On certain Differential Operators, and their use to form a Complete System of Seminvariants of any Degree, or any Weight.

By Prof. ELLIOTT, F.R.S. Received and read February 14th, 1895.

1. There is a one-to-one correspondence between seminvariants in the unending series of letters $a, b, c, d, e, \dots$, and products of these letters which have been called by MacMahon power enders, *i.e.*, products which when arranged from the beginning in alphabetical order end in a higher power of a letter than the first. Thus $abc^2, c^3, ad^2, b^3ef^2, \dots$ are power ending products.

This fact has been exhibited in various lights by Major MacMahon and the late Professor Cayley—alas! that we must now say the late—in a number of papers in the *American Journal of Mathematics*. The main object of this short paper is to exhibit the fact in another light by showing that a complete system of seminvariants may be deduced from a complete system of power ending products, one from one, by differential operations only. Two ways of doing this will be arrived at.

2. We regard $a, nb, \frac{n(n-1)}{1.2}c, \dots$ as the coefficients in a binary quantic of infinite order n , or, as will do equally well, of a binary quantic of order n which exceeds w , the weight of gradients of the type with which we are at any time dealing. It is well known that, if any seminvariant, obtained for such a quantic, involves the first $j+1$ letters only, then it is equally a seminvariant of the j -ic whose coefficients are $a, jb, \frac{j(j-1)}{1.2}c, \dots$.

$$\begin{aligned} \text{Let} \quad \Omega &= a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \dots \text{ to } \infty, \\ O_1 &= b\partial_a + c\partial_b + d\partial_c + e\partial_d + \dots \text{ to } \infty, \\ O_2 &= c\partial_b + 2d\partial_c + 3e\partial_d + \dots \text{ to } \infty, \end{aligned}$$

where, for instance, ∂_b means $\frac{d}{db}$. We have, if the operators stop at weight n ,

$$nO_1 - O_2 \equiv 0,$$

where O is the second annihilator of invariants of the binary n -ic.

We at once see that

$$\begin{aligned}\Omega O_1 - O_1 \Omega &= a\partial_a + b\partial_b + c\partial_c + \dots \text{ to } \infty \\ &= i,\end{aligned}$$

when the operation is on a homogeneous function of degree i ,

$$\begin{aligned}\Omega O_1^2 - O_1^2 \Omega &= (\Omega O_1 - O_1 \Omega) O_1 + O_1 (\Omega O_1 - O_1 \Omega) \\ &= iO_1 + O_1 i = 2iO_1, \\ \Omega O_1^3 - O_1^3 \Omega &= (\Omega O_1 - O_1 \Omega) O_1^2 + O_1 (\Omega O_1 - O_1 \Omega) O_1 + O_1^2 (\Omega O_1 - O_1 \Omega) \\ &= 3iO_1^2,\end{aligned}$$

and generally $\Omega O_1^k - O_1^k \Omega = kiO_1^{k-1}$.

Thus, if u be any homogeneous function of order i ,

$$\begin{aligned}(\Omega O_1 - O_1 \Omega) u &= iu, \\ (\Omega O_1^2 - O_1^2 \Omega) \Omega u &= 2iO_1 \Omega u, \\ (\Omega O_1^3 - O_1^3 \Omega) \Omega^2 u &= 3iO_1^2 \Omega^2 u, \\ (\Omega O_1^4 - O_1^4 \Omega) \Omega^3 u &= 4iO_1^3 \Omega^3 u, \\ &\&c. \quad \&c.,\end{aligned}$$

and if u be rational and integral and of weight w , *i.e.*, be a gradient of type w, i , these equalities from the $(w+2)^{\text{th}}$ onwards are mere identities of zeroes, for $\Omega^{w+1}u = 0$.

Hence, by addition of multiples of the above, so chosen that terms on the right except the first cancel against second terms on the left,

$$\left\{ \frac{1}{i} \Omega O_1 - \frac{1}{1 \cdot 2 \cdot i^2} \Omega O_1^2 \Omega + \frac{1}{1 \cdot 2 \cdot 3 \cdot i^3} \Omega O_1^3 \Omega^2 - \dots \right\} u = u,$$

in which the series on the left certainly terminates in consequence of

$$\Omega^{-1}u = 0,$$

so that no doubt arising from questions of convergency can enter.

