Note on Pure Periodic Continued Fractions.

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(1) There is a short paper * in a recent volume of the *Comptes Rendus* of the French Academy of Sciences which deserves notice if only in order that the attention of the author and others may be drawn to previous work on the same subject and to more effective methods of treatment.

(2) It is well known that by the solution of a quadratic equation we can show that the periodic continued fraction

$$= \frac{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_n + \dots}}}}{\sqrt{\{(a_1, \dots, a_n) - (a_2, \dots, a_{n-1})\}^2 + 4(a_1, \dots, a_{n-1})(a_2, \dots, a_n) + \{(a_1, \dots, a_n) - (a_2, \dots, a_{n-1})\}}} \quad (I)$$

where the expressions $(a_1, \dots, a_n)$, etc., on the right are continuants explained by the example,

$$(a_1, a_2, a_3, a_4) \equiv \begin{vmatrix} a_1 & 1 & . & . \\ -1 & a_2 & 1 & . \\ . & -1 & a_3 & 1 \\ . & . & -1 & a_4 \end{vmatrix}.$$

Taking the elements of the cycle $a_1, a_2, \dots, a_n$ in the reverse order we have

$$= \frac{a_n + \frac{1}{a_{n-1} + \dots + \frac{1}{a_2 + \frac{1}{a_1 + \dots}}}}{\sqrt{\{(a_n, \dots, a_1) - (a_{n-1}, \dots, a_2)\}^2 + 4(a_n, \dots, a_2)(a_{n-1}, \dots, a_1) + \{(a_n, \dots, a_1) - (a_{n-1}, \dots, a_2)\}}}.$$

the numerator here being the same as before because of the fact

* CRELIER, Sur le développement de certaines irrationnelles en fraction continue. *Comptes Rendus* . . . (Paris), cxxviii. pp. 229-231 (year 1899).

that a continuant is unaltered by reversing the order of its diagonal elements. We are thus led at once to the results

$$\frac{a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \cdots}{a_n + \frac{1}{a_{n-1}} + \cdots + \frac{1}{a_1} + \cdots} = \frac{(a_1, \dots, a_{n-1})}{(a_2, \dots, a_n)}, \quad (\text{II})$$

$$\begin{aligned} &= \frac{(a_1, \dots, a_{n-1}) \div (a_2, \dots, a_{n-1})}{(a_n, \dots, a_2) \div (a_{n-1}, \dots, a_2)}, \\ &= \frac{a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}}}{a_n + \frac{1}{a_{n-1}} + \cdots + \frac{1}{a_2}}. \end{aligned} \quad (\text{III})$$

(3) From (I) it is easy to formulate the conditions which must hold in order that a given irrational expression $(\sqrt{H} + R)/D$ may be transformable into a pure periodic continued fraction: it is more convenient, however, to put this quasi-converse proposition in the following form:—

The irrational expression $(\sqrt{A^2 + 4\alpha\beta} + A)/2\alpha$ is transformable into a pure periodic continued fraction, if positive integers $a_1, \dots, a_n$ are determinable so that

$$\begin{aligned} \frac{A}{\alpha} &= \frac{(a_1, \dots, a_n) - (a_2, \dots, a_{n-1})}{(a_2, \dots, a_n)}, \\ \text{and } \frac{\beta}{\alpha} &= \frac{(a_1, \dots, a_{n-1})}{(a_2, \dots, a_n)}. \end{aligned} \quad (\text{IV})$$

