.

The symbols α , β , γ , thus interpreted, will be analogous to, but not quite identical with, ϵ , η , θ . For example, ϕ^a asserts that ϕ is *always true*, but it does not expressly state *why*; whereas the symbol ϕ^s asserts that ϕ is always true *because it would be an inconsistency of language, or a contradiction to our definitions, to say that it is false*. In my Fifth Paper I have drawn no distinction between ϕ^a and ϕ^s , using the former symbol in both senses. With regard to many problems and investigations this may safely and conveniently be done; but cases might arise in which it would be necessary to differentiate between the two significations. When such cases present themselves we shall always have the formulæ $A^a : A'$, $A^s : A'$, $A^v : A'$; but *not* always the converse statements $A' : A^a$, $A' : A^s$, $A' : A^v$. For example, $A^s : A'$ may be read: "If the statement A is *self-contradictory*, it asserts an *impossibility*"; whereas $A' : A^s$ may be read: "If A asserts an *impossibility*, it is *self-contradictory*, which is not a necessary or legitimate inference. The symbol A^v , but not the symbol A^s , would apply to such a statement as "He swam across the Atlantic in five minutes"; the symbols A^s and A^v would each apply to such a statement as "The whole is less than its part." Briefly, A^s asserts that A is a *formal impossibility*; while A^v simply asserts that A is an impossibility, asserting nothing as to the *kind* of impossibility.

In connexion with these symbolic interpretations I may here reply to some criticism on my Fifth Paper. It has been objected that it is incorrect to speak of a statement as "sometimes true and sometimes

false"; that a statement once made belongs either to the class of truths or to the class of falsehoods once for all, and can never again pass from that class to its opposite. As this criticism evidently proceeds from a misapprehension of my meaning, it seems necessary to explain that meaning a little more fully. My point of view is this, that, just as the implication $A : B$ may be true or false according to the nature of the statements denoted by A and B , and is, as a matter of fact, true in the case of some statements and false in the case of others, so certain statements, such as "Mrs. Brown is not at home," are sometimes used to convey, and do convey, a true information, while at other times they are used (*in the very same form of words*) to convey, and do convey, a false information. This class of statements I call *variables*. Let A denote a statement of this class, and (to fix our ideas) suppose it to have been made on five different occasions: *truly* on the *first, third, and fifth* occasions; *falsely* on the *second and fourth*. Let the symbols A_1, A_2, A_3, A_4, A_5 denote these five separate statements, each expressed *in the same form of words*. The general symbol A (not any of the *particular* symbols A_1, A_2, A_3, A_4, A_5) denotes this form of words; and, therefore, I hold that, though none of the statements A_1, A_2 , &c., can be spoken of separately as *sometimes true and sometimes false*, yet their general representative A , which may denote any of them, whether true or false, can be thus spoken of. When A represents A_1 , then A is true because A_1 is true; when A represents A_2 , then A is false because A_2 is false; and similarly for the other cases.

In this it seems to me that I am only imitating a common and useful convention among mathematicians, who speak of the points $(x_1, y_1), (x_2, y_2)$, &c., as *fixed*, their fixed positions being determined by the *constants* x_1, y_1 , or x_2, y_2 , &c.; while they speak of the point (x, y) as a *current* or *moving* point whose position *varies* with the *varying* values of its coordinates x and y .

DEF. 2.—The symbol $A^{..}$ (especially when A is a complicated expression) may be used as a convenient abbreviation for $A''A'$. When thus used it must not be confounded with the symbol A'' , which (as already defined) means $(A'')'$, nor with $A^{..}$, which means $A'' + A'$. Thus, if κ, λ, μ respectively denote (1) statements *known and admitted to be true, or provable from the data*; (2) statements *known and admitted to be false, or provable false from the data*; (3) statements *belonging neither to the class κ nor to the class λ , and therefore doubtful*, we get $A^{\lambda..} = A^{\lambda}A' = \eta$; but $A^{\lambda\kappa} = (A^{\lambda})'$, which merely asserts that A is always known to be false.

