

true. It may be that energy passes from the potential into the kinetic form in the æther itself, and not merely on the surface of the molecules. Kinetic energy may consist of the motion of the whole system of energy-cells. This would lead us very near the theory which regards the molecule as being nothing but the mathematical centre from which forces proceed, or perhaps, from another point of view, as having infinite extension.

XXXV. *On a Simple Form of Harmonic Analyser.* By
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“The subject of the decomposition of an arbitrary function into the sum of functions of special types has many fascinations. No student of mathematical physics, if he possess any soul at all, can fail to recognize the poetry that pervades this branch of mathematics.”—OLIVER HEAVISIDE.

§ 1. **A**BOUT a year ago several instruments for determining the coefficients of a Fourier Series expressing the equation to a given curve were described before this Society by Professor Henrici †. One of them, Professor Henrici's shifting-table analyser, used a planimeter as the integrator; an arrangement that seemed to me very noteworthy from the point of view of simplicity and cheapness. The analyser I am going to describe also uses a planimeter: consequently it can also only give the value of one coefficient at a time.

§ 2. Let P Q R be the curve to be analysed. Let the base P R range from $x = -l$ to $x = +l$, and the equation to the curve in terms of a Fourier Series be

$$y = \frac{1}{2}A_0 + A_1 \cos \theta + A_2 \cos 2\theta + \dots \\ + B_1 \sin \theta + B_2 \sin 2\theta + \dots,$$

where

$$\theta = \pi x/l$$

and $\frac{1}{2}A_0$ is the mean ordinate of the curve. The values of the other coefficients are given by

* Communicated by the Physical Society: read March 8, 1895.

† Phil. Mag. xxxviii., July 1894; also Catalogue of the Mathematical Exhibition at Munich (1892-93).

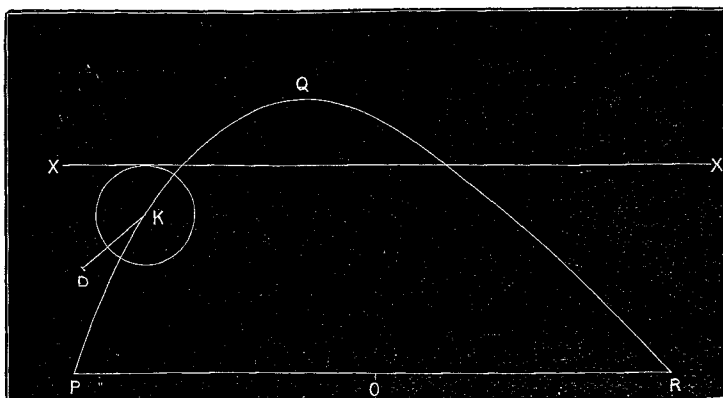
$$A_n = \frac{1}{l} \int_{-l}^{+l} y \cos n\theta \, d\theta,$$

$$B_n = \frac{1}{l} \int_{-l}^{+l} y \sin n\theta \, d\theta.$$

These are the integrals which any harmonic analyser has to evaluate.

Now suppose we have a circular disk, centre K (fig. 1),

Fig. 1.



constrained to keep in contact with a straight line XX parallel to PR , and capable of rolling along XX without slip. Further, let XX be capable of motion in the plane in a vertical but not a horizontal direction, so that every point fixed in it describes a perpendicular to PR . Then we can make the point K trace out any arbitrary curve by moving XX and rolling the disk along it.

Bring K over P , and then mark any point D on the horizontal diameter at a distance r from the centre (not necessarily inside the disk). Starting from P carry K right round PQR and back to P again by the motion just described. Supposing the circumference of the disk to be an aliquot part of PR , say $2/n$, let us find the area of the curve traced out by D during this operation. As the disk turns through an angle $2n\pi$ in rolling along a length $2l$ of XX , it will turn through $n\pi x/l$ in a distance x ; so if x, y be the coordinates of K at some point on its journey, the corresponding coordinates of D will be

$$\begin{aligned} x - r \cdot \cos n\pi \cdot \cos n\theta, \\ y + r \cdot \cos n\pi \cdot \sin n\theta, \end{aligned}$$

where, to fix the sign, we have assumed D to lie initially to the left of K and X X to lie above the disk, as in the figure.

