XXXVI. A new analytical demonstration of the "Parallelogram of forces."

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less of the grosser æther, in proportion of the finer, than in the regions of the air; and that yet the grosser æther in the air affects the upper regions of the earth, and the finer æther in the earth the lower regions of the air, in such a manner, that from the top of the air to the surface of the earth, and again from the surface of the earth to the centre thereof, the æther is insensibly finer and finer. Imagine now any body suspended in the air, or lying on the earth, and the æther being by the hypothesis grosser in the pores, which are in the upper parts of the body, than in those which are in its lower parts, and that grosser æther being less apt to be lodged in those pores than the finer æther below, it will endeavour to get out and give way to the finer æther below, which cannot be, without the bodies descending to make room above for it to go out into.

From this supposed gradual subtilty of the parts of æther some things above might be further illustrated and made more intelligible; but by what has been said, you will easily discern whether in these conjectures there be any degree of probability, which is all I aim at. For my own part, I have so little fancy to things of this nature, that had not your encouragement moved me to it, I should never, I think, have thus far set pen to paper about them. What is amiss, therefore, I hope you will the more easily pardon in

Your most humble servant and honourer,

Cambridge, Feb. 28, 1678-9.

ISAAC NEWTON.

XXXVI. *A New Analytical Demonstration of the "Parallelogram of Forces."* By THOMAS WEDDLE, Esq., Newcastle-on-Tyne*.

THE following investigation of this fundamental proposition, whatever may be its defects, has at least one advantage—the most general case is discussed at once. In all the analytical proofs that I have seen, a particular case only is established analytically; thus Laplace (*Mécanique Céleste*) and Pontécoulant (*Système du Monde*) consider the forces to act at right angles; and Poisson (*Mécanique*) first finds the resultant of two equal forces, and afterwards thence deduces, by *geometrical* considerations, that of any two forces. I have here attempted to conduct the whole investigation analytically, and to do so without first establishing a particular case.

I shall assume that the resultant of two equal forces bisects

* Communicated by the Author.

the angle contained by their directions, and is never greater than their sum.

Let the forces P, Q, R , acting upon the point O in the directions OP, OQ, OR , be in equilibrium, so that R , acting in the opposite direction RO , is the resultant of P and Q . Denote the angles QOR, ROP , and POQ by α, β , and γ . Make $\angle ROQ' = \angle ROQ = \alpha$, and $\angle ROP' = \angle ROP = \beta$; and to the point O apply in the direction OP' , a force $P' = P$; in OQ' , a force $Q' = Q$; and in OR , a force $R' = R$; hence the forces P', Q', R' are respectively equal to P, Q, R , and the angles contained by the directions of the former forces are equal to those contained by the directions of the latter; now P, Q, R are in equilibrium, hence P', Q', R' are also in equilibrium; wherefore the forces P, P', Q, Q' , and $R + R'$, whose directions are OP, OP', OQ, OQ' , and OR , must balance each other, and consequently the resultant of the first four forces must be equal and opposite to $R + R'$.

Moreover, since $\angle ROP = \angle ROP' = \beta$, RO (produced, if necessary) will bisect the angle POP' ; hence the resultant R_1 of the equal forces P and P' will act in RO , and R_1 will be a function of P and β ; now, since the numerical values of R_1 and P vary with the unit of force, and that of β is independent of it, the ratio $\frac{R_1}{P}$ must be independent of this unit, and consequently a function of β only; denote this function by $2 \phi \beta$, $\therefore R_1 = 2 P \cdot \phi \beta$.

Now R_1 cannot, for any value of β , exceed $P + P'$ or $2 P$, wherefore if $\psi \beta = \cos^{-1} \phi \beta$, the value of $\psi \beta$ will always be real, and we may therefore assume

$$R_1 = 2 P \cdot \cos \psi \beta.$$

Hence also, if R_2 be the resultant of the equal forces Q and Q' , we must have

$$R_2 = 2 Q \cdot \cos \psi \alpha.$$

Moreover, since R_1 and R_2 act in the same straight line, $R_1 + R_2$ is the resultant of P, P', Q, Q' ; and consequently $R_1 + R_2 = R + R' = 2 R$,

