



VIII. On the frequency ranges of non-generating force exerting cumulative influence

Andrew Stephenson

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Multiply by r , differentiate with regard to r , and divide by r , and (25) becomes

$$\frac{\partial^2}{\partial t^2} \frac{1}{r} \frac{\partial(ur)}{\partial r} = g \int_0^h \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial(ur)}{\partial r} dz + \nu \frac{\partial^2}{\partial t \partial z^2} \frac{1}{r} \frac{\partial(ur)}{\partial r}. \quad (26)$$

Substitute

$$\frac{1}{r} \frac{\partial(ur)}{\partial r} = J_0(mr)Z,$$

where Z is a function of z alone. Then

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial(ur)}{\partial r} &= \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial(ur)}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial(ur)}{\partial r}, \\ &= Z \left(\frac{\partial^2 J_0(mr)}{\partial r^2} + \frac{1}{r} \frac{\partial J_0(mr)}{\partial r} \right), \\ &= -m^2 J_0(mr)Z. \end{aligned}$$

Hence $J_0(mr)$ cancels out of equation (26). If we assume a time-factor $e^{i\sigma t}$ we have for Z

$$\sigma^2 Z - gm^2 \int_0^h Z dz + \nu i \sigma \frac{\partial^2 Z}{\partial z^2} = 0, \quad \dots \quad (27)$$

which is the same as equation (7).

VIII. On the Frequency Ranges of Non-generating Force exerting Cumulative Influence. By ANDREW STEPHENSON*.

1. **P**ERIODIC non-generating force acting on a system in oscillation about a position of stable equilibrium, exerts a cumulative action in intensifying or diminishing the amplitude if its frequency is contained within any one of a number of ranges lying in the vicinity of 2μ , $2\mu/2$, $2\mu/3$..., where μ is the natural frequency of the system†.

The limits of the leading ranges have been obtained for small non-generating force. It is our object here to give a general method of finding the ranges in magnitude and position when the force is finite, and in particular to obtain the numerical values of the range limits about the double frequency for various intensities.

* Communicated by the Author.

† "On a Class of Forced Oscillations," *Quart. Journ. of Mathematics*, No. 168, 1906.

2. The complete solution of the equation of motion

$$\ddot{x} + \mu^2(1 + 2 \sum_1^{\infty} \alpha_r \cos rnt)x = 0 \quad . \quad . \quad . \quad (i.)$$

is given by

$$x = \sum_{-\infty}^{\infty} A_r \sin \{(c - rn)t + \epsilon\}, \quad . \quad . \quad . \quad (ii.)$$

where ϵ is arbitrary and

$$A_r \{\mu^2 - (c - rn)^2\} + \mu^2 \{\alpha_1(A_{r-1} + A_{r+1}) + \alpha_2(A_{r-2} + A_{r+2}) + \dots\} = 0.$$

On eliminating the A 's we obtain the infinite determinant

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha_2 & \alpha_1 & [-1] & \alpha_1 & \alpha_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha_2 & \alpha_1 & [0] & \alpha_1 & \alpha_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha_2 & \alpha_1 & [1] & \alpha_1 & \alpha_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0, \quad . \quad (1)$$

where $[r]$ denotes $\{\mu^2 - (c - rn)^2\}/\mu^2$. This equation determines c , and the roots are all included in the form $\pm c_0 - rn$, where r has the zero or any positive or negative integral value*.

Considering n as a variable parameter we observe that the roots can become equal in pairs only through c_0 becoming equal to zero or half of some multiple of n . But the values of n leading to equal roots separate the ranges of n giving a real ' c ' from those giving a complex ' c .' Hence the determination of the ranges of cumulative action is equivalent to the problem of obtaining the values of n which make the roots of the above determinant equal in pairs; i. e. to the problem of finding the roots of each of the two equations in n obtained from (1) by putting $c=0$ and $\frac{1}{2}n$ respectively.

3. When $c=0$, $[-r]=[r]$, and when $c=\frac{1}{2}n$, $[-(r-1)]=[r]$: on account of the symmetry thus introduced the idea suggests itself that for these values of c equation (1) admits of reduction. We shall investigate this question by direct examination of the motion in the limiting cases.