Hence the area traced out by D is

$$R_1 = \int y dx - r \cdot \cos n\pi \int y d(\cos n\theta) \\ + r \cdot \cos n\pi \int \sin n\theta dx - r^2 \int \sin n\theta d(\cos n\theta).$$

The last two integrals vanish on taking them round a closed curve. Nothing is added to either of the first two by continuing the integration from R back to P, as y is then zero. Therefore, calling the area of the whole curve P Q R α , we have

$$R_1 = \alpha + \cos n\pi \cdot \frac{rn\pi}{l} \int_{-l}^{+l} y \sin n\theta \cdot dx.$$

Similarly, if D had been initially on the vertical instead of on the horizontal diameter, and below K, we should have had

$$R_2 = \alpha + \cos n\pi \cdot \frac{rn\pi}{l} \int_{-l}^{+l} y \cos n\theta \cdot dx.$$

It will evidently be convenient to take r some multiple of $1/\pi$ units of length, say 10. We then have, rewriting the last two equations,

when $r = 10/\pi$

$$R_1 = \alpha + \cos n\pi \cdot 10nB_n, \quad . \quad . \quad . \quad . \quad (1)$$

$$R_2 = \alpha + \cos n\pi \cdot 10nA_n, \quad . \quad . \quad . \quad . \quad (2)$$

Care must be taken with regard to the sign on the right-hand side if any other initial position of disk and line be assumed than that dealt with above.

§ 3. These two equations contain the whole theory of my instrument; they show how to construct a curve the area of which gives the required coefficients. The geometrical mechanism seems to me to be somewhat interesting, and to be possibly capable of generalization by the use of noncircular disks.

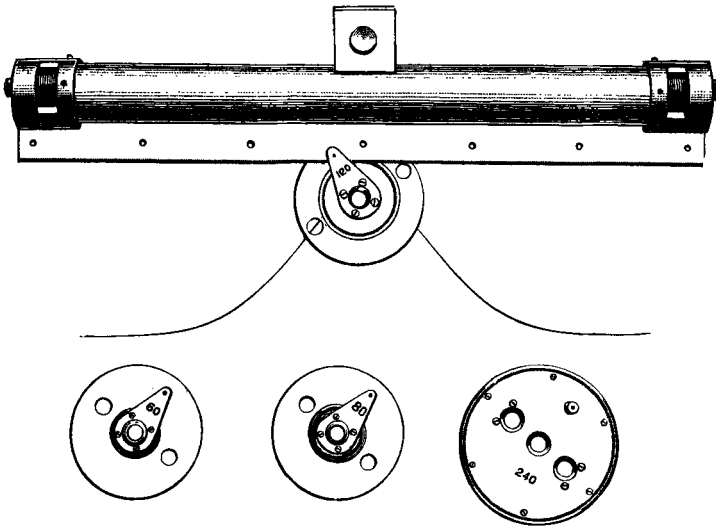
The area of the D-curve (in dealing with a material instrument) might be obtained in two ways. We might put a pencil through the disk at D, draw the curve, and integrate it afterwards: or we might attach the pointer of an integrator to D and let the integrating go on simultaneously with the following of the curve.

It is the latter alternative that I have adopted. The former method would have some advantages, but would be slow and would lead to mechanical difficulties.

§ 4. The first analyser on this principle was made for me last summer by Mr. James Hicks. Though suffering from sundry defects (due entirely to my own design) it proved a really useful and workable analyser, but required too much care and patience in use. The construction has now been entirely revised. The present instrument, designed by Mr. Horace Darwin and made by the Cambridge Scientific Instrument Company, is the final outcome. For several suggestions I am indebted to Professor Karl Pearson.

The ruler XX of fig. 1 is a rolling parallel ruler with a rack cut along its front edge (fig. 2). The weight of the

Fig. 2.



rack is counterbalanced by a block projecting from the back of the rule. Normally this block swings just clear of the paper, but it may be held down when one wants to keep the ruler still.

Corresponding to the disk of fig. 1 we have a series of toothed wheels; the number of teeth in the successive sizes being 240, 120, 80, and so on. Four of these disks have actually been made; they would probably be workable up to the sixth. The analyser is intended to work to a base-length of 30 centim.; the rack being cut 8 teeth to a centimetre. The ruler itself is made longer for the sake of stability.

