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XV. *Contributions to the Theory of the Resolving Power of Objectives.* By PROFESSOR J. D. EVERETT, F.R.S.*

IN high class objectives, both of telescopes and microscopes, the practical limit to the power of separating close points (called *resolving power* or *separating power*) depends upon the blurring due to diffraction. Owing to diffraction, the image formed of a bright point is not a point, but a spot, brightest in the central part, and falling off without any discontinuity from the centre to the margin. In favourable circumstances this spot is surrounded by a succession of bright rings. The phenomenon is seen in its greatest perfection when small aperture is combined with good definition. Blocking out the central portion of the objective makes the spot smaller and the surrounding rings relatively brighter.

Dawes (Mem. R. Ast. Soc. xxxv. p. 158) made very elaborate observations on double stars for the purpose of investigating the separating power of telescopes; and arrived at the conclusion that the angular distance between the two components, when they are nearly equal in magnitude, and are just separated, is given by the formula—

4.56 seconds, divided by diameter of objective in inches.

The first calculation of the relative brightness at different points of the spot and rings, which constitute the diffraction image of a point formed by a lens symmetrical round an axis, was published by Airy in 1836 (Camb. Trans. v. p. 283), in a very clear and readable paper. His basis of procedure is the very direct and intelligible one of considering the concave wave-front which advances from the objective to the focus, and computing, for its initial position, the “disturbance” which it produces (according to Huygens’ principle) at any given small distance measured laterally from the geometrical focus.

Another principle of calculation, less obviously correct but leading to precisely the same result, is employed in Mascart’s *Optique* and in Preston’s *Light*.

Both methods of procedure lead to one and the same infinite series for the “disturbance” at given lateral distance from the geometrical focus; and this series is a Bessel’s function of the first order. It is in fact $\frac{2J_1(m)}{m}$, m denoting $\frac{2\pi R}{\lambda f}b$, where R is radius of aperture, f focal length, b lateral distance, and λ wave-length. The calculation assumes identity of disturbance both in degree and in kind at all points of the wave-front.

* Communicated by the Physical Society; read Feb. 28, 1902.

A simple calculation (given at p. 277 of my *Deschanel*, Part iv.) shows that the extreme difference of optical path, for disturbances coming from different points of a concave wave-front to a point at lateral distance b from the geometrical focus (the centre of the sphere to which the wave-front belongs), is $2b \sin \alpha$, α denoting the angular radius of the wave-front as seen from the focus. When the extreme difference of path is λ , we have therefore

$$b = \frac{\lambda}{2 \sin \alpha}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Comparison with observation shows that this value of b represents with fair accuracy the limit of separation. The angle subtended by the distance b at the second nodal point of the objective, which is identical with the angle subtended by the corresponding distance in the object, as seen from the first nodal point, is

$$\frac{b}{f} = \frac{\lambda}{2f \sin \alpha} = \frac{\lambda}{D}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

f being the focal length, and D the diameter of the objective. This formula λ/D for the least distance between the components of a double star, agrees with Dawes's value above quoted, if we put $\lambda = .000022$ inch = .56 micron. The wavelength for the brightest rays is usually taken as .55 micron, which is as good an agreement as could be desired.

Passing now to the case of the microscope, and supposing the same formula for the minimum distance b in the image to be still applicable, we may conveniently transform it by means of the equation. (which we shall discuss later)

$$\mu_1 y_1 \sin \theta_1 = \mu_2 y_2 \sin \theta_2 \quad . \quad . \quad . \quad . \quad (3)$$

applicable to any optical system which gives sharp flat images.

In this equation,

y_1, y_2 denote the distances of a point of the object, and the corresponding point of the image, from the axis of the system ;

μ_1, μ_2 the indices of the first and last media ;

θ_1, θ_2 the angles made with the axis by any incident ray and the corresponding emergent ray.

The ratio $(\mu_1 \sin \theta_1)/(\mu_2 \sin \theta_2)$ is equal to the magnification y_2/y_1 , and is therefore the same for all values of θ . This constancy is called by Abbe the *sine condition*.