(4) If these conditions be satisfied we obtain by division

$$\frac{A}{\beta} = \frac{(a_1, \dots, a_n) - (a_2, \dots, a_{n-1})}{(a_1, \dots, a_{n-1})},$$

and therefore

$$\begin{aligned} \frac{+4\alpha\beta + A}{2\beta} &= \sqrt{\left\{ \frac{(a_1, \dots, a_n) - (a_2, \dots, a_{n-1})}{2(a_1, \dots, a_{n-1})} \right\}^2 + \frac{(a_2, \dots, a_n)}{(a_1, \dots, a_{n-1})} + \frac{(a_1, \dots, a_n) - (a_2, \dots, a_{n-1})}{2(a_1, \dots, a_{n-1})}} \\ &= \frac{\left\{ \sqrt{(a_1, \dots, a_n) - (a_2, \dots, a_{n-1})} \right\}^2 + 4(a_1, \dots, a_{n-1})(a_2, \dots, a_n) + \{(a_1, \dots, a_n) - (a_2, \dots, a_{n-1})\}}{2(a_1, \dots, a_{n-1})} \\ &= a_n + \frac{1}{a_{n-1}} + \cdots + \frac{1}{a_1} + \cdots \end{aligned}$$

We thus have the theorem :—

If the irrational expression $(\sqrt{A^2 + 4a\beta} + A)/2a$ be representable by a pure periodic continued fraction, so also is $(\sqrt{A^2 + 4a\beta} + A)2/\beta$, and the cycle of partial denominators for the one is the reverse of that for the other. (V)

With this may be compared the theorem which it is the object of M. Crelier's paper to establish.

(5) Irrational expressions of M. Crelier's form,

$$— (\sqrt{A^2 + a\beta} + A) \div a,$$

are not all representable as pure periodic continued fractions. In fact, if we have an expression of this kind which is so representable, say

$$\frac{\sqrt{A^2 + a\beta} + A}{a} = \underset{*}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n} + \dots$$

then, t being any integer less than A/a , it follows that

$$\begin{aligned} (a_1 - t) + \underset{*}{\frac{1}{a_2}} + \dots + \underset{*}{\frac{1}{a_n}} + \underset{*}{\frac{1}{a_1}} + \dots &= \frac{\sqrt{A^2 + a\beta} + A - at}{a}, \\ &= \frac{\sqrt{(A - at)^2 + a(\beta + 2tA - at^2)} + (A - at)}{a}, \end{aligned}$$

where the period of the continued fraction now begins at a different point, but the equivalent irrational expression is still of the form referred to. An example of this is—

$$\frac{\sqrt{5^2 + 3 \times 18 + 5}}{3} = 4 + \underset{*}{\frac{1}{1}} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots$$

Even here, however, the twin expression $(\sqrt{5^2 + 3 \times 18 + 5}) \div 18$ is representable as a periodic continued fraction whose period is the reverse of that just written, viz., we have

$$\frac{\sqrt{5^2 + 3 \times 18 + 5}}{18} = \frac{1}{1} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \underset{*}{\frac{1}{5}} + \dots$$

(6) The one thing necessary to have recalled in connection

with the whole matter is the following theorem,* viz.: If a quadratic equation have real irrational roots, and these be transformed into continued fractions of the type $a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}$, or its reciprocal, where $a_1, a_2, \dots$ are positive integers, then both continued fractions are periodic, and † the one period is the reverse of the other.

Applying this to the quadratic equation

$$ax^2 - 2Ax - \beta = 0,$$

we first note that its roots are

$$\frac{\sqrt{A^2 + a\beta + A}}{a}, \quad - \frac{\sqrt{A^2 + a\beta + A}}{a},$$

and that the second of these is transformable into

$$- \frac{\beta}{\sqrt{A^2 + a\beta + A}};$$

and we thus find that we need not go further.

(7) There is an important matter connected with the fundamental result of § 2, which, apparently, has not hitherto been noticed. This result, if we write p_n for $(a_1, \dots, a_n)$, q_n for $(a_2, \dots, a_n)$, etc., is

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_n + \dots}}} = \frac{\sqrt{(p_n - q_{n-1})^2 + 4p_{n-1}q_n} + (p_n - q_{n-1})}{2q_n}.$$

Now the number under the root-sign being non-quadrate, its square root must be expressible as a recurring continued fraction, and the problem is thus suggested of finding this fraction. By a

* Unfortunately omitted from most modern text-books. See Serret's *Cours d'algèbre supérieure*, 4^e éd. (1877), i. p. 49.