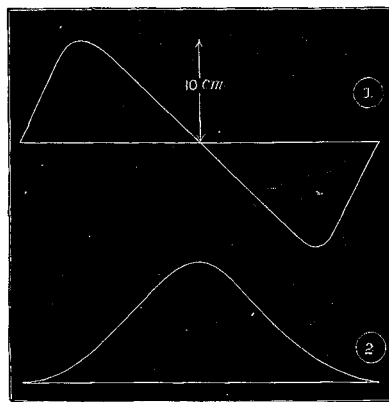
The largest disk is the simplest. It is a flat disk of brass with teeth cut round the edge. Three windows are cut through it. One, in the centre, is glazed, and the centre of

the glass is marked (on the side next the paper) with a black dot, the "tracing dot." The two remaining windows are provided with reference marks that give with the centre dot a base-line for setting the disk in any desired position. A small conical hole is made in the top of the disk, on a radius perpendicular to its base-line at a distance $10/\pi$ centimetres from the centre. This hole serves to receive the tracing-point of an ordinary Amsler planimeter, which performs the integration. The smaller disks are built up in three layers—a flat bottom plate, the toothed wheel, and a projecting crank, in the top of which is the planimeter hole. This hole is outside the circumference of the toothed wheel, so a crank or some such device is necessary. The crank swings clear above the rack when the wheel is in gear. The windows are arranged as in the first disk.

§ 5. The ordinary pattern Amsler planimeter with an arm about 16 centim. long does very well, but the tracer must be made vertically adjustable. The alteration is easily made without risk of damage. The planimeter remains of course available for its ordinary purposes, and for the determination of the absolute term.

The travel of the ruler is limited in two ways: first by the "reach" of the planimeter, secondly by the risk of running the reading wheel into the rack. Curves to be analysed must consequently be drawn to a moderate scale, but the permissible magnitude varies with the type of curve. If the type be anything like (1) of fig. 3 (sine type), 10 centim.

Fig. 3.



amplitudes can be taken in comfortably; but if the type be that of (2) in the same figure, a considerably smaller scale

must be used. In cases of physical curves (*e.g.* E.M.F. curves of alternators, conduction-of-heat curves) one is generally free to choose the limits of the period so as to bring the curve to the desired type.

The accuracy of the instrument will be measured by the accuracy of the planimeter. This cannot be fairly stated in percentages, as an error of unity in the vernier reading is never difficult, and may be anything per cent. in a small total. I strongly recommend drawing curves on cardboard; it is much more favourable to the planimeter than drawing-paper. The following tests may be taken as typical of the results that are obtained with care: the curves were drawn on card:—

(1) Actual curve,

$$3\cdot13 + 4\cdot60 \cos \theta + 1\cdot82 \cos 2\theta + \cdot39 \cos 3\theta + \cdot045 \cos 4\theta.$$

Analysed,

$$3\cdot14 + 4\cdot58 \cos \theta + 1\cdot84 \cos 2\theta + \cdot39 \cos 3\theta + \cdot042 \cos 4\theta.$$

(2) Actual curve (sloping straight line),

$$6\cdot37 \sin \theta - 3\cdot18 \sin 2\theta + 2\cdot12 \sin 3\theta - 1\cdot59 \sin 4\theta.$$

Analysed,

$$6\cdot39 \sin \theta - 3\cdot20 \sin 2\theta + 2\cdot11 \sin 3\theta - 1\cdot58 \sin 4\theta.$$

The units are centimetres.

§ 6. So much for the accuracy and range of the instrument.

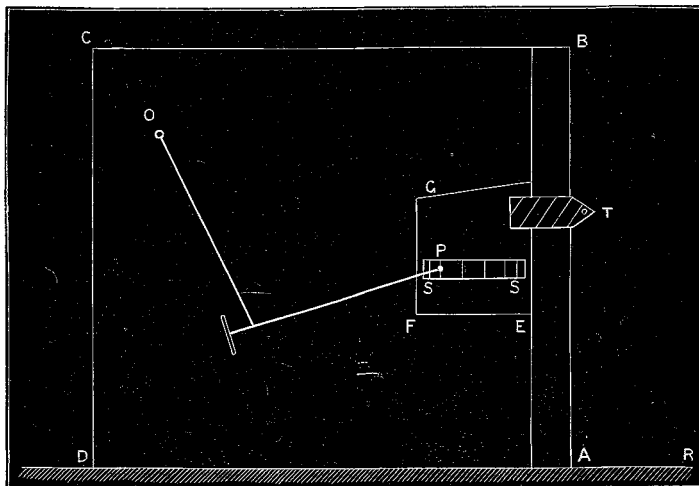
To get any desired coefficient, the ruler is set with its edge parallel to the curve-base and with the proper disk in gear. Ruler and disk are then adjusted till the tracing-dot stands over the point P (fig. 1), and the base-line of the disk is either vertical or horizontal according as a sine or cosine term is wanted. The planimeter-point is finally dropped into the hole provided for it, and the tracing-dot carried completely round the curve. The resulting planimeter reading will be the area of the curve plus or minus $10n$ times the desired coefficient.