DEF. 3.—Let P, Q denote two *individuals* of the same class; and, to fix our ideas, suppose them to be *points* limited to some line, area, or volume. Then, if A, B, C , &c., denote *statements*, the symbols $P_A, P_{\phi(A, B)}$, &c., will assert that the statements $A, \phi(A, B)$, &c., are true *when applied to the point P* . The symbol P_A'' means $(P_A)''$, and asserts (in accordance with previous definitions) that the statement P_A belongs to the class of statements denoted by u . Thus P'_A or P'_A asserts that A is false when applied to the point P , no assertion being made as to whether P is fixed or variable; P'_A asserts that A is true for all permissible positions of the varying point P ; P''_A asserts that A is false for all permissible positions of the point P ; and P^o_A asserts that A is true for some permissible positions of the point P , but not for all.

From these definitions we get several self-evident formulæ, such as

$$P_{AB'} = P_A P'_{B'}, \quad P_{A+B'} = P_A + P'_{B'}, \quad P_{A:B} = P_A : P_B,$$

and, generally, $P_{\phi(A, B)} = \phi(P_A, P_B)$.

When we have to speak of the mutual relations of two or more points P, Q , &c., such statements as $P_A + Q_B$, $P_A : Q'_B$, &c., explain their own meaning, and require no fresh definition. When there is no risk of ambiguity, P_a may replace P_A .

To give some geometrical illustrations, let the symbols $\alpha'', \alpha^o, \alpha''''$ respectively assert (as before) that α is *positive*, that α is *negative*, that α is *imaginary*; and let α^o (or some other appropriate symbol in the few cases in which there might be danger of ambiguity) assert that α is *zero*. Then, assuming our *universe of points* to be restricted to the same plane, adopting the usual conventions as to rectangular axes, and putting ϕ or $\phi(x, y)$ for $x^2 + y^2 - r^2$, the symbol

$$P\phi'' Q\phi^o R\phi^o S\phi''''$$

necessarily asserts that P is *outside* the circle ϕ^o (i.e., the circle $x^2 + y^2 - r^2 = 0$), that Q is *inside*, that R is *on the circumference*, and that S does not belong to our *universe of points*; i.e., that, if the point S exists at all, it must be in some other plane. With the same symbolic conventions, we should have the easily verified formula

$$(PQ - 2r)'' : (P\phi^o : Q\phi'') : P\phi'' + Q\phi'',$$

which may be read: "If PQ is greater than $2r$, then, if P is inside the circle ϕ^o , Q must be outside this circle; a conditional assertion

which implies the alternative that either P is *not* inside or else that Q is outside." We should also have

$$(PQ - 2r)^0 : (P\phi^0 : Q\phi'' + Q\phi^0),$$

which may be read: "If PQ is equal to the diameter, then, if P is on the circumference, Q must be either *outside* the circle ϕ^0 or else on its circumference."

As another example, let P be a point taken at random in the accompanying square, the equation to the enclosed circle being

$$x^2 + y^2 - r^2 = 0,$$

or, briefly, ϕ^0 , in which ϕ denotes

$$x^2 + y^2 - r^2.$$

FIG. 1.

Then we have $P^0\phi''P^0\phi^0$, in which $P^0\phi''$ and $P^0\phi^0$ are abbreviations for $(P\phi'')^0$ and $(P\phi^0)^0$. The factor $(P\phi'')^0$ asserts that the statement ϕ'' in reference to the point P may be true or false; and the factor $(P\phi^0)^0$ asserts that the statement ϕ^0 in reference to P may also be true or false.