The result may be written

$$\left\{ 1 - \frac{1}{i} \frac{\Omega O_1}{1} + \frac{1}{i^2} \frac{\Omega O_1^2 \Omega}{1 \cdot 2} - \frac{1}{i^3} \frac{\Omega O_1^3 \Omega^2}{1 \cdot 2 \cdot 3} + \dots \right\} u = 0,$$

$$\text{or, again, } u = \Omega O_1 \left\{ \frac{1}{i} - \frac{1}{i^2} \frac{O_1 \Omega}{2} + \frac{1}{i^3} \frac{O_1^2 \Omega^2}{2 \cdot 3} - \dots \right\} u,$$

in which form it shows that any gradient u is of the form Ωv . Moreover, a possible v corresponding to any u is of the form $O_1 W$.

This result might have been obtained by putting $n = \infty$ in a paper of my own (*Proceedings*, Vol. xxiii., pp. 298, &c.) on the exactness of Cayley's number of seminvariants of a given type. It was there proved that, when the number $n+1$ of $a, b, c, d, \dots$ is finite, any gradient in them can be derived by operation with Ω on another gradient in them if $n-2w$ is positive, as is, of course, the case when n is infinite. We now see that in all cases a gradient can be derived by operation with Ω from another gradient, if the latter gradient be allowed to involve a longer series of the letters $a, b, c, d, e, \dots$ than the former does.

3. Again, returning to the definitions in § 2,

$$\Omega O_1 - O_1 \Omega = 2 (b\partial_b + 2c\partial_c + 3d\partial_d + \dots \text{to } \infty)$$

$$= 2w,$$

$$\Omega O_1^2 - O_1^2 \Omega = (\Omega O_1 - O_1 \Omega) O_1 + O_1 (\Omega O_1 - O_1 \Omega)$$

$$= 2 (w+1) O_1 + O_1 \cdot 2w,$$

since, if w is the weight of u , $w+1$ is that of $O_1 u$,

$$= 2 (2w+1) O_1,$$

$$\Omega O_1^3 - O_1^3 \Omega = (\Omega O_1 - O_1 \Omega) O_1^2 + O_1 (\Omega O_1 - O_1 \Omega) O_1 + O_1^2 (\Omega O_1 - O_1 \Omega)$$

$$= 2 (w+2) O_1^2 + O_1 \cdot 2 (w+1) O_1 + O_1^2 \cdot 2w$$

$$= 3 (2w+2) O_1^2,$$

and generally

$$\Omega O_1^k - O_1^k \Omega = \{2 (w+k-1) + 2 (w+k-2) + \dots + 2w\} O_1^{k-1}$$

$$= k (2w+k-1) O_1^{k-1}.$$

Thus, u being, as before, a gradient of weight w ,

$$(\Omega O_1 - O_1 \Omega) u = 2wu,$$

$$(\Omega O_1^2 - O_1^2 \Omega) \Omega u = 2 (2w-1+1) O_1 \Omega u,$$

for Ωu is of weight $w-1$,

$$= 2 (2w-1) O_1 \Omega u,$$

$$(\Omega O_1^3 - O_1^3 \Omega) \Omega^2 u = 3 (2w-2+2) O_1^2 \Omega^2 u$$

$$= 3 (2w-2) O_1^2 \Omega^2 u,$$

&c. &c.,

and, generally,

$$\begin{aligned} (\Omega O_3^k - O_3^k \Omega) \Omega^{k-1} u &= k (2w - k + 1 + k - 1) O_2^{k-1} \Omega^{k-1} u \\ &= k (2w - k + 1) O_2^{k-1} \Omega^{k-1} u. \end{aligned}$$

Here for the value $w+2$ of k , and for higher values, both sides vanish. Moreover, the value $w+1$ of k makes the factor $2w-k+1$ equal to w , which does not vanish when w is positive, as it must be except for gradients a' involving a only. Also all lower values of k make this positive, and so non-vanishing. Thus, eliminating right-hand sides except the first,

$$\left\{ 1 - \frac{\Omega O_2}{1.2w} + \frac{\Omega O_2^2 \Omega}{1.2.2w(2w-1)} - \frac{\Omega O_2^3 \Omega^2}{1.2.3.2w(2w-1)(2w-2)} + \dots \right. \\ \left. + (-1)^{w+1} \frac{\Omega O_2^{w+1} \Omega^w}{1.2.3 \dots (w+1).2w(2w-1) \dots w} \right\} u = 0,$$

which holds for all gradients u except those of weight zero.