The general solution (ii.) may be written

$$\begin{aligned} x = & \cos \epsilon \left(\sin ct \sum_{-\infty}^{\infty} A_r \cos rnt - \cos ct \sum_{-\infty}^{\infty} A_r \sin rnt \right) \\ & + \sin \epsilon \left(\cos ct \sum_{-\infty}^{\infty} A_r \cos rnt + \sin ct \sum_{-\infty}^{\infty} A_r \sin rnt \right). \end{aligned}$$

Now in the limit when $c=0$, either $A_{-r}=A_r$ or $A_{-r}=-A_r$,

* Thus far we follow the analysis introduced by G. W. Hill in his lunar theory.

according to which end of the range n approaches. On the former alternative one particular solution is given by

$$x = \sum_{-\infty}^{\infty} A_r \cos rnt \\ = A_0 + 2 \sum_1^{\infty} A_r \cos rnt.$$

and the other by

$$x = \text{Lt}_{c=0} \left\{ t \sum_{-\infty}^{\infty} A_r \cos rnt - \frac{1}{c} \sum_{-\infty}^{\infty} A_r \sin rnt \right\} \\ = \text{Lt}_{c=0} \left\{ t(A_0 + 2 \sum_1^{\infty} A_r \cos rnt) - 2 \sum_1^{\infty} \frac{A_r - A_{-r}}{c} \sin rnt \right\}.$$

Thus the solutions can be expressed by convergent series of the form

$$x = \sum_0^{\infty} B_r \cos rnt \\ \text{and} \quad x = t \sum_0^{\infty} B_r \cos rnt + \sum_1^{\infty} C_r \sin rnt.$$

By direct substitution

$$B_0 + \alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \dots = 0, \\ 2\alpha_1 B_0 + \{1 - (n/\mu)^2 + \alpha_2\} B_1 + (\alpha_1 + \alpha_3) B_2 + (\alpha_2 + \alpha_4) B_3 + \dots = 0, \\ 2\alpha_2 B_0 + (\alpha_1 + \alpha_3) B_1 + \{1 - (2n/\mu)^2 + \alpha_4\} B_2 + (\alpha_1 + \alpha_5) B_3 + \dots = 0, \\ 2\alpha_3 B_0 + (\alpha_2 + \alpha_4) B_1 + (\alpha_1 + \alpha_5) B_2 + \{1 - (3n/\mu)^2 + \alpha_6\} B_3 + \dots = 0; \\ \dots \dots \dots$$

also

$$\{1 - (n/\mu)^2 - \alpha_2\} C_1 + (\alpha_1 - \alpha_3) C_2 + (\alpha_2 - \alpha_4) C_3 + \dots = 2n \cdot B_1, \\ (\alpha_1 - \alpha_3) C_1 + \{1 - (2n/\mu)^2 - \alpha_4\} C_2 + (\alpha_1 - \alpha_5) C_3 \dots = 2n \cdot 2B_2, \\ (\alpha_2 - \alpha_4) C_1 + (\alpha_1 - \alpha_5) C_2 + \{1 - (3n/\mu)^2 - \alpha_6\} C_3 + \dots = 2n \cdot 3B_3. \\ \dots \dots \dots$$

The eliminant of the former set gives

$$\begin{vmatrix} 2 & 2\alpha_1 & 2\alpha_2 & 2\alpha_3 & \dots \\ 2\alpha_1 & 1 - \left(\frac{n}{\mu}\right)^2 + \alpha_2 & \alpha_1 + \alpha_3 & \alpha_2 + \alpha_4 & \dots \\ 2\alpha_2 & \alpha_1 + \alpha_3 & 1 - 4\left(\frac{n}{\mu}\right)^2 + \alpha_4 & \alpha_1 + \alpha_5 & \dots \\ 2\alpha_3 & \alpha_2 + \alpha_4 & \alpha_1 + \alpha_5 & 1 - 9\left(\frac{n}{\mu}\right)^2 + \alpha_6 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0, \quad (2)$$

where the term in the $(r+1)$ th row and $(s+1)$ th column is $\alpha_{r+s} + \alpha_{|r-s|}$, unless $r=s$, in which case the term is $1 - s^2(n/\mu)^2 + \alpha_{2s}$.