In the mathematical applications of symbolic logic, and especially in probability, the number of ways in which we can economize time, space, and brain labour (without sacrificing one iota of clearness and accuracy), by varying the meanings we give to our symbols of relation (indices, subscripts, &c.), is almost unlimited. In my last two papers the symbol A^ϵ asserts that the chance of the statement A being true (assuming the condition ϵ) is more than zero and less than unity. But there is no reason in the world why the symbol θ should not, when possible and convenient, be replaced by the *exact value* of this chance. Thus $A^{\frac{2}{3}}$ would be equivalent to the equation

$$\frac{A}{\epsilon} = \frac{2}{3},$$

and would assert that the chance of A being true (the condition ϵ being assumed throughout) was $\frac{2}{3}$. It being understood that A is a *statement* (and in certain investigations it might be convenient to reserve capitals for statements), this meaning could not possibly be confounded with the customary meaning of the same symbol when A is a number or ratio, for the simple reason that to talk of the

square of the cube root of a *statement*, or of the cube root of its square, would be to talk nonsense. To show the working of this convention, I will take the following definitions :—

DEF. 4.—The symbol A^x is short for $\frac{A}{e} = x$, and asserts that the chance of A being true (on the understood assumption e) is x .

DEF. 5.—The symbol $\frac{B}{A^x}$ (in conformity with the notation of my last paper) denotes the *chance* that the statement B is true on the assumption that the statement A^x is true. Thus we get

$$\left(\frac{B}{A^x} = y\right) = A^x : B^y = \left(\frac{B^y}{A^x} = 1\right);$$

each of the three equivalences asserting that when the chance of A being true is x the chance of B being true is y .

From these definitions we readily get several formulæ, such as

$$(1) \quad A^a B^b (AB)^c : (A+B)^{a+b-c},$$

$$(2) \quad A^a B^b (AB)^c : \left(\frac{A}{B} = \frac{c}{b}\right),$$

$$(3) \quad A^a B^b (AB)^c : \left(\frac{B}{A} = \frac{c}{a}\right),$$

$$(4) \quad A^a B^b : \left(\frac{A}{B} = \frac{a}{b} \frac{B}{A}\right) \left(\frac{B}{A} = \frac{b}{a} \frac{A}{B}\right),$$

$$(5) \quad A^a B^b : \left\{ \frac{A}{B} - \frac{A}{B'} = \frac{1}{1-b} \left(\frac{A}{B} - a\right) \right\}.$$

In the last formula, when A and B are independent, we have

$$\frac{A}{B} = \frac{A}{e} = a,$$

$$\text{so that we get} \quad \frac{A}{B} - \frac{A}{B'} = 0,$$

as we ought to have.

DEF. 6.—The symbol a' , when a denotes a *chance*, means $1-a$; so

that when a, b, c , &c., are respectively used to denote $\frac{A}{e}, \frac{B}{e}, \frac{C}{e}$, &c., then a', b', c' , &c., will denote $\frac{A'}{e}, \frac{B'}{e}, \frac{C'}{e}$, &c.

Using the notation of Def. 6, we prove Formula 5 as follows:—

$$\begin{aligned}
 \frac{A}{B} - \frac{A}{B'} &= \frac{A}{B} - \frac{a}{b'} \cdot \frac{B'}{A} \quad (\text{by Formula 4}) \\
 &= \frac{A}{B} - \frac{a}{b'} \left(1 - \frac{B}{A}\right) \\
 &= \frac{A}{B} - \frac{a}{b'} \left(1 - \frac{b}{a} \cdot \frac{A}{B}\right) \quad (\text{by Formula 4}) \\
 &= \frac{A}{B} - \frac{a}{b'} + \frac{b}{b'} \cdot \frac{A}{B} \\
 &= \left(1 + \frac{b}{b'}\right) \frac{A}{B} - \frac{a}{b'} \\
 &= \frac{1}{b'} \left(\frac{A}{B} - a\right),
 \end{aligned}$$

since

$$b + b' = 1.$$

To prove Formula 4, we have

$$\frac{B}{A} = \frac{AB}{e} \div \frac{A}{e} = \frac{B}{e} \cdot \frac{A}{B} \div \frac{A}{e} = b \cdot \frac{A}{B} \div a = \frac{b}{a} \cdot \frac{A}{B}.$$

In connexion with the preceding formulæ, 4 and 5, I may take a problem which Dr. Keynes, in his *Formal Logic* (Third Edition, p. 374), gives as an example of “numerical logic”; but it may fairly be called a problem in probability:—

“Show that if the proportion of B ’s out of the class A is greater than out of the class *not-A*, then the proportion of A ’s out of the class B will be greater than that out of the class *not-B*.”

