On Class-Invariants. By Professor G. B. MATHEWS.

[Read Feb. 13th, 1890.]

The modular function usually denoted by $J(\omega)$ is of such fundamental importance, especially in connexion with the arithmetical aspect of the modular-function theory, that it seems desirable to discuss at least some cases of complex multiplication with direct reference to the (J, J') modular equations. This is what I have endeavoured to do in the following pages in the cases enumerated by Klein (*Math. Annalen*, t. xiv.*), where the equation defining J' or $J(\omega/n)$ as a function of $J(\omega)$ is of "Geschlecht" zero; that is to say, when $n = 2, 3, 4, 5, 7, 13$, respectively. The notation of the paper just quoted has been adopted, so far as possible, throughout; in some respects it would perhaps be more convenient to express the results in terms of $j(\omega) = 1728 J(\omega)$; but there is no difficulty in making the alteration if desired. Since the interest of this paper is, I think, principally algebraical, only so much of the general theory of modular functions has been prefixed as seems necessary to show the connexion of the results with the arithmetical theory of definite quadratic forms.†

If ω is any complex quantity with a positive imaginary part, then, putting $e^{i\omega} = q$, the function $J(\omega)$ is defined by

$$J(\omega) = \frac{\left(\frac{1}{12} + 20 \sum m^3 \frac{q^{2m}}{1 - q^{2m}}\right)^3}{q^2 \prod (1 - q^{2m})^{24}}. \quad (m = 1, 2, 3 \dots \infty.)$$

The fundamental property of J is that, ω', ω being different complex quantities,

$$J(\omega') = J(\omega)$$

if, and only if,

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta},$$

* A summary of this paper will be found in *Proc. Lond. Math. Soc.*, Vol. ix., p. 123.

† Perhaps the most elementary introduction to the theory of modular functions is Hurwitz's memoir, *Math. Ann.*, xviii. (1881), p. 528, where further references will be found.

where $\alpha, \beta, \gamma, \delta$ are real integers such that

$$\alpha\delta - \beta\gamma = 1.$$

If, on the other hand, $\alpha\delta - \beta\gamma = n$,

an integer different from unity, then $J(\omega')$ is an algebraical function of $J(\omega)$, and the equation connecting $J(\omega')$ and $J(\omega)$, or say J' and J , may be called the (J, J') modular equation of the n^{th} order.

Now, suppose ω is a root (with positive imaginary part) of a primitive quadratic form (a, b, c) , so that

$$a\omega^2 + 2b\omega + c = 0;$$

then, according as the form is properly or improperly primitive, this gives

$$a\omega + 2b = -\frac{c}{\omega},$$

or
$$\frac{1}{2}a\omega + b = -\frac{\frac{1}{2}c}{\omega};$$

from which we infer $J(a\omega) = J(c\omega)$,

or $J(\frac{1}{2}a\omega) = J(\frac{1}{2}c\omega)$,

respectively.

In particular, the forms $(1, b, n)$, $(2, b, 2n)$ lead to the relation

$$J(\omega) = J(n\omega);$$

so that, if in the (J, J') modular equation of the n^{th} order we put $J' = J$, one of the roots of the resulting equation will be a class-invariant for the form considered; and the values of J thus obtained will all belong to forms of determinants $-\Delta$ for which Δ is positive, and of the form $n - b^2$ or $4n - b^2$, respectively. It will be seen, then, that the method is substantially identical with that of Hermite, in his "*Théorie des Equations Modulaires*," the difference being that the irrational quantities $\sqrt[4]{\kappa}$, $\sqrt[4]{\kappa'}$ do not appear; instead of them, we have the quantity which Klein calls τ , an algebraic function of J in terms of which (for the special values of n here considered) all other modular functions can be rationally expressed (*cf. Klein, loc. cit.*, p. 127).

