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On a Mathematical Formula to express the Rate of Mortality throughout the whole of Life, tested by a Series of Observations made use of by the Danish Life Insurance Company of 1871. By Dr. T. N. THIELE, of Copenhagen. Translated from the Danish by T. B. SPRAGUE, M.A., Vice-President of the Institute of Actuaries.

THE formula for the law of mortality which I am about to explain and apply to a practical case, is not a merely empirical formula, but is based on a presumed property of the causes of death. The value of the formula is however a question which is in a great measure independent of the correctness of my hypothesis; for it is well known that even a false hypothesis may be of great service. While I shall certainly be very much interested to learn the opinion entertained by actuaries as to the theoretical part of the subject, yet the practical application of the formula is the point to which I attach most importance, and to which I would by preference invite criticism.

I venture the supposition that the causes of death naturally fall into three or four large groups, which can be known by the ways in which they operate upon different ages; three of the groups dividing the whole of life between them, and operating principally, or almost exclusively, upon childhood, middle age, and old age, respectively. The fourth group will accordingly include

the causes of death which operate upon all ages indifferently with about the same force; but it appears to me that such causes of death are of comparatively small importance.

The force of mortality, $\mu(x)$, for a given age x , must therefore be represented by the sum of three (or four) terms, each of which is a function of the age x , so that

$$\mu(x) = \mu_1(x) + \mu_2(x) + \mu_3(x) \quad . \quad . \quad . \quad (1)$$

Taking $\mu_3(x)$ to represent the mortality of old age, it is natural to choose the form of the function so that that part of the mortality becomes very small for small values of x , but increases rapidly and continuously with it. My choice agrees on this point with the best formulas which have hitherto been used to represent the aggregate mortality at the higher ages, namely, Gompertz's and Makeham's. I put

$$\mu_3(x) = a_3 b_3^x \quad . \quad . \quad . \quad (2)$$

where ε is the base of the natural logarithms, while a_3 , b_3 , are two constants to be determined separately for each series of observations. These, as well as all the other constants contained in my formula, always have positive values.

For the mortality of middle age, $\mu_2(x)$, I believe I have found a suitable expression in the law of error. I put

$$\mu_2(x) = a_2 \varepsilon^{-b_2^2(x-c)^2} \quad . \quad . \quad . \quad (3)$$

Here are three constants to be determined by reference to experience; namely, c , the age for which this part of the mortality is a maximum; a_2 , the amount of that maximum; and b_2 , which shows how fast this part of the mortality diminishes as the difference between the ages x and c increases, so that $\frac{2}{b_2}$ is a measure of the space of time in which the mortality of middle age makes its influence felt.

In my first attempt at graduating tables of mortality I assumed the number surviving at each age, out of a given number born, to be represented by the sum of a large number of terms of the form μ_2 .

The determination of the true rate of mortality in childhood, $\mu_1(x)$, is a difficult problem, and the more so, because the observations that can be used for the purpose are comparatively few and difficult to submit to strict treatment. However, that problem is of less importance, since insurances are seldom effected at young ages, so that for life insurance purposes we can be well satisfied

with a less accurate approximation. The general conditions which $\mu_1(x)$ must satisfy are that for $x=0$, $\mu_1(x)$ must be extremely large; and that as x increases, $\mu_1(x)$ must diminish rapidly and vanish. The only important previous investigation of this point that I know of is that of Professor Oppermann, explained in the *Insurance Record* for 11th February, 1870. He gives a formula for the force of mortality at young ages, that is up to about 20,