$$\therefore R = P \cos \psi \beta + Q \cos \psi \alpha. \quad (1.)$$

$$\text{Similarly,} \quad Q = P \cos \psi \gamma + R \cos \psi \alpha, \quad (2.)$$

$$\text{and} \quad P = Q \cos \psi \gamma + R \cos \psi \beta, \quad (3.)$$

α, β and γ being connected by the equation

$$\alpha + \beta + \gamma = 2\pi. \quad (4.)$$

Eliminating P and Q from (1. 2. 3.), R will disappear at the same time, and there results,

$$\left. \begin{aligned} 1 - \cos^2 \psi \alpha - \cos^2 \psi \beta - \cos^2 \psi \gamma \\ - 2 \cos \psi \alpha \cdot \cos \psi \beta \cdot \cos \psi \gamma = 0. \end{aligned} \right\} \quad (5.)$$

Solve this equation for $\cos \psi \gamma$,

$$\begin{aligned} \therefore \cos \psi \gamma &= -\cos \psi \alpha \cdot \cos \psi \beta \pm \sqrt{(1 - \cos^2 \psi \alpha)(1 - \cos^2 \psi \beta)} \\ &= -\cos \psi \alpha \cdot \cos \psi \beta \pm \sin \psi \alpha \cdot \sin \psi \beta \\ &= -\cos \{\psi \alpha \pm \psi \beta\}; \end{aligned}$$

$$\therefore (2m+1)\pi \mp \psi \gamma = \psi \alpha \pm \psi \beta,$$

$$\text{or} \quad \psi \alpha \pm \psi \beta \pm \psi \gamma = (2m+1)\pi, \quad (6.)$$

m (as well as n, p, q and r employed below) being a positive or negative whole number. Again, when $\alpha = \beta = \gamma = \frac{2}{3}\pi$, we have

$$(1. 2. 3.), \cos \psi \left(\frac{2}{3}\pi \right) = \frac{1}{2} = \cos \left(2n \pm \frac{1}{3} \right) \pi,$$

$$\therefore \psi \left(\frac{2}{3}\pi \right) = \left(2n \pm \frac{1}{3} \right) \pi. \quad (7.)$$

Now the left-hand member of (6.) may be written four different ways: $\psi \alpha + \psi \beta + \psi \gamma$, $\psi \alpha + \psi \beta - \psi \gamma$, $\psi \alpha - \psi \beta + \psi \gamma$, or $\psi \alpha - \psi \beta - \psi \gamma$; but when $\psi \alpha = \psi \beta = \psi \gamma = \psi \left(\frac{2}{3}\pi \right)$, the last three give

$$\psi \left(\frac{2}{3}\pi \right) = (2m+1)\pi, \text{ or } -(2m+1)\pi,$$

which is inconsistent with (7.); hence we must have

$$\psi \alpha + \psi \beta + \psi \gamma = (2m+1)\pi. \quad (8.)$$

Differentiate (8.) first with respect to α , and then with respect to β , observing that (4.) $\frac{d\gamma}{d\alpha} = \frac{d\gamma}{d\beta} = -1$,

$$\therefore \psi' \alpha = \psi' \beta = \psi' \gamma, \quad (9.)$$

where, as usual, $\psi' \alpha = \frac{d}{d\alpha} \cdot \psi \alpha$. Now two of the quantities α, β, γ being arbitrary (4.), it is evident that (9.) cannot be satisfied but by $\psi' \alpha = \text{constant} = c$; hence integrating

$$\psi \alpha = c\alpha + c_1. \quad (10.)$$

To determine the constants c, c_1 , we observe that when

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$\alpha = 0$ and $\beta = \gamma = \pi$, the forces all act in the same straight line, P in one direction and Q and R in the opposite; hence $P = Q + R$: and we derive from (1. 2. 3.),

$$\cos \psi 0 = -1 = \cos (2p + 1)\pi \text{ and } \cos \psi \pi = 1 = \cos 2q\pi, \\ \therefore \psi 0 = (2p + 1)\pi \text{ and } \psi \pi = 2q\pi;$$

hence (10.) gives

$$c_1 = (2p + 1)\pi, \text{ and } c\pi + c_1 = 2q\pi, \text{ or } c = 2(q - p - 1) \\ + 1 = 2r + 1 \text{ (suppose),}$$

$$\therefore (10.) \text{ becomes } \psi \alpha = (2r + 1)\alpha + (2p + 1)\pi,$$

and

$$\cos \psi \alpha = \cos \{(2r + 1)\alpha + (2p + 1)\pi\} = -\cos (2r + 1)\alpha. \quad (11.)$$