Hence

$$u = \Omega O_2 \left\{ \frac{1}{1.2w} - \frac{O_2 \Omega}{1.2.2w(2w-1)} + \frac{O_2^2 \Omega^2}{1.2.3.2w(2w-1)(2w-2)} - \dots \right\} u,$$

in which the series certainly terminates in $w+1$ terms at most, no difficulty arising in consequence of vanishing of denominators or questions of convergency.

We have thus again the same result as before, excluding the case of weight zero, for which no information is this time given, that a gradient u is necessarily of the form Ωv . We have also the new information that a possible v for any u is of the form $O_2 W$. This last is not, of course, the case when u is of zero weight, *i.e.*, is a' , so that

$$v = \frac{1}{i} a^{i-1} b,$$

there being no seminvariant of unit weight to add; for every term in an $O_2 W$ must contain either c or d or $\&c$.

4. Now, take any gradient whatever of type w, i , and call it u . Ωu is then a gradient of type $w-1, i$, and may, as follows from either of the last two articles, be any gradient whatever of that type. Put Ωu

for u in the two results proved. They become

$$(1) \Omega \left\{ 1 - \frac{1}{i} \frac{O_1 \Omega}{1} + \frac{1}{i^2} \frac{O_1^2 \Omega^2}{1.2} - \frac{1}{i^3} \frac{O_1^3 \Omega^3}{1.2.3} + \dots \right\} u = 0,$$

$$(2) \Omega \left\{ 1 - \frac{1}{2w-2} \frac{O_1 \Omega}{1} + \frac{1}{(2w-2)(2w-3)} \frac{O_1^2 \Omega^2}{1.2} \right. \\ \left. - \frac{1}{(2w-2)(2w-3)(2w-4)} \frac{O_1^3 \Omega^3}{1.2.3} + \dots \right. \\ \left. \dots + (-1)^w \frac{1}{(2w-2)(2w-3) \dots (w-1)} \frac{O_1^w \Omega^w}{1.2.3 \dots w} \right\} u = 0,$$

the first being general, and the second applying when w is 2 or a greater number. Write these

$$(1) \Omega \{1 - \mathfrak{S}_1\} u = 0,$$

$$(2) \Omega \{1 - \mathfrak{S}_2\} u = 0,$$

where $\mathfrak{S}_1, \mathfrak{S}_2$ are differential operators whose full expressions are given. We have thus two conclusions:—

(1) u being any gradient whatever, $(1 - \mathfrak{S}_1) u$ either vanishes or is a seminvariant.

(2) u being any gradient of weight 2 or more, $(1 - \mathfrak{S}_2) u$ either vanishes or is a seminvariant.

Also, regarding the forms of $\mathfrak{S}_1, \mathfrak{S}_2$, which are

$$O_1 \{ \dots \} u,$$

$$O_2 \{ \dots \} u,$$

we draw the following conclusions:—

(1) If $(1 - \mathfrak{S}_1) u$ is not a seminvariant but vanishes, u is of the form $O_1 v$.

(2) If $(1 - \mathfrak{S}_2) u$ is not a seminvariant but vanishes, u is of the form $O_2 v$.

5. Now, we know that the number of aszygetic seminvariants of type w, i is

$$(w; i, \infty) - (w-1; i, \infty),$$

where $(w; i, \infty)$ is the number of products of $a, b, c, d, e, \dots$ of weight w and degree i . Now, this difference is also the number of power ending products of type w, i of $a, b, c, d, e, \dots$. For every non-

power ender of type w, i can be obtained from a product of type $w-1, i$ by replacing its last letter, once only, by the next most advanced letter, and no two of type $w-1, i$ thus altered give the same one of type w, i , but every one of the first type gives one of the second.