In further explanation of the following results it may be added that, for $n = 2, 3, 4, 5, 7, 13$, Klein expresses J in the form $\phi(\tau)/\psi(\tau)$, where $\phi(\tau)$, $\psi(\tau)$ are integral functions of τ , and J' , that is, $J(\omega/n)$, is the same function of τ' , where τ and τ' are connected by a linear

equation of the form

$$\tau\tau' = O \quad \text{or} \quad \tau + \tau' = O,$$

where O is a constant. By putting $J' = J$, and eliminating τ' , an equation in τ is obtained by solving which the singular values of τ , and thence of J , are found. In addition to the values of τ and J , I have determined in almost all cases the classes of quadratic forms to which they belong.

From the point of view of the complex multiplication of elliptic functions,

$$J = \frac{4}{27} \cdot \frac{(1 - \kappa^2 \kappa'^2)^3}{\kappa^4 \kappa'^4},$$

where κ, κ' are Legendre's moduli; and ω , the root of any one of the corresponding quadratic forms, is the ratio of two elementary periods of the elliptic functions connected with these particular moduli. Thus, for instance, $J(1, 0, n)$ is connected with that case of complex multiplication for which $K'/K = \sqrt{n}$. The arithmetical, or modular function theory, seems, however, more fundamental, and the results have therefore been exhibited in their connexion therewith.

$$\text{I. } n = 2.$$

Here

$$27J = \frac{(4\tau - 1)^3}{\tau} = \frac{(4 - \tau)^3}{\tau^3},$$

therefore

$$\tau(4\tau - 1)^3 - (4 - \tau)^3 = 0,$$

or

$$(\tau^3 - 1)(64\tau^3 - 47\tau + 64) = 0;*$$

and the values of τ and J are as follows:—

$$\tau = 1, \quad J = 1,$$

corresponding to the class $(1, 0, 1) \sim (1, 1, 2)$;

$$\tau = -1, \quad J = \frac{5^3}{3^3},$$

corresponding to $(1, 0, 2)$.

As a verification, we know that in the latter case

$$\kappa = \sqrt{2} - 1, \quad \kappa'/\kappa = \sqrt{2};$$

whence

$$\kappa^2 = (\sqrt{2} - 1)^2, \quad \kappa'^2 = 2(\sqrt{2} - 1),$$

$$1 - \kappa^2 \kappa'^2 = 5(3 - 2\sqrt{2}) = 5(\sqrt{2} - 1)^2,$$

* Here and elsewhere the sign $\sim$ denotes equivalence.

and
$$J = \frac{4}{27} \frac{(1 - \kappa^3 \kappa'^3)^3}{\kappa^4 \kappa'^4} = \frac{5^3}{3^3}.$$

The other values of τ are

$$\tau = \frac{47 \pm 45i\sqrt{7}}{128},$$

giving
$$4\tau - 1 = \frac{15(1 \pm 3i\sqrt{7})}{32},$$

$$\begin{aligned} J &= \frac{15^3 \cdot 128 (1 \pm 3i\sqrt{7})^3}{27 \cdot 32^3 (47 \pm 45i\sqrt{7})} \\ &= -\frac{15^3 \cdot 128 \cdot 4}{27 \cdot 32^3} = -\frac{5^3}{2^3}, \end{aligned}$$

and the corresponding class is $(2, 1, 4)$.

II. $n = 3$.

$$-64J = \frac{(r-1)(9r-1)^3}{r} = \frac{(1-r)(9-r)^3}{r^3},$$

$$(r-1) \{r^3 (9r-1)^3 - (r-9)^3\} = 0,$$

or $(r-1)(r+1) \left\{ 729 \left(r + \frac{1}{r} \right)^3 - 972 \left(r + \frac{1}{r} \right) - 460 \right\} = 0;$

hence

$$r = 1, \quad J = 0,$$

$$r = -1, \quad J = \frac{5^3}{2^3},$$

$$r = \frac{23 \pm 10i\sqrt{2}}{27}, \quad J = \frac{5^3}{3^3},$$

$$r = \frac{-5 \pm 8i\sqrt{11}}{27}, \quad J = -\frac{2^3}{3^3}.$$

The corresponding classes are

$$(2, 3, 6) \sim (2, 1, 2), (1, 0, 3), \quad (1, 1, 3) \sim (1, 0, 2), (2, 1, 6).$$

(See also Greenhill, *Proc. Lond. Math. Soc.*, Vol. xix., pp. 306, 351.)