† It would be better to insert here the words "if the periods be different"; for they may be really identical, the mode of writing in the case of certain periods being deceptive on this point. Thus, when $A=7$, $a=2$, $\beta=15$, we

$$\text{have } \frac{\sqrt{7^2 + 2 \cdot 15 + 7}}{2} = 7 + \frac{1}{1 + \frac{1}{16 + \frac{1}{1} + \dots}}, \quad \frac{\sqrt{7^2 + 2 \cdot 15 + 7}}{15} = \frac{1}{*} + \frac{1}{16 + \frac{1}{1 + \frac{1}{*} + \dots}}$$

where the periods, although seemingly different, are not really so. A still better instance is got from $A=7$, $a=1$, $\beta=30$.

well-known property of continuants (v. *Math. Gazette*, ii. pp. 58-59)

$$(p_n - q_{n-1})^2 + 4p_{n-1}q_n = (p_n + q_{n-1})^2 - 4(-1)^n,$$

and therefore, if ω be odd and ϵ be even,

$$= (p_\omega + q_{\omega-1})^2 + 4 \quad \text{or} \quad (p_\epsilon + q_{\epsilon-1})^2 - 4.$$

Also as regards the square roots of these forms it is known that

$$\sqrt{(2M)^2 + 4} = 2M + \frac{1}{M} + \frac{1}{4M} + \dots$$

$$\sqrt{(2M+1)^2 + 4} = (2M+1) + \frac{1}{M} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4M+2} + \dots$$

$$\sqrt{(2M+4)^2 - 4} = (2M+3) + \frac{1}{1} + \frac{1}{M} + \frac{1}{1} + \frac{1}{4M+6} + \dots$$

$$\sqrt{(2M+3)^2 - 4} = (2M+2) + \frac{1}{1} + \frac{1}{M} + \frac{1}{2} + \frac{1}{M} + \frac{1}{1} + \frac{1}{4M+4} + \dots$$

Consequently, taking the first case only, viz., where $p_\omega + q_{\omega-1}$ is even, we have

$$\sqrt{(p_\omega + q_{\omega-1})^2 + 4} = (p_\omega + q_{\omega-1}) + \frac{1}{\frac{1}{2}(p_\omega + q_{\omega-1})} + \frac{1}{2(p_\omega + q_{\omega-1})} + \dots$$

whence, by addition of $p_\omega - q_{\omega-1}$ to both sides,

$$2q_\omega \left\{ \frac{a_1}{*} + \frac{1}{a_2} + \dots + \frac{1}{a_\omega} + \dots \right\} = 2p_\omega + \frac{1}{\frac{1}{2}(p_\omega + q_{\omega-1})} + \frac{1}{2(p_\omega + q_{\omega-1})} + \dots$$

and finally

$$q_\omega \left\{ \frac{a_1}{*} + \frac{1}{a_2} + \dots + \frac{1}{a_\omega} + \dots \right\} = p_\omega + \frac{1}{p_\omega + q_{\omega-1}} + \dots \quad (\text{VI})$$

For example,

$$7 \left\{ \frac{1}{*} + \frac{1}{2} + \frac{1}{3} + \dots \right\} = 10 + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \dots$$

Even although we do not take an infinite number of periods, the identity still holds, provided that the same number of periods be taken on both sides; thus—

$$7 \left\{ 1 + \frac{1}{2} + \frac{1}{3} \right\} = 10,$$

$$7 \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right\} = 10 + \frac{1}{12},$$

.