This equation determines one limit of each of the ranges of n associated with $\mu, \mu/2, \dots \mu/r \dots$. For the other limit $A_{-r} = -A_r$ in (ii.), and the particular solutions are

$$x = \sum_1^{\infty} B_r \sin rnt,$$

$$x = t \sum_1^{\infty} B_r \sin rnt - \sum_0^{\infty} C_r \cos rnt,$$

where

$$\begin{aligned} \{1 - (n/\mu)^2 - \alpha_2\} B_1 + (\alpha_1 - \alpha_3) B_2 + (\alpha_2 - \alpha_4) B_3 + (\alpha_3 - \alpha_5) B_4 + \dots &= 0, \\ (\alpha_1 - \alpha_3) B_1 + \{1 - (2n/\mu)^2 - \alpha_4\} B_2 + (\alpha_1 - \alpha_5) B_3 + (\alpha_2 - \alpha_6) B_4 + \dots &= 0, \\ (\alpha_2 - \alpha_4) B_1 + (\alpha_1 - \alpha_5) B_2 + \{1 - (3n/\mu)^2 - \alpha_6\} B_3 + (\alpha_1 - \alpha_7) B_4 + \dots &= 0, \\ (\alpha_3 - \alpha_5) B_1 + (\alpha_2 - \alpha_6) B_2 + (\alpha_1 - \alpha_7) B_3 + \{1 - (4n/\mu)^2 - \alpha_8\} B_4 + \dots &= 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

$$\begin{aligned} C_0 + \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3 + \dots &= 0, \\ 2\alpha_1 C_0 + \{1 - (n/\mu)^2 + \alpha_2\} C_1 + (\alpha_1 + \alpha_3) C_2 + (\alpha_2 + \alpha_4) C_3 + \dots &= 2n \cdot B_1, \\ 2\alpha_2 C_0 + (\alpha_1 + \alpha_3) C_1 + \{1 - (2n/\mu)^2 + \alpha_4\} C_2 + (\alpha_1 + \alpha_5) C_3 + \dots &= 2n \cdot 2B_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Hence,

$$\begin{vmatrix} 1 - \left(\frac{n}{\mu}\right)^2 - \alpha_2 & \alpha_1 - \alpha_3 & \alpha_2 - \alpha_4 & \alpha_3 - \alpha_5 & \cdot \\ \alpha_1 - \alpha_3 & 1 - 4\left(\frac{n}{\mu}\right)^2 - \alpha_4 & \alpha_1 - \alpha_5 & \alpha_2 - \alpha_6 & \cdot \\ \alpha_2 - \alpha_4 & \alpha_1 - \alpha_5 & 1 - 9\left(\frac{n}{\mu}\right)^2 - \alpha_6 & \alpha_1 - \alpha_7 & \cdot \\ \alpha_3 - \alpha_5 & \alpha_2 - \alpha_6 & \alpha_1 - \alpha_7 & 1 - 16\left(\frac{n}{\mu}\right)^2 - \alpha_8 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0, (3)$$

where the term in the r th row and s th column is $\alpha_{|r-s|} - \alpha_{r+s}$, if $r \neq s$; if $r=s$ the term is $1 - s^2(n/\mu)^2 - \alpha_{2s}$. This equation determines the other limits.

The case $c = \frac{1}{2}n$ admits of similar treatment. When c has this value we find that either

$$\begin{vmatrix} 1 - \left(\frac{n}{2\mu}\right)^2 + \alpha_1 & \alpha_1 + \alpha_2 & \alpha_2 + \alpha_3 & . \\ \alpha_1 + \alpha_2 & 1 - 9\left(\frac{n}{2\mu}\right)^2 + \alpha_3 & \alpha_1 + \alpha_4 & . \\ \alpha_2 + \alpha_3 & \alpha_1 + \alpha_4 & 1 - 25\left(\frac{n}{2\mu}\right)^2 + \alpha_5 & . \\ . & . & . & . \end{vmatrix} = 0, \quad (4)$$

or

$$\begin{vmatrix} 1 - \left(\frac{n}{2\mu}\right)^2 - \alpha_1 & \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & . \\ \alpha_1 - \alpha_2 & 1 - 9\left(\frac{n}{2\mu}\right)^2 - \alpha_3 & \alpha_1 - \alpha_4 & . \\ \alpha_2 - \alpha_3 & \alpha_1 - \alpha_4 & 1 - 25\left(\frac{n}{2\mu}\right)^2 - \alpha_5 & . \\ . & . & . & . \end{vmatrix} = 0, \quad (5)$$

where the term in the r th row and s th column is

$$\alpha_{|r-s|} \pm \alpha_{r+s+1},$$

except in the case $r=s$, for which it has the form

$$1 - (2s-1)^2(n/2\mu)^2 \pm \alpha_{2s-1}.$$

Hence the ranges of n associated with $2\mu, \dots, 2\mu/r$ are obtained.

Thus when $c=0$ equation (1) splits up into (2) and (3); and when $c=\frac{1}{2}n$, into (4) and (5).