In the present case θ_2 is α , y_2 is b , μ_2 is 1 ; and if α_1 denote the obliquity of an extreme incident ray, the equation gives

$$\mu_1 y_1 \sin \alpha_1 = b \sin \alpha, \\ y_1 = \frac{\sin \alpha}{\mu_1 \sin \alpha_1} b = \frac{\sin \alpha}{\mu_1 \sin \alpha_1} \cdot \frac{\lambda}{2 \sin \alpha} = \frac{\lambda}{2 \mu_1 \sin \alpha_1} = \frac{\lambda_1}{2 \sin \alpha_1}. \quad (4)$$

λ_1 denoting the wave-length in the first medium which corresponds to λ in air. This value for y_1 , the distance between points or lines which can be barely separated, has been extensively adopted. Helmholtz in the *Jubelband* of Pogg. *Ann.* 1874, p. 557 adopts it in the form last written. Abbe calls $\mu_1 \sin \alpha_1$ the *numerical aperture* of the objective, and adopts the formula *wave-length divided by twice the numerical aperture*. Drude (*Lehrbuch der Optik*) adopts it as the limiting distance for oblique illumination, and its double as the limiting distance for direct illumination.

Microscopic test-objects are not self-luminous like double stars, but are viewed by transmitted light. If no condensing arrangement is employed, the pencil of light sent by a point of the object to the objective consists of rays from different parts of the source, that is, in effect, from different sources. An orthogonal section of such a pencil does not possess the characteristic properties of a wave-front. Different portions of it have no definite relation of phase, and are incapable of mutual interference. Our formulæ are therefore no longer applicable. Practically we may regard such an orthogonal section as made up of a number of small parts, each of which is a wave-front, giving by reason of its smallness a very large diffraction image of a point of the object. These separate images of the same point overlap without interference, and as they do not exactly coincide, compose a larger and more blurred image of the point represented.

The cure for this evil is furnished by employing a condenser of high quality, to throw upon the part of the object under examination a very sharp image of the source of illumination. If the image were perfectly sharp, each point of the object would get its light from its own special point of the source, and the effect would be to make the object act as if it were self-luminous. Each point of the object would send to the objective a pencil whose orthogonal sections would be true wave-fronts, to which our previous reasoning would be applicable; so that the diffraction spot which represents a point would have the small size due to the largeness of the entire aperture.

This appears to me to be the chief benefit conferred by sharply focussing the source on the object; but it has not so far as I am aware been pointed out by any writer on the microscope hitherto. Abbe in his great paper on microscopic perception (*Archiv für mikr. Anat.* ix. p. 413, 1873) regards the condenser merely from the point of view of geometrical optics, and recommends the use of one which is not achromatic. Microscopic observers long ago ascertained, as an empirical fact, that achromatic condensers gave better results than non-

achromatic ; while mathematicians refused to believe them, and maintained that achromatism could be of no advantage, seeing that the sole purpose of a condenser was to give wide pencils of strong light.

There is another advantage from sharp focussing by the condenser, which may be regarded as the complement of that above indicated. If the focussing were perfectly sharp, the waves from one point of the object could not interfere with waves from another point. Such interference gives rise to spurious diffraction patterns, liable to be mistaken for structures existing in the object. The two components of a double star exhibit no mutual interference in a telescope ; and different points of a microscopic object cannot produce mutual interference if they send light which has come from completely distinct sources. Lord Rayleigh (*Phil. Mag.* xlii. 1896) was, I believe, the first to indicate this advantage. Abbe, in his paper on microscopic perception, makes no allusion either to the focussing of the source on the object, or to the finite size of the spot which (with its surrounding rings) is the diffraction image of a single isolated point.

The following explanation of the advantage of oblique illumination is, I believe, new.

Perfect sharpness of focussing by a condenser is unattainable ; and two points of the object which are not further apart than twice the limiting distance of separability will inevitably have a portion of the source in common, as regards their illumination. Let β denote the obliquity of the illumination, the two object-points in question being supposed to be in a plane which contains the illuminating rays and the axis of the objective. The difference of optical path for rays coming from the same point of the source to the two object-points is $s \sin \beta$, s denoting the distance between the two points. The best condition for separation is, that this difference of path shall be half a wave-length in the medium in which the object is immersed (say $\frac{1}{2}\lambda_1$), for this gives the most complete extinction in the overlapping portion of the two diffraction spots which are the images of the two points. Putting then

$$s \sin \beta = \frac{1}{2}\lambda_1, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and assigning to s the value $\lambda_1/\sin \alpha_1$, which being double of the accepted minimum value may be taken as representing an ordinary test, we deduce

$$\sin \beta = \frac{1}{2} \sin \alpha_1. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we put s equal to the accepted minimum itself, we obtain

$$\sin \beta = \sin \alpha_1. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These conclusions agree with the received view among

microscopists, that the obliquity of illumination should be rather less than the obliquity of the extreme rays of the incident pencil.