(8) Taking the two periodic continued fractions in § 2, and subtracting from the first the reciprocal of the second, we obtain

$$\alpha_1 + \frac{1}{\alpha_2 + \dots + \frac{1}{\alpha_n + \dots}} = \frac{p_n - q_{n-1}}{q_n} + \frac{1}{\alpha_n + \alpha_{n-1} + \dots + \alpha_1 + \dots} \quad (\text{VII})$$

If with a view to obtaining from this an instance of the equality of a pure and a mixed periodic continued fraction we impose the condition that $(p_n - q_{n-1})/q_n$ be a positive integer, we are led to a result which makes the one side an exact copy of the other, viz.,

$$\frac{p_n - q_{n-1}}{q_n} = a_1, \quad a_2 = a_n, \quad a_3 = a_{n-1}, \quad \dots$$

(9) Another matter worthy of attention is connected with the theorem* that every mixed recurring continued fraction is equal to an irrational expression of the form $(U + V\sqrt{N}) \div W$.

Putting

$$a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_m + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n + \dots}}}}} \equiv X,$$

$$\text{and} \quad \frac{a_1}{\alpha_1 + \frac{1}{\alpha_2 + \dots + \frac{1}{\alpha_n + \dots}}} \equiv x,$$

we have

$$\begin{aligned} X &= \frac{(a_1, a_2, \dots, a_m, x)}{(a_2, \dots, a_m, x)}, \\ &= \frac{x(a_1, a_2, \dots, a_m) + (a_1, a_2, \dots, a_{m-1})}{x(a_2, \dots, a_m) + (a_2, a_3, \dots, a_{m-1})}, \\ &= \frac{x\pi_m + \pi_{m-1}}{x\rho_m + \rho_{m-1}}, \text{ say.} \end{aligned}$$

If now we insert the value of x given in § 2, writing it for the moment in the form $(\sqrt{H} + R) \div D$, we obtain

* Serret's *Cours d'algèbre supérieure*, i. (1877), p. 46, or Chrystal's *Algebra*, ii. (1889), p. 430.

$$X = \frac{\pi_m \sqrt{H} + (\pi_m R + \pi_{m-1} D)}{\rho_m \sqrt{H} + (\rho_m R + \rho_{m-1} D)};$$

and this after rationalisation of the denominator will be found to be simplifiable by the removal of the common factor D from numerator and denominator, $\sqrt{H}$ being thus left in the numerator with the coefficient 1. This simplification is the main matter deserving of note, but the full result is—

If the irrational expression which is the equivalent of

$$a_1 + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \dots$$

be denoted by $(\sqrt{H} + R) \div D$, then the like expression for

$$a_1 + \frac{1}{a_2} + \dots + \frac{1}{a_m} + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \dots$$

is $(\sqrt{H} + R') \div D'$, where

$$R' = (p_{n-1}, -p_n + q_{n-1}, -q_n \text{ } \S \text{ } 2\pi_m \rho_m, \pi_{m-1} \rho_m + \pi_m \rho_{m-1}, 2\pi_{m-1} \rho_{m-1})$$

and

$$D' = (\text{ } \S \text{ } 2\rho_m^2, 2\rho_m \rho_{m-1}, 2\rho_{m-1}^2),$$

π_m/ρ_m and p_n/q_n being the m^{th} and n^{th} convergents to $a_1 + \frac{1}{a_2} + \dots + \frac{1}{a_m}$

and $a_1 + \frac{1}{a_2} + \dots + \frac{1}{a_n}$ respectively. (VIII)

*D', it will be observed, is got from R' by writing ρ_m, ρ_{m-1} for π_m, π_{m-1} .**

* It is extremely probable that among the papers of the late Mr C. E. Bickmore, Fellow of New College, Oxford, who was an earnest and capable student of this branch of algebra, there are to be found quite a number of results as yet unpublished and worthy of being made known. In a letter dated 29th April 1895, in which some noteworthy theorems are given from a paper prepared for the Oxford Mathematical Society, he says: "Our Society is too poor to print anything." It would appear that French mathematicians are not similarly hampered (v. *Lond. Math. Soc. Proc.*, xxxiv. p. 129).