Both hands must be used in guiding the disk, as rack and disk have to be held together while the latter is turned. The operator forms, in fact, an essential link of our mechanism which without him is unconstrained. It is this liberal use of the operator that enables me to dispense with slides, carriages, and other expensive things, and thus gain in simplicity.

§ 7. This analyser arose from a simple form of step-by-step integrator or "adder" which may be worth a brief description. The instrument is shown diagrammatically in fig. 4: it was made from materials at hand and I describe it as made. A B C D is a square sheet of card with a foot-rule glued along one edge A B. A set square F E G can be slid up and down

along this rule : it is provided with a tracer T and some scales of sines SS. The whole sheet of card is guided parallel to

Fig. 4.



the axis of x by a T-square D A R clamped to the drawing-board: if the card is pulled forward by the corner A the friction keeps it set against the square. O P is a planimeter with its pole fixed to the card and its pointer P resting on one of the scales of sines, which must stand parallel to the axis of x .

Mark off along the base of the curve to be analysed a number of equal divisions, *e. g.* 6° each, and erect the ordinates y_1, y_2 , &c. at the centres of each of these elements. Suppose the pointer P of the planimeter to rest initially at the zero of the scale, and the tracer T attached to the set square to stand over the origin of the curve. Pull the card forward, carry T to the top of y_1 , and then shift P to 6° on the scale. Pull the card forward again, carry T to the top of y_2 , and then shift P to 12° . Continue this procedure right on to the end of the curve.

P will then have come back to its starting-point on the scale, after describing on the card a certain curve or stepped polygon. The area of the polygon is

$$\begin{aligned} & y_1 (\sin 6^\circ - \sin 0^\circ), \\ & + y_2 (\sin 12^\circ - \sin 6^\circ), \\ & + y_3 (\sin 18^\circ - \sin 12^\circ), \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \\ & + y_4 (\sin 360^\circ - \sin 354^\circ), \end{aligned}$$

a quantity which (as the elements are small) approximates to

$$\int_0^{2\pi} y d(\sin \theta).$$

Thus the planimeter-reading after this procedure gives us the first cosine coefficient. I need not enter at length into the mode of getting the others. For the second coefficient one would have to shift the tracer P 12° at a time instead of 6° , and so on.

In my case I actually used steps of 6° , as above; and there were three scales on E F G going by steps of 6° , 12° , and 18° respectively for the first three terms, the separation of the scales helping to avoid confusion. The results were good: for example, in one test the actual coefficients were 4.82, 1.09, .009: the instrument gave 4.86, 1.08, .01. The chief objection to such a non-automatic integrator is of course that one is liable to forget to shift the planimeter pointer at some stage of the proceedings. The chief advantage is that your curve need not be drawn to any particular base-length. Whatever the base, it is only necessary to divide it into the proper number of equal parts, and erect the ordinates at the centres of these elements.

Suppose we wished to make the arrangement automatic. We might substitute for the harmonic motion of P along SS a circular motion round the centre of SS. This merely amounts to giving P another harmonic motion (perpendicular to SS), a proceeding which adds nothing to the planimeter-reading if the integration be continued completely round the curve.

But this is not an easy motion to obtain mechanically. The difficulty is obviated at once if we remove the card ABCD altogether and fix the pole of the planimeter in the drawing-board. If we give P the same circular motion as before we have the "disk" analyser, which I described in the first part of this paper. The area of the curve analysed is added to the integral given by the planimeter, but that is all.

Evidently in the instrument of fig. 4 we have only taken a special case in making the scales of sines. We might have used scales graduated proportionally to x^n and got moments. Any other integral could be obtained approximately if a proper scale could be drawn.