$$\mu(x) = ax^{-3} + b + cx^3$$

This formula obviously assumes that the force of mortality at the instant of birth is infinitely large. The structure of the formula also contains a definite intimation that with respect to the first years of life we shall make more progress by taking the square root of the time, instead of the time itself, as the independent variable of the mortality. It is my conviction that Professor Oppermann has gained for himself lasting credit by this formula. Even if a more natural expression for the mortality at young ages should be found, yet it is in the highest degree improbable that a partial formula will be found better adapted for calculation. This formula is of the greatest value for calculating the force of mortality in the first months and years of life from the rough observations by means of a correction corresponding to differentiation. That the force of mortality just at the beginning of life should be infinitely large, appears at first sight self-contradictory; but I entertain no serious doubt that in reality the fact is so. At any rate, the experiences which I have tested with this object do not contradict that proposition. Unfortunately, however, that formula does not permit of being inserted as the first term $\mu_1(x)$ of my formula, in the same way as Gompertz's formula forms its last term. If anything is done in that way, it will at all events be necessary to reject the third term cx^3 of Professor Oppermann's formula, and to substitute for it the sum of $\mu_2(x)$ and $\mu_3(x)$ as given by (3) and (2). Such a formula, consisting of four terms, the analytical forms of which enable them to represent respectively my four hypothetical groups of causes of death, appears at first sight admissible for my purpose. But in order to agree with the observed mortality of childhood, the constant b in Professor Oppermann's formula would frequently have to be taken as negative; and not to mention that a negative part of the force of mortality is a contradiction in terms, it would be difficult as regards the higher ages to satisfy the observations, if this term had a negative value, unless it were very small. At all events, I am

more inclined to keep only the first term of Professor Oppermann's formula, giving it the more general form

$$\mu_1(x) = x^{-\phi},$$

where ϕ is a function of x which is positive for all positive values of x , and such that $x=0$ gives $\phi=\frac{1}{2}$. A more precise determination of the form of this function would however be premature. It appears to me to be a necessary preliminary step, and also to be of importance for life insurance calculations, that a formula should be formed which can satisfy the series of observations for the whole of life except the earliest part of childhood; and such I hope to produce by putting provisionally

$$\mu_1(x) = a_1 \epsilon^{-b_1 x} \dots \dots \dots (4)$$

so that
$$\mu(x) = a_1 \epsilon^{-b_1 x} + a_2 \epsilon^{-\frac{1}{2} b_2^2 (x-c)^2} + a_3 \epsilon^{b_3 x} \dots \dots \dots (5)$$

Here, then, there are in all seven constants to be determined from the observations. It may possibly be sometimes necessary to add to the right-hand side of (5) a constant term answering to the part of the force of mortality common to all ages; but hitherto I have not met with such a case. Where agreement is only required for higher ages than, say, 10 or 20, it may be practicable to omit entirely the term $\mu_1(x)$.

The observations cannot possibly themselves indicate the force of mortality; but they answer more strictly to certain integrals of it: in particular, $l(x)$, the proportion of those who attain the age x out of a number of persons observed from birth, is here of importance.

But
$$\log_e l(x) = \int_x^0 \mu(x) dx \dots \dots \dots (6)$$

On $l(x)$ depends the probability of surviving a year, $p(x) = \frac{l(x+1)}{l(x)} \cdot *$

But

$$\begin{aligned} \log_e \frac{l(x)}{l(x+1)} &= -\log_e p(x) \\ &= \int_x^{x+1} \mu(x) dx \\ &= \frac{a_1}{b_1} (1 - \epsilon^{-b_1}) \epsilon^{-b_1 x} + a_2 \int_x^{x+1} \epsilon^{-\frac{1}{2} b_2^2 (x-c)^2} dx + \frac{a_3}{b_3} (\epsilon^{b_3} - 1) \epsilon^{b_3 x} \dots \dots (7) \end{aligned}$$

* It may be useful to remind the reader that most continental writers use the functional form of notation, where English writers are in the habit of using the index, or subscript, form. Thus in Dr. Thiele's paper, $l(x)$ and $p(x)$, which are considered as functions of the age x , mean exactly the same thing as the l_x and p_x commonly used in England.—
ED. J. I. A.