III. $n = 4$.

$$108J = \frac{(16 - 16r + r^3)^3}{r^4(1-r)} = \frac{(1 + 14r + r^3)^3}{r(1-r)^4},$$

$$r(r-1) \{ (r-1)^3 (r^3 - 16r + 16)^3 + r^3 (r^3 + 14r + 1)^3 \} = 0.$$

$r = 0$ or 1 gives $J = \infty$, which has to be rejected. Taking the other factor, we have either

$$(i.) \quad (r-1)(r^3-16r+16)+r(r^3+14r+1) = 0,$$

$$\text{that is,} \quad (2r-1)(r^3-r+16) = 0;$$

$$\text{or} \quad (ii.) \quad r^6-3r^5+756r^4-1507r^3+1761r^2-1008r+256 = 0,$$

$$\text{that is,} \quad (r^3-r+1) \{ (r^3-r)^3 + 752 (r^3-r) + 256 \} = 0.$$

Solving for r , we get

$$r = \frac{1}{2}, \quad J = \frac{11^3}{2^3},$$

$$r = \frac{1 \pm 3i\sqrt{7}}{2}, \quad J = -\frac{5^3}{2^3},$$

$$r = \frac{1 \pm i\sqrt{3}}{2}, \quad J = \frac{5^3}{2^3},$$

$$r = \frac{1 \pm i\sqrt{3}(16+7\sqrt{5})}{2},$$

$$J = \frac{5\sqrt{5}}{2^{10}} (\sqrt{5}-2)^3 (4-\sqrt{5})^3 (\sqrt{5}+1)(\sqrt{5}+3)^3$$

$$= \frac{5\sqrt{5}}{2^8} (4-\sqrt{5})^3 (\sqrt{5}-1)^3,$$

$$r = \frac{1 \pm i\sqrt{3}(16-7\sqrt{5})}{2},$$

$$J = \frac{5\sqrt{5}}{2^{10}} (\sqrt{5}+2)^3 (4+\sqrt{5})^3 (\sqrt{5}-1)(\sqrt{5}-3)^3$$

$$= -\frac{5\sqrt{5}}{2^8} (4+\sqrt{5})^3 (\sqrt{5}+1)^3.$$

The reduction in the last two cases is very curious. Putting

$$r^3-r+8(47+21\sqrt{5}) = 0,$$

one value of r is

$$\begin{aligned} r &= \frac{1+i\sqrt{1503+672\sqrt{5}}}{2} \\ &= \frac{1+i\sqrt{3}(16+7\sqrt{5})}{2}. \end{aligned}$$

This gives

$$\begin{aligned}\frac{1+14\tau+\tau^2}{1-\tau} &= \frac{3(16+7\sqrt{5})(-7\sqrt{5}+5i\sqrt{3})}{1-i\sqrt{3}(16+7\sqrt{5})} \\ &= \frac{960(16+7\sqrt{5})}{(-1+i\sqrt{3})(112\sqrt{5}+240)} \\ &= \frac{60(16+7\sqrt{5})}{(-1+i\sqrt{3})(7\sqrt{5}+15)}.\end{aligned}$$