This is equivalent to proving

$$\left(\frac{B}{A} - \frac{B}{A'}\right)^u : \left(\frac{A}{B} - \frac{A}{B'}\right)^u,$$

in which the index u asserts (as in previous conventions) that the

expression affected by it is real and positive. We have (by Formula 5)

$$\begin{aligned}
 \left(\frac{B}{A} - \frac{B}{A'}\right)^u : \left(\frac{A}{B} - \frac{A}{B'}\right)^u &= \left\{ \frac{1}{a'} \left(\frac{B}{A} - b\right) \right\}^u : \left\{ \frac{1}{b'} \left(\frac{A}{B} - a\right) \right\}^u \\
 &= \left(\frac{B}{A} - b\right)^u : \left(\frac{A}{B} - a\right)^u \\
 &= \left(\frac{b}{a} \frac{A}{B} - b\right)^u : \left(\frac{A}{B} - a\right)^u \text{ (by Formula 4)} \\
 &= \left(\frac{A}{B} - a\right)^u : \left(\frac{A}{B} - a\right)^u = \epsilon;
 \end{aligned}$$

for
$$\left(\frac{b}{a} \frac{A}{B} - b\right)^u = \left\{ \frac{a}{b} \left(\frac{b}{a} \frac{A}{B} - b\right) \right\}^u = \left(\frac{A}{B} - a\right)^u.$$

"The chance that, if A is true, B is true" is a somewhat ambiguous expression. It might mean $\frac{A : B}{\epsilon}$, or $\frac{A' + B}{\epsilon}$, or $\frac{B}{A}$. I showed in my last paper that the second of these is generally greater and never less than the first. I will now show that the second is also generally greater and never less than the third. We have

$$\begin{aligned}
 \frac{A' + B}{\epsilon} - \frac{B}{A} &= 1 - \frac{AB'}{\epsilon} - \frac{B}{A} = 1 - \frac{A}{\epsilon} \frac{B'}{A} - \frac{B}{A} \\
 &= 1 - \frac{A}{\epsilon} \left(1 - \frac{B}{A}\right) - \frac{B}{A} = \left(1 - \frac{A}{\epsilon}\right) \left(1 - \frac{B}{A}\right).
 \end{aligned}$$

Now, since $\frac{A}{\epsilon}$ and $\frac{B}{A}$ are chances, neither of these two bracket factors is negative; so that the chance $\frac{A' + B}{\epsilon}$ can never be less than the chance $\frac{B}{A}$. But the two chances may be equal, firstly, when A is a certainty, and, secondly, when A implies B .

I have in my Fifth Paper used the symbol $W\phi$ to denote the weakest datum or antecedent (expressed in terms of ϵ or η or θ as indices or predicates) from which we can infer ϕ , and $S\phi$ to denote the strongest conclusion we can draw from ϕ as our only datum. I now propose, as an almost necessary complement to these symbols, to employ another symbol $V\phi$ to denote the antecedents from which (without further data) we can infer neither ϕ nor ϕ' . For example, if ϕ denote

$\phi(A, B, O)$, the symbol $V\phi$ will denote the sum (or the simplified equivalent of the sum) of the terms from which we cannot (in the absence of further data) infer either ϕ or ϕ' ; the total number of terms of which these V -terms form a part being the 27 given by the product $A^{n,n} B^{n,n} C^{n,n}$; and the remaining terms being those belonging to $W\phi$ (from each of which we can infer ϕ) and those belonging to $W\phi'$ (from each of which we can infer ϕ'). It is evident that $V\phi = V\phi'$, so that we have