If the unit of time is not taken greater than a year, a sufficient approximation will in almost all cases be obtained by writing

$$\log_{\epsilon} \frac{l(x)}{l(x+1)} = -\log_{\epsilon} p(x) = a'_1 \epsilon^{-b_1 x} + a_2 \epsilon^{-\frac{1}{2} b_2^2 (x-c')^2} + a'_3 \epsilon^{b_3 x} \quad (7a)$$

where the right-hand side is of the same form as the expression for the mortality itself, but three of the constants have different values, namely.

$$a'_1 = \frac{a_1}{b_1} (1 - \epsilon^{-b_1}), \quad c' = c - \frac{1}{2}, \quad \text{and} \quad a'_3 = \frac{a_3}{b_3} (\epsilon^{b_3} - 1).$$

In practice it will be more suitable to transform (7a) so that the natural logarithms and powers of ϵ , are replaced by Briggs's logarithms and powers of 10. Thus,

$$\text{Log} \frac{l(x)}{l(x+1)} = -\text{Log} p(x) = a_1 10^{-\beta_1 x} + a_2 10^{-\beta_2 (x-c')^2} + a_3 10^{\beta_3 x} \quad (7b)$$

$$\text{where} \quad a_1 = a'_1 \text{Log } \epsilon, \quad a_2 = a_2 \text{Log } \epsilon, \quad a_3 = a'_3 \text{Log } \epsilon \\ \beta_1 = b_1 \text{Log } \epsilon, \quad \beta_2 = \frac{1}{2} b_2^2 \text{Log } \epsilon, \quad \beta_3 = b_3 \text{Log } \epsilon$$

(Log denoting the common logarithm.)

In many important cases, such as the example mentioned below, the observations show for each year of age in how many completed observations of single lives death has occurred within the term of a full year. If D denote the deaths and X the number of observations at the age x , then

$$\frac{X-D}{X} = p(x) = \frac{l(x+1)}{l(x)} \quad (8)$$

is the observed probability of surviving a year at age x .

From seven trustworthy observations the seven constants of the formula can now be calculated if for each observation we first calculate

$$-\text{Log} p(x) = \text{Log } X - \text{Log} (X-D)$$

then substitute in equation (7b), and determine the constants by trial. Provided that the observations are well divided among the different ages; that is, if 2 fall below the age of 20, 3 at ages between 20 and 50, and 2 at higher ages; that indirect calculation is not very difficult, since changes of a_1 and β_1 exercise material influence only on the values from the two youngest observations; those of a_2 , β_2 , and c' , only on the three middle observations; and those of a_3 , β_3 , only on the observations from the two highest ages.

But however carefully the observations may be made, it results from the nature of the case, that we must be prepared for such large deviations, that a calculation of the constants based upon observations for single years of age cannot be expected to give even a rough approximation to the true values of the constants. We must therefore of necessity determine the constants by a simultaneous employment of the whole series of observations, each taken according to its weight according to the rules of the calculus of probabilities. The mean error to be expected in the number of deaths D , out of X persons alive, will, if $p(x)$ is the true probability of surviving a year, be

$$\sqrt{p(x)\{1-p(x)\}X},$$

or will in most cases be not much less than the square root of D itself. But the weights, which in absolute magnitude are equal to $\frac{1}{(\text{mean error})^2}$, can also be expressed as functions of the sought constants of the formula together with the numbers X and the age x . For the number, D , of deaths, the weight is

$$w_x = \frac{1}{p(x)\{1-p(x)\}X} \cdot \cdot \cdot \cdot \cdot \quad (9)$$

We have likewise an exact expression for the weights of the probabilities of surviving a year, that is, of $\frac{l(x+1)}{l(x)}$, as calculated from the observations by means of equation (8),

$$\begin{aligned} w'_x &= \frac{X}{p(x)\{1-p(x)\}} \\ &= X^2 w_x \cdot \cdot \cdot \cdot \cdot \quad (10) \end{aligned}$$

For the weight, however, of the number, $-\text{Log } p(x)$, as deduced from the observations and destined for use in equation (7b), we have only approximately, but with sufficient accuracy in almost all cases, except at high ages,

$$\begin{aligned} w''_x &= w'_x \left\{ \frac{p(x)}{\text{Log } \epsilon} \right\}^2 \\ &= \frac{p(x)X}{\{1-p(x)\}(\text{Log } \epsilon)^2} \cdot \cdot \cdot \cdot \quad (11) \end{aligned}$$

According to the method of least squares, that system of values of the constants must be regarded as the most probable which gives a minimum value to the sum of the products formed by

multiplying each weight into the square of the difference between the corresponding observed and computed results. The problem of finding this system of values cannot be solved directly; but we may conceive it to be solved indirectly by repeated applications of the method of least squares, using the rules of that method to obtain successive corrections of the assumed values of the constants. But that mode of proceeding also in its strict form is here impracticable, since the uncertainty of the observations renders it so difficult to find a sufficiently correct system of values of the constants to enable us to escape a frequent repetition of laborious calculation.