Now

$$16+7\sqrt{5} = (4-\sqrt{5})(9+4\sqrt{5}),$$

$$7+3\sqrt{5} = (3-\sqrt{5})(9+4\sqrt{5}),$$

and the expression reduces to

$$\frac{3\sqrt{5}(4-\sqrt{5})}{3-\sqrt{5}}(-1-i\sqrt{3}).$$

Also, $\tau(1-\tau) = 8(47+21\sqrt{5}) = 8(\sqrt{5}+2)^3(\sqrt{5}-1),$

and $3-\sqrt{5} = (\sqrt{5}-2)(\sqrt{5}+1);$

therefore

$$\begin{aligned}108J &= \frac{1}{8(\sqrt{5}-1)(\sqrt{5}+2)^3} \cdot \frac{27 \cdot 5\sqrt{5}(4-\sqrt{5})^3}{(\sqrt{5}-2)^3(\sqrt{5}+1)^3} \cdot 8, \\ J &= \frac{5\sqrt{5}(4-\sqrt{5})^3}{4^2(\sqrt{5}+1)^2} \\ &= \frac{5\sqrt{5}}{2^8}(4-\sqrt{5})^3(\sqrt{5}-1)^2.\end{aligned}$$

The cases $n = 7$ and $n = 13$ present some reductions of the same character, but even more complicated; I have not thought it worth while to give them explicitly.

The classes corresponding to the different values of J are

$$(1, 0, 4), (2, 1, 4), (1, 1, 4) \sim (1, 0, 3), (2, 1, 8), (4, 1, 4).$$

I have not determined which of the irrational values of J corresponds to $(2, 1, 8)$, and which to $(4, 1, 4)$.

IV. $n = 5$.

$$-1728J = \frac{(r^2-10r+5)^3}{r} = \frac{(r'^2-10r'+5)^3}{r'},$$

where

$$rr' = 125;$$

or, writing $5t, 5t'$ for r, r' , so that $tt' = 5$,

$$-\frac{1728J}{25} = \frac{(5t^3 - 10t + 1)^3}{t} = \frac{(t^3 - 50t + 125)^3}{5t^3},$$

whence

$$625t^{10} - 3750t^9 + 7875t^8 - 6500t^7 + 1574t^6 - 7870t^5 + 162500t^3 - 984375t^2 \\ + 2343750t - 1953125 = 0.$$

This is

$$(t^3 - 5)(625t^8 - 3750t^7 + 11000t^6 - 25250t^5 \\ + 56574t^4 - 126250t^3 + 275000t^2 - 468750t + 390625) = 0.$$

Putting $t = y\sqrt{5}$ in the second factor, we get a reciprocal equation in y ; and writing

$$u = \frac{5\sqrt{5}}{2} \left(y + \frac{1}{y} \right) = \frac{5}{2} \left(t + \frac{5}{t} \right),$$

the equation for u is

$$u^4 - 15u^3 - 15u^2 + 775u - 1386 = 0,$$

that is,

$$(u-2)(u+7)(u-9)(u-11) = 0.$$

Hence the values of t are given by

$$t^3 - 5 = 0,$$

$$5t^3 - 4t + 25 = 0,$$

$$5t^3 + 14t + 25 = 0,$$

$$5t^3 - 18t + 25 = 0,$$

$$5t^3 - 22t + 25 = 0.$$

Calculating the values of J , we have—

$$t = \sqrt{5}, \quad J = -\frac{5\sqrt{5}}{2^3 \cdot 3^3} (\sqrt{5}-1)^3 (\sqrt{5}-2)^3 (\sqrt{5}+4)^3,$$

$$t = -\sqrt{5}, \quad J = +\frac{5\sqrt{5}}{2^3 \cdot 3^3} (\sqrt{5}+1)^3 (\sqrt{5}+2)^3 (4-\sqrt{5})^3,$$