Moreover, if

$$\alpha = \frac{\pi}{2r + 1}, \text{ and } \beta = \gamma = \pi - \frac{1}{2r + 1} \cdot \frac{\pi}{2},$$

we shall have (11.),

$$\cos \psi \alpha = 1 \text{ and } \cos \psi \beta = \cos \psi \gamma = 0,$$

$$\therefore (1. 2. 3.) Q = R \text{ and } P = 0;$$

hence the equal forces Q, R acting at an angle $\frac{\pi}{2r + 1}$, balance each other; but this is impossible, unless $r = 0$, when they act in the same straight line in opposite directions. Make therefore $r = 0$ in (11.), and we have finally,

$$\cos \psi \alpha = -\cos \alpha. \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

The equations (1, 2, 3.) therefore become,

$$R = -P \cos \beta - Q \cos \alpha. \quad . \quad . \quad . \quad . \quad (13.)$$

$$Q = -P \cos \gamma - R \cos \alpha. \quad . \quad . \quad . \quad . \quad (14.)$$

$$P = -Q \cos \gamma - R \cos \beta. \quad . \quad . \quad . \quad . \quad (15.)$$

Multiply these equations by R, Q and P respectively, and we readily get

$$R^2 = P^2 + Q^2 + 2PQ \cos \gamma. \quad . \quad . \quad . \quad (16.)$$

Eliminate P, and then Q, from any two of the equations (13. 14. 15.), and reduce by means of (4.),

$$\therefore \frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma}. \quad . \quad . \quad . \quad . \quad (17.)$$

After having (by any means) obtained the form of $\phi \alpha$ ($= \cos \psi \alpha$), the formula (17.) may be found in the following manner:—

Produce RO to R', draw GOH perpendicular to RO R',

and make $\angle R'OP' = \angle ROP$, and $\angle R'OQ' = \angle ROQ$. In the directions OP' , OQ' and OR' , apply the forces P' , Q' and R' equal to P , Q and R respectively; these six forces will, as before, be in equilibrium, and R and R' being equal and opposite may be removed; and hence P , P' , Q and Q' will balance each other. Now since

$$\angle POH = \angle P'OH = \frac{3\pi}{2} - \beta,$$

and $\angle QOG = \angle Q'OG = \frac{3\pi}{2} - \alpha,$

the resultant of the equal forces

$$P \text{ and } P' = 2P \cdot \phi\left(\frac{3\pi}{2} - \beta\right),$$

and that of Q and $Q' = 2Q \cdot \phi\left(\frac{3\pi}{2} - \alpha\right);$

hence, as there is equilibrium, we must have

$$2P \cdot \phi\left(\frac{3\pi}{2} - \beta\right) = 2Q \cdot \phi\left(\frac{3\pi}{2} - \alpha\right);$$

$$\therefore \frac{P}{\phi\left(\frac{3\pi}{2} - \alpha\right)} = \frac{Q}{\phi\left(\frac{3\pi}{2} - \beta\right)};$$

similarly, it is found that

$$\frac{Q}{\phi\left(\frac{3\pi}{2} - \beta\right)} = \frac{R}{\phi\left(\frac{3\pi}{2} - \gamma\right)},$$

$$\therefore \frac{P}{\phi\left(\frac{3\pi}{2} - \alpha\right)} = \frac{Q}{\phi\left(\frac{3\pi}{2} - \beta\right)} = \frac{R}{\phi\left(\frac{3\pi}{2} - \gamma\right)}.$$

Since $\phi\alpha = -\cos\alpha$, this agrees with (17.).

XXXVII. On the Action of Hyponitric Acid upon Aqueous Solutions of Bromine and Chlorine. By Dr. C. F. SCHÖENBEIN*.

THE hydrobromic and hydrochloric acids being decomposed by nitric acid into water, hyponitric acid and romine or chlorine, it appears little likely that by the com-

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