$$\phi = W\phi + \phi \cdot V\phi,$$

$$\phi' = W\phi' + \phi' \cdot V\phi,$$

$$\epsilon = W\phi + W\phi' + V\phi.$$

For example, let it be required to find $W\phi$, $W\phi'$, $S\phi$, $S\phi'$, and $V\phi$, when we have

$$\phi = A_{OB} : A_{B'+C}.$$

By simple inspection, we get

$$\phi! A^n + B^n + C^n;$$

and therefore

$$\phi' : A^n B^n C^n,$$

two considerations which will enable us to shorten the operations of finding $W\phi$, $W\phi'$, &c. These may be found by any of the methods in my Fifth Paper, and, when reduced to their simplest forms, are

$$W\phi = A^n + B^n + C^n,$$

$$W\phi' = A^n B^n C^n,$$

$$S\phi = W'\phi' = (W\phi')' = A^n + B^n + C^n,$$

$$S\phi' = W'\phi = (W\phi)' = A^n B^n C^n,$$

$$V\phi = V\phi' = (A^n C^n + A^n C^n) B^n.$$

Though we can infer neither ϕ nor ϕ' from any of the three terms in $V\phi$, we can, nevertheless, draw other conclusions, as follows:—

$$A^n C^n B^n \phi = A^n B^n C^n (A_{OB} : A_{B'+C})$$

$$= A^n B^n C^n (A_B + A_{B'+C});$$

for, A , B , C being understood to be *singulars*, which implies that each is always of the same class, be it ϵ or η or θ , it follows that A_{OB} and

$A_{B'+C}$ are constants. Hence

$$\begin{aligned} A^{\circ}B^{\circ}C^{\circ}\phi &= A^{\circ}B^{\circ}C^{\circ}(A_B + A_{B'+\eta}) \\ &= A^{\circ}B^{\circ}C^{\circ}(A_B + A_{B'}). \end{aligned}$$

Also
$$\begin{aligned} A^{\circ}B^{\circ}C^{\circ}\phi &= A^{\circ}B^{\circ}C^{\circ}(\epsilon_B + \epsilon_{B'+C}) \\ &= A^{\circ}B^{\circ}C^{\circ}B_C; \end{aligned}$$

for
$$B^{\circ}\epsilon_B = \eta, \text{ and } \epsilon_{B'+C} = B_C.$$

But from the third of the three terms which constitute $V\phi$ we can draw no conclusion; we only get

$$A^{\circ}B^{\circ}C^{\circ}\phi = A^{\circ}B^{\circ}C^{\circ}(A_B + A_{B'+C}) = A^{\circ}B^{\circ}C^{\circ}\phi,$$

which is a truism.

All these results may be illustrated diagrammatically. Take, for example,

$$W\phi' = A^{\circ}B^{\circ}C^{\circ},$$

which asserts that ϕ is false when A is a *certainty*, B a *variable*, and C an *impossibility*. We draw a boundary A , including the *whole universe of discourse* E ; then a boundary B , including a *part, but not the whole*, of the universe of discourse E ; and, finally, a boundary C , *wholly excluded* from the universe of discourse E . This may be done in various ways of which the accompanying figure (Fig. 2) is one.

FIG. 2.

The statement ϕ denotes $A_B + A_{B'+C}$, which is equivalent to $A_B + (AB)_C$.

and asserts an alternative, namely, that any point taken at random in the collection E must either be in B if it is in A , or else must be in C if it is in both A and B . Both terms of the alternative are false, so that ϕ is false. Or, putting ϕ into the (here) equivalent form $A_{OB} : (AB)_C$, and employing the phraseology of the ordinary logic, the implication may be read: "If some A is not B , then all AB is C "—a false conditional, since its antecedent is true and consequent false.

Next, take $W\phi = A^* + B^* + C^*$.

The middle term is equivalent to $B^* + B^*$, so that the equation asserts that each of the four terms A^* , B^* , B^* , C^* implies ϕ . To illustrate the case B^* , we may draw the boundaries A and C as we please, provided the points in the collection B (if such there be) are *wholly excluded from our universe of discourse, the collection E* .