$$t = \frac{2 \pm 11i}{5}, \quad J = \frac{11^3}{2^3},$$

$$t = \frac{-7 \pm 2i\sqrt{19}}{5}, \quad J = -2^0,$$

$$t = \frac{9 \pm 2i\sqrt{11}}{5}, \quad J = -\frac{2^0}{3^3},$$

$$t = \frac{11 \pm 2i}{5}, \quad J = 1.$$

As a verification of the second value of J , we have, for $\Delta = 5$,

$$\begin{aligned}\kappa\kappa' &= \frac{\sqrt{5}-2}{2}, \\ J &= \frac{4}{27} \cdot \frac{(1-\kappa^3\kappa'^3)^3}{\kappa^4\kappa'^4} \\ &= \frac{4}{27} \cdot \frac{\left\{1 - \frac{9-4\sqrt{5}}{4}\right\}^3}{\frac{161-72\sqrt{5}}{16}} \\ &= \frac{1}{27} \cdot \frac{(-5+4\sqrt{5})^3}{161-72\sqrt{5}} \\ &= \frac{5\sqrt{5}(4-\sqrt{5})^3}{27} (161+72\sqrt{5}) \\ &= \frac{1}{8 \cdot 27} \cdot 5\sqrt{5}(4-\sqrt{5})^3(\sqrt{5}+1)^3(\sqrt{5}+2)^3.\end{aligned}$$

The corresponding classes for the last five values of J are

$$\begin{aligned}(1, 0, 5), \quad (1, 1, 5) \sim (1, 0, 4), \quad (2, 1, 10), \\ (2, 3, 10) \sim (2, 1, 6), \quad (1, 2, 5) \sim (1, 0, 1),\end{aligned}$$

and for the first, presumably, $(2, 1, 3)$.

$$\text{V. } n = 7.$$

Writing $7t, 7t'$ for Klein's τ, τ' ,

$$\begin{aligned}1728J &= \frac{(7t^3+13t+7)(49t^3+35t+1)^3}{t} \\ &= \frac{(7t^3+13t+7)(t^3+35t+49)^3}{t^7}.\end{aligned}$$

$$\text{If } 7t^3+13t+7=0, \quad J=0.$$

Next, put

$$t^6(49t^3+35t+1)^3 - (t^3+35t+49)^3 = 0;$$

that is,

$$\begin{aligned}(49t^4+35t^3-35t-49) \\ \times (2401t^8+3430t^7+1372t^6+1820t^5+3629t^4 \\ +1820t^3+1372t^2+3430t+2401) = 0.\end{aligned}$$

The first factor gives

$$(t^2-1)(49t^2+35t+49)=0;$$

putting $7\left(t+\frac{1}{t}\right)=y$ in the other,

$$y^4+10y^2-168y^2-1210y+5687=0,$$

$$\text{or} \quad (y-11)(y+11)(y^2+10y-47)=0,$$

and the remaining equations for t are

$$7t^2-11t+7=0,$$

$$7t^2+11t+7=0,$$

$$7t^2+(5+6\sqrt{2})t+7=0,$$

$$7t^2+(5-6\sqrt{2})t+7=0.$$

Hence we derive the following values for J :—

$$t=1, \quad J=\frac{5^3 \cdot 17^3}{2^6},$$

$$t=-1, \quad J=-\frac{5^3}{2^6},$$

$$t=\frac{-5 \pm 3i\sqrt{19}}{14}, \quad J=-2^9,$$

$$t=\frac{11 \pm 5i\sqrt{3}}{14}, \quad J=-\frac{2^9 \cdot 5^3}{3^2},$$

$$t=\frac{-11 \pm 5i\sqrt{3}}{14}, \quad J=\frac{5^3}{2^9},$$

$$t=\frac{-(5+6\sqrt{2}) \pm i\sqrt{3}(5-2\sqrt{2})}{14}, \quad J=(\sqrt{2}-1)^3(5-2\sqrt{2})^3,$$

$$t=\frac{-(5-6\sqrt{2}) \pm i\sqrt{3}(5+2\sqrt{2})}{14}, \quad J=(\sqrt{2}+1)^3(5+2\sqrt{2})^3,$$