4. When the non-generating force is simply periodic

$$\ddot{x} + \mu^2(1 + 2\alpha \cos nt)x = 0.$$

If α is so small that its square may be neglected, it is evident from (4) and (5) that the range of cumulative effect about the double frequency, 2μ , is $2\mu(1 \pm \frac{1}{2}\alpha)$. To a second approximation the limits are given by

$$\begin{vmatrix} 1 - \left(\frac{n}{2\mu}\right)^2 \pm \alpha & \alpha \\ \alpha & 1 - 9\left(\frac{n}{2\mu}\right)^2 \end{vmatrix} = 0,$$

whence

$$n = 2\mu(1 \pm \frac{1}{2}\alpha - \frac{1}{16}\alpha^2). \quad . \quad . \quad . \quad (6)$$

For the limits of the range about μ , from (2) and (3)

$$\begin{vmatrix} 1 & \alpha & \\ 2\alpha & 1 - \left(\frac{n}{\mu}\right)^2 & \alpha \\ & \alpha & 1 - 4\left(\frac{n}{\mu}\right)^2 \end{vmatrix} = 0,$$

and $\begin{vmatrix} 1 - \left(\frac{n}{\mu}\right)^2 & \alpha & \\ \alpha & 1 - 4\left(\frac{n}{\mu}\right)^2 & \end{vmatrix} = 0,$

i. e. $\left. \begin{aligned} n &= \mu(1 - \frac{5}{8}\alpha^2) \\ n &= \mu(1 + \frac{1}{8}\alpha^2) \end{aligned} \right\} \dots \dots \dots (7)$

Again, for the range associated with $2\mu/3$,

$$\begin{vmatrix} 1 - \left(\frac{n}{2\mu}\right)^2 \pm \alpha & \alpha & \\ \alpha & 1 - 9\left(\frac{n}{2\mu}\right)^2 & \alpha \\ & \alpha & 1 - 25\left(\frac{n}{2\mu}\right)^2 \end{vmatrix} = 0,$$

whence $n = \frac{2\mu}{3} (1 - \frac{9}{32}\alpha^2 \pm \frac{81}{128}\alpha^3) \dots \dots \dots (8)$

These results are in agreement with those obtained previously by another method*.

5. From the physical point of view interest attaches to the change in the breadth and position of the range about the double frequency when the intensity of the force is increased. For this range the roots approach rapidly towards their limiting values if α is not too large, and we readily find:—

α .	Range of $\left(\frac{n}{2\mu}\right)^2$.	Range of $\frac{n}{2\mu}$.
0.25	0.7607, 1.2560	0.872, 1.121
0.50	0.5619, 1.5197	0.749, 1.232
0.75	0.4423, 1.7873	0.665, 1.337
1.00	0.4007, 2.0571	0.633, 1.434
1.25	0.3983, 2.3284	0.631, 1.526

and so on.

* *Loc. cit.* § 3.

If Z_n denotes the determinant obtained by taking the terms common to the first n rows and columns of either (4) or (5), when all the α 's except α_1 are zero,

$$Z_n = \{1 - (2n-1)^2 z\} Z_{n-1} - \alpha^2 Z_{n-2},$$

where $z = \left(\frac{n}{2\mu}\right)^2$. Hence if z_0 is a root of $Z_{n-1} = 0$, and if the corresponding root of $Z_n = 0$ differs from z_0 by a small quantity s , then approximately

$$s = \frac{\alpha^2 Z_{n-2}}{\{1 - (2n-1)^2 z\} \frac{dZ_{n-1}}{dz}}, \text{ when } z = z_0.$$

This affords a ready method of testing the approximation given by z_0 .

To the degree of accuracy shown in the table the last three lower limits follow from $Z_3 = 0$; the rest of the roots exhibited are given by $Z_2 = 0$, with the small correction s if necessary.

6. The cumulative effect of nongenerating force of long period, being comparatively feeble, is likely to escape notice, except in the case of a system in which the motional resistance is inappreciable and the time of observation extended. As both of these conditions hold for the solar system, it is natural to inquire whether any outstanding discrepancies between astronomical calculations and observations could be traced to the neglect of such slow cumulative action. Records of solar eclipses are available over a long period, and it is found that the calculated paths exhibit a regularly varying error which becomes more marked with the remoteness of the eclipse*. The regularity of variation is held to confirm the accuracy of the record; and the question remains as to how the discrepancy arises.

I would suggest that it may be due to the cumulative action of some lunar disturbance of relatively long period--an action requiring for its detection a higher approximation, perhaps, than has been attained in existing calculations.

* See, for example, "The Moon's Motion," *Nature*, Oct. 1908, p. 599.