What will be proved is that there is a one-to-one correspondence between the seminvariants and the power enders; in fact, that every power ender when operated on by either $1-\mathfrak{S}_1$ or $1-\mathfrak{S}_2$ gives a seminvariant, and that no two give the same; or, in other words, that the most general seminvariant of type w, i is obtained by operating either with $1-\mathfrak{S}_1$ or with $1-\mathfrak{S}_2$ on the most general linear function of the $(w; i, \infty) - (w-1; i, \infty)$ power enders.

This will follow if we can show that no power ending product or linear function of power ending products can be of the form $O_1 v$, or equally if we can show that none is of the form $O_2 v$. If none is of the form $O_1 v$, then, by (1) of the preceding article, every one when operated on by $1-\mathfrak{S}_1$ gives a seminvariant and not a zero. Moreover, all the seminvariants thus given are linearly independent if no linear function of the power ending products is of the form $O_1 v$; for, if $P, P', P'', \dots$ be the power ending products, and $S, S', S'', \dots$ the derived seminvariants, so that

$$(1-\mathfrak{S}_1)P = S, \quad (1-\mathfrak{S}_1)P' = S', \quad (1-\mathfrak{S}_1)P'' = S'', \dots,$$

and if $\lambda S + \lambda' S' + \lambda'' S'' + \dots = 0$,

then $(1-\mathfrak{S}_1)(\lambda P + \lambda' P' + \lambda'' P'' + \dots) = 0$,

and this would require that $\lambda P + \lambda' P' + \lambda'' P'' + \dots$ be of the form $O_1 v$.

And quite similarly with regard to $\mathfrak{S}_2$ and O_2 .

6. We have then only to prove, either that no linear function of power ending products is of the form $O_1 v$, or that none is of the form $O_2 v$. Either fact will suffice; but both are proved with equal ease as follows.

Take v any gradient whatever. Call the most advanced letter which occurs in it a_r , so that

$$v = A + a_r B + a_r^2 C + a_r^3 D + \dots,$$

where $A, B, C, D, \dots$ are free from a_r , and $B, C, D, \dots$ do not all vanish. Then

$$\begin{aligned} O_1 v &= \left(\dots + a_{r+1} \frac{d}{da_r} \right) v \\ &= a_{r+1} (B + 2a_r C + 3a_r^2 D + \dots) + \text{terms free from } a_{r+1}. \end{aligned}$$

Thus, $B, C, D, \dots$ not all vanishing, and being free from a_r and a_{r+1} , $O_1 v$ must contain terms which end in the single power a_{r+1} , and so are not power enders. Thus

$$\lambda P + \lambda' P' + \lambda'' P'' + \dots$$

cannot be of the form $O_1 v$.

Neither in like manner can it be of the form $O_2 v$.

It is then completely established, and in two ways, that all the linearly independent seminvariants of type w_i are obtained from all the different power ending products of that type, one from each, by differential operation. Operation either with $1 - S_1$ or $1 - S_2$ suffices, the former applying to all cases whatever, the latter to all cases where the weight is 2 or more.

In fact, the latter gives all seminvariants just as much as the former. For there are no power ending products, nor seminvariants, of unit weight, and there is no failure for the zero weight of a^i .

Notes on the Theory of Groups of Finite Order. By W. BURNSIDE.

Read February 14th, 1895. Received February 18th, 1895.

[June 21st, 1895.]

Shortly after the following notes had been communicated to the Society, I became acquainted with a paper by Herr G. Frobenius, "Ueber auflösbare Gruppen," *Berliner Sitzungsberichte*, May, 1893. In this paper Herr Frobenius has completely anticipated the result of my Note III. I have, however, with the consent of the referees, allowed the note to stand in its original form; as, had I replaced it by a reference to Herr Frobenius' work, the remainder of my paper would not have been self-contained, and in places would have been barely intelligible. In the paper mentioned Herr Frobenius makes two other statements that bear on the present communication. He says that on some other occasion he will prove that simple groups whose orders are of the forms

$$p_1^m p_2 \quad \text{and} \quad p_1 p_2 \dots p_{n-2} p_{n-1}^2 p_n^m,$$

where $p_1 < p_2 < \dots < p_n$, and the symbols all represent primes, do not exist. I cannot find that he has yet done so.

Secondly, he states his *belief* that the only non-soluble groups (*i.e.*, groups whose factors of composition do not consist entirely of primes), whose orders are the products of five primes, are two groups