$$t=\frac{-13 \pm 3i\sqrt{3}}{14}, \quad J=0.$$

The classes corresponding to the first five of these are

$$(1, 0, 7), \quad (2, 1, 4), \quad (2, 3, 14) \sim (2, 1, 10),$$

$$(2, 1, 14), \quad (1, 2, 7) \sim (1, 0, 3);$$

to the last belongs

$$(2, 5, 14) \sim (2, 1, 2).$$

Hence we conjecture that the remaining two correspond to $(1, 0, 6)$ and $(2, 0, 3)$; and this may be verified by means of Weber's results contained in *Math. Ann.*, XXXIII. (1889), p. 402. He finds that, for the principal class $(1, 0, 6)$,

$$2 \sqrt{\frac{\kappa'^3}{\kappa}} = 2 \sqrt{2} (\sqrt{2} + 1),$$

or
$$\frac{\kappa'^2}{\kappa} = (2 + \sqrt{2})^2 = (6 + 4\sqrt{2}).$$

Now, if $\frac{\kappa'^2}{\kappa} = x$, it is easily seen that

$$J = \frac{4}{27} \cdot \frac{(1+x^2)^3}{x^3},$$

and this gives, in the present case,

$$\begin{aligned} J &= \frac{4}{27} \frac{(69 + 48\sqrt{2})^3}{(2 + \sqrt{2})^3} = (23 + 16\sqrt{2})^3 (\sqrt{2} - 1) \\ &= \{(5 + 2\sqrt{2})(3 + 2\sqrt{2})\}^3 (\sqrt{2} - 1)^4 \\ &= (5 + 2\sqrt{2})^3 (\sqrt{2} + 1)^3 (\sqrt{2} - 1)^4 \\ &= (5 + 2\sqrt{2})^3 (1 + \sqrt{2})^3, \text{ as above.} \end{aligned}$$

VI. $n = 13$.

Putting $t\sqrt{13}$, $t'\sqrt{13}$ for Klein's τ , τ' ,

$$\begin{aligned} &1728 \sqrt{13} \cdot J \\ &= \frac{(13t^2 + 5\sqrt{13} \cdot t + 13)(169t^4 + 91\sqrt{13} \cdot t^3 + 260t^2 + 19\sqrt{13} \cdot t + 1)^3}{t} \\ &= \frac{(13t^2 + 5\sqrt{13} \cdot t + 13)(t^4 + 19\sqrt{13} \cdot t^3 + 260t^2 + 91\sqrt{13} \cdot t + 169)^3}{t^{13}}. \end{aligned}$$

Hence the following alternative solutions :—

(i.) $13t^2 + 5\sqrt{13} \cdot t + 13 = 0$, that is, $r^2 + 5r + 13 = 0$;

(ii.) $169t^4 + 91\sqrt{13} \cdot t^3 + 260t^2 + 19\sqrt{13} \cdot t$

$$-19\sqrt{13}t^3 - 260t^2 - 91\sqrt{13} \cdot t - 169 = 0;$$

that is,

$$(t^2-1)\{169t^6+91\sqrt{13}\cdot t^5+29t^4+110\sqrt{13}\cdot t^3+429t^2+91\sqrt{13}\cdot t+169\}=0.$$

In the latter factor put

$$y = \sqrt{13}\left(t + \frac{1}{t}\right) = r + \frac{13}{r};$$

then

$$y^3+7y^2-6y-72=0,$$

or

$$(y-3)(y+4)(y+6)=0.$$

Thus the equations for r are

$$r^2-13=0,$$

$$r^3-3r+13=0,$$

$$r^3+4r+13=0,$$

$$r^3+6r+13=0.$$

(iii.) The remaining equation in t is reciprocal, and as follows:—

$$\begin{aligned} 28561t^{16}+30758\sqrt{13}\cdot t^{15}+195533t^{14}+53742\sqrt{13}\cdot t^{13} \\ +113061t^{12}+13364\sqrt{13}\cdot t^{11}+71890t^{10} \\ +44036\sqrt{13}\cdot t^9+208510t^8+\dots=0. \end{aligned}$$