Instead therefore of investigating the most probable system of values of the constants by the strict but almost impracticable manner, I use the method employed in Astronomy in similar cases, and called the method of Normal Places. This name I have retained in the present case, altho, as nothing local is observed, a modification of the term might be desirable. This method is nothing but such an extension of the method of arithmetical means as makes it applicable to observations of phenomena that vary during the observation according to any law. It is supposed that we know a function $Q(x)$ of the age x , which shows no considerable deviation when compared with the series of observations $P(x)$, $P(x+1)$, $P(x+n)$. The analytical form of this function is of little or no consequence. The best course will be to use a previous adjustment by the same formula; but we may also for groups of adjacent observations use partial adjustments by other formulas; for instance, by an integral rational function of the age; even graphical adjustment is allowable. Then from the observed numbers $P(x)$, $P(x+1)$, $P(x+n)$, we subtract the corresponding theoretical value, $Q(x)$, $Q(x+1)$, $Q(x+n)$, and compute for each the corresponding weight W_x , W_{x+1} , W_{x+n} . Then we calculate $\Delta\xi$ by the formula

$$\Delta\xi = \frac{W_x\{P(x)-Q(x)\} + W_{x+1}\{P(x+1)-Q(x+1)\} + \dots + W_{x+n}\{P(x+n)-Q(x+n)\}}{W_x + W_{x+1} + \dots + W_{x+n}} \dots (12)$$

Also the corresponding mean age under observation by the formula

$$\xi = \frac{xW_x + (x+1)W_{x+1} + \dots + (x+n)W_{x+n}}{W_x + W_{x+1} + \dots + W_{x+n}} \dots (13)$$

Then is calculated the value $Q(\xi)$ corresponding to the age ξ , and $\Delta\xi$ is added, giving

$$P(\xi) = Q(\xi) + \Delta\xi,$$

which may be presumed to be an improved observation (normal place) corresponding to the age ξ ; the weight of this normal place being

$$W_x + W_{x+1} + \dots + W_{x+n} \dots (14)$$

Instead of the exact value ξ found from (13) we may almost always safely take the nearest integral age.

When the whole series of observations, divided into suitable groups, has thus been used for the computation of normal places, these are used instead of the observations. Thus a double advantage is gained, the normal places being less in number as well as more trustworthy than the original observations. The determination of the seven constants in formula (7b) requires of course a division of the observation into seven groups, each of which gives its normal place. It is further to be observed that five of the normal places belong to ages below 50, since only two of the constants are important with regard to the higher ages. If, however, the number of lives observed upon was very great, it would be preferable to form a greater number of normal places, say 15 or 20, and then to treat them according to the method of least squares. But in common cases it will suffice to form only seven

normal places, $\text{Log } \frac{l(x)}{l(x+1)}$ being regarded as the observed magnitude, and to determine the seven constants by trial from the seven normal places. When the values of the constants are determined, we have to compare the observed and computed values of $\text{Log } \frac{l(x)}{l(x+1)}$, to calculate the weights afresh, and to form new normal places, using the same grouping of the observations as before. If the new normal places thus found differ much from the old ones, the whole calculation must be performed again with new normal places. Lastly, it is necessary to investigate how far we may regard the observations as satisfied by the assumed formula, and the calculated constants. For this purpose we are not confined to the common processes, founded on a consideration of the manner in which the errors are grouped (in the table of errors) with regard to sign and magnitude; but since we can calculate *à priori* as well as *à posteriori* the mean error of an observation of given weight, we can use a comparison of these numbers to make an estimate of the goodness of the adjustment. If the weight of an observation calculated by (9), (10), or (11) is 1, then the mean error *à priori* is = 1; *à posteriori* we find the mean error is

$$\sqrt{\frac{\sum w_x (D_x - D'_x)^2}{n - k}},$$

where D_x is the observed number of deaths at age x , D'_x the computed number, w_x the weight according to