Let the four boundaries be drawn as in the accompanying figure (Fig. 3). The antecedent A_{OB} of the implication $A_{OB} : (AB)_C$ asserts

FIG. 3.

that a point taken at random in E may be in the collection A without being in the collection B , which is true; while the consequent $(AB)_C$ asserts that a point taken at random in E cannot be in both A and B and excluded from C , which is also true since such a point cannot be in B . Hence ϕ in this case holds good.

In my Fifth Paper I referred to Dr. Schroder's error, that in Formula 13 of my Third Paper, namely, the formula

$$A_x + B_x : (AB)_x,$$

the sign of implication ($:$) should be the sign of equality ($=$). The accompanying figure (Fig. 4) will make the error evident. For, speaking of a point taken at random in the universe of discourse E ,

FIG. 4.

to say that if it falls in A it will fall in x , or that if it falls in B it will fall in x , is to assert an *impossibility*; while to say that its falling in both A and B (i.e., in the non-existent area common to both), but not in x , is an impossibility (which is the consequent assertion $AB : x$ or its equivalent $ABx' : \eta$) is to affirm a *certainity*, since

$$AB = \eta, \quad \text{and} \quad \eta : x = \epsilon.$$

Thus Formula 13 in the case here illustrated reduces to the form $\eta : \epsilon$, which always $= \epsilon$; whereas, if the sign of equality replaced the sign of implication in the formula, it would become $\eta = \epsilon$, which always $= \eta$.

Of course, if my symbol $\alpha : \beta$ were always synonymous with my symbol $\alpha' + \beta$, then, since $A_x + B_x$ would in that case be synonymous with $(A' + x) + (B' + x)$, while $(AB)_x$ would be synonymous with $(AB)' + x$, that is, with $A' + B' + x$, the antecedent and consequent of my implication $A_x + B_x : (AB)_x$ would be equivalent, and the symbol $(=)$ might be substituted for the symbol $(:)$; but I have always insisted in my previous papers published years ago in the *Proceedings*, *Mind*, and elsewhere, that, though $\alpha : \beta$ always implies $\alpha' + \beta$, it is *not*

equivalent to it. To account for the divergence of my view from that of most other logicians in regard to this point I may here repeat, what I long ago stated in my Third Paper, that my logical calculus owes its birth to a question in probability (No. 3440) proposed by Mr. Stephen Watson twenty-six years ago in the *Educational Times*. The method having thus sprung from probability, it is not surprising that in its subsequent developments it should throughout bear the marks of its origin. While the majority of other logicians, starting from the traditional logic, naturally thought of their *universes*, their *classes* or *properties*, &c., I, from the mere accident of circumstances, entering upon the subject by a different road, necessarily regarded it from a different point of view, and thought of *occurrences* with their causes and consequences; of *statements* with their implications and equivalences; of *chances* always restricted between the limits of unity and zero. Hence the fact that, though in my words I may, like others, have sometimes confounded the *true* with the *certain*, the *false* with the *impossible*, I always kept clear of these ambiguities in my symbolic formulæ.

[*A Correction.*—By actual application of the experimental method described in Note 7, p. 569, of my Sixth Paper, I have been led to doubt the correctness of the chance, .18906, therein given, and revision of my calculation has shown this doubt to have been well founded. The true chance is .25441. I had made the mistake of taking $\log 2$ as meaning $\log_{10} 2$, whereas it, of course, means $\log_2 2$. The mistake is repeated on p. 573. This numerical error does not, of course, invalidate the algebraic formulæ or general principle; and the matter only deserves notice as an example of the advantage of testing theory by experiment in probability, as in other branches of science. The actual result was 35 quadratics with real roots in a total of 125 quadratics; a result which, by the formula $M / \left(M + \frac{1}{n+1} \right)$, gives a probability of about 2 to 1 *against* the correctness of .18906, and about 8 to 1 *in favour* of the correctness of .25441.]