If we write $r + \frac{13}{r} = y$, as before, we find

$$\begin{aligned} y^8+14y^7-15y^6-956y^5-2893y^4+13482y^3 \\ +70959y^2+57348y-43092=0; \end{aligned}$$

that is,

$$(y+2)(y+7)(y^2-27)(y^2-y-38)(y^2+6y-3)=0.$$

Hence the remaining equations for r are

$$r^2+2r+13=0,$$

$$r^2+7r+13=0,$$

$$r^2\pm 3\sqrt{3}\cdot r+13=0,$$

$$r^2-\left(\frac{1\pm 3\sqrt{17}}{2}\right)r+13=0,$$

$$r^2+(3\pm 2\sqrt{3})r+13=0.$$

All the values of τ , and the corresponding values of J , are contained in the following table:—

$$\begin{aligned} \tau &= \pm \sqrt{13}, \\ J &= \frac{5^3}{2^3 \cdot 3^3} (\pm \sqrt{13} - 2)(\pm \sqrt{13} + 4)(\pm \sqrt{13} + 5)^3 (\pm \sqrt{13} + 6)^3 \\ &= \frac{5^3}{2^3} (\pm \sqrt{13} + 3)^3 (\pm \sqrt{13} + 6)^3, \\ \tau &= \frac{-5 \pm 3i\sqrt{3}}{2}, \quad J = 0, \\ \tau &= \frac{3 \pm i\sqrt{43}}{2}, \quad J = -2^{12} \cdot 5^3, \\ \tau &= -2 \pm 3i, \quad J = \frac{11^3}{2^3}, \\ \tau &= -3 \pm 2i, \quad J = 1, \\ \tau &= -1 \pm 2i\sqrt{3}, \quad J = -\frac{2^9 \cdot 5^3}{3^3}, \\ \tau &= \frac{-7 \pm i\sqrt{3}}{2}, \quad J = \frac{5^3}{2^3}, \\ \tau &= \frac{3\sqrt{3} \pm 5i}{2}, \quad J = \frac{5^3}{2^4} (4 - 3\sqrt{3})^3 (3 - 2\sqrt{3})^3, \\ \tau &= \frac{-3\sqrt{3} \pm 5i}{2}, \quad J = \frac{5^3}{2^4} (4 + 3\sqrt{3})^3 (3 + 2\sqrt{3})^3, \\ \tau &= \left(\frac{1 \pm i\sqrt{51}}{2} \right) \left(\frac{1 \mp i\sqrt{3}}{2} \right), \quad J = -2^8 (7 - \sqrt{17})(4 + \sqrt{17})^4, \\ \tau &= \left(\frac{1 \pm i\sqrt{51}}{2} \right) \left(\frac{1 \pm i\sqrt{3}}{2} \right), \quad J = -2^8 (7 + \sqrt{17})(4 - \sqrt{17})^4, \\ \tau &= (3 \pm 2i) \left(\frac{-1 \pm i\sqrt{3}}{2} \right), \\ J &= -\frac{\sqrt{3}}{2^2 \cdot 3^3} (\sqrt{3} - 1)(\sqrt{3} - 1)^3 (3\sqrt{3} - 2)^3 (2\sqrt{3} - 1)^3 \\ &= -\frac{\sqrt{3}}{2^2 \cdot 3^2} (\sqrt{3} - 1)^4 (3\sqrt{3} - 2)^3 (2\sqrt{3} - 1)^3, \\ \tau &= (3 \pm 2i) \left(\frac{-1 \mp i\sqrt{3}}{2} \right), \\ J &= \frac{\sqrt{3}}{2^2 \cdot 3^2} (\sqrt{3} + 1)^4 (3\sqrt{3} + 2)^3 (2\sqrt{3} + 1)^3. \end{aligned}$$