Note on Hockin's proof of the Sine Condition.

Various proofs have been given of the sine condition expressed by equation (3), which must be fulfilled in every case in which a sharp image, in a plane perpendicular to the axis of the instrument, is formed of a small flat object whose plane is perpendicular to the axis. By far the simplest is that given in an article "On the Estimation of Aperture in the Microscope," published after the author's death in the *Journal of the Royal Microscopical Society* (1881, ser. 2, iv. p. 337), where he is described as the late Mr. Charles Hockin, junr., an electrician and mathematician of repute. Appreciative notes by Abbe are inserted in the article. Strange to say, the proof does not seem to have been reproduced in any English publication, though it is to be found, modified for the worse, in German optical treatises. In Müller-Pouillet it is erroneously described, and the author's name is given as John Hockin. These circumstances, in conjunction with the great importance of the theorem itself, are my reasons for reproducing it. I have corrected a clerical error of $-$ for $+$ in the two principal equations.

Let PP' in the figure represent the axis of an optical

system which gives the linear image $P'Q'$ of the small object-line PQ , both the lines PQ and $P'Q'$ being perpendicular to the axis. The incident pencils may be of large angle; and the image is supposed to be *aplanatic*, that is to say, all rays sent by P pass through P' , and all rays sent from Q pass through Q' . Let PS be any one of the rays sent from P , and QS a ray from Q intersecting it at S .

Since PQ is small, the angle PSQ is small, and the plane pencil bounded by PS , QS will give an emergent pencil bounded by $P'S'$, $Q'S'$, the optical path from S to S' having the same value for all the rays of the pencil; denote this value by (SS') .

Then, if μ be the index of the first and μ' that of the last medium, the optical path from P to P' is

$$\mu \cdot PS + (SS') - \mu' \cdot P'S',$$

and is constant for all rays that go from P to P'. Similarly the optical path from Q to Q' is

$$\mu \cdot QS + (SS') - \mu' \cdot Q'S',$$

and is constant. Subtracting, and denoting the difference of the two constants by c , we have

$$\mu(PS - QS) - \mu'(P'S' - Q'S') = c,$$

or (calling the obliquities θ, θ'),

$$\mu \cdot PQ \sin \theta - \mu' P'Q' \sin \theta' = c.$$

But θ and θ' vanish together, therefore c is zero; and we have

$$\mu \cdot PQ \sin \theta = \mu' \cdot P'Q' \sin \theta'. \quad . \quad . \quad (3)$$

The ratio of $\mu \sin \theta$ to $\mu' \sin \theta'$ has therefore the constant value $P'Q'/PQ$ for all the rays by which the image is formed.

The present paper has been framed with a view to supplementing two papers by Lord Rayleigh, one of them (in two parts) in the *Phil. Mag.* for the second half of 1879, and the other (which has been already quoted) in *Phil. Mag.* vol. xlii. 1896. They contain a much fuller treatment of the theory of resolution than I have met with elsewhere.

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XVI. Notices respecting New Books.

Histoire de l'Observatoire de Paris de sa fondation à 1793. Par C. WOLF, Membre de l'Institut, Astronome Honoraire de l'Observatoire. Paris: Gauthier-Villars, Imprimeur-Libraire.

THIS is the first part of a work in which it is proposed to give a history, not of the astronomical observations and results which have been accomplished at the famous Observatory of Paris, which was founded about eight years earlier than that at Greenwich, but of its buildings, instruments, and the *personnel* composing its staff, with the successive modifications and additions in these. It is well known that the Paris Observatory, for more than the first century and a quarter of its existence, was directed by four generations of the family of Cassini, the first of whom was invited from Italy by Colbert, and eventually appointed to take charge of the new establishment. The fourth, also J. D. Cassini, was in charge at the time when the great Revolution transformed not only the rule of the Observatory, but that of all France. Before leaving his post, Cassini formed the project of writing a complete history of the establishment, and requested of the Director of the royal buildings that a search should be made in the public archives for the documents relating to the foundation of the Observatory and its early annals. Though the answer gave him very little encouragement in his scheme, he took the trouble to gather together all the accessible papers which were left by his ancestors, particularly his great-grandfather: these he presented in 1811 to the Bureau des Longitudes, and they form one of the principal sources of information