The corresponding classes are

$(1, 0, 13), (2, 1, 7), (2, 7, 26) \sim (2, 1, 2),$
 $(2, 3, 26) \sim (2, 1, 22), (1, 3, 13) \sim (1, 0, 4),$
 $(1, 0, 1), (2, 5, 26) \sim (2, 1, 14), (1, 0, 3),$
 $(1, 1, 13) \sim (1, 0, 12), (3, 0, 4), (2, 1, 26),$
 $(6, 3, 10), (2, 1, 5), (1, 2, 13) \sim (1, 0, 9).$

I may add that I have verified almost all the foregoing results by comparison with those given in the papers already quoted, as well as in Weber's memoirs in the *Acta Mathematica*, Vols. vi. and xi.

Thursday, March 13th, 1890.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

The following communications were made :—

Perfect Numbers : Major Macmahon, R.A.

The Relation of Distortion in Prismatic Images to Dispersion :

Dr. J. Larmor.

On the Satellite of a Line relatively to a Cubic : The President (Prof. Greenhill in the Chair).

An Approximate Relation connecting Successive Terms of the Expansion for $\tan x$: G. Heppel.

The following presents were received :—

"Proceedings of the Royal Society," Nos. 285, 286.

"Educational Times," for March.

"Proceedings of the Physical Society of London," Vol. x., Part III.

"Proceedings of the Royal Irish Academy," Third Series, Vol. i., No. 2 ; Dublin, 1889.

"Transactions of the Royal Irish Academy," Vol. xxix., Part XII. ; Dublin, 1889.

"Chess Telegraphic Codes," by Edwin Anthony, 4to pamphlet ; London, 1890.

"Annual Report of the Canadian Institute," Session 1888-9 ; 8vo ; Toronto, 1889.

"Bulletin des Sciences Mathématiques," Tome xiv., Fev. 1890.

"Beiblätter zu den Annalen der Physik und Chemie," Band xiv., Stück 2.

- "Rendiconti del Circolo Matematico di Palermo," Fasc. 1 e 2; 1890.
 "Atti della R. Accademia dei Lincei—Rendiconti," Vol. v., Fasc. 11–13.
 "Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 99, 100.
 "Acta Mathematica," xiv., 1.
 "Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxiv., Livr. 1^{re}.
 "Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. iii., Fasc. 1–12; Napoli.
 "American Journal of Mathematics," Vol. xii., No. 3; Baltimore, March, 1890.
 "Haverford College Studies," published by the Faculty of Haverford College, No. 1, Pa.

On the Satellite of a Line relatively to a Cubic.

By J. J. WALKER, M.A., F.R.S.

[Read March 13th, 1890.]

Introduction.

1. In a "Memoir," of some length, "On the Diameters of a Plane Cubic" (*Phil. Trans.*, 1888 A., pp. 166–170), the equation of the Satellite of a line was incidentally arrived at in a form (48)(38)(37) [(25) of the present paper], involving the square of the primitive line as an implicit factor. It did not come within the scope of that memoir to discuss the reduction of this form with the generality which its importance authorised, nor were the subsidiary results, on which that general reduction is based, at that time published. I therefore limited myself to verifying the reduction given by reference to special forms of the cubic.

The results referred to having since appeared, as quoted hereinafter, I show in the present paper how the general reduction may be effected; the method of which appears to have some novelty and interest *per se*; and to save the necessity of reference to the "Memoir," as well as to obtain the equation in a slightly simplified form at once—viz., one in which only the first power of the line is implicitly involved—I revert to what is, in fact, the original investigation by which I arrived at it. Four forms of the Satellite Line are deduced (24), (25), (26), (27), in the last two of which the product of the "Quippian" of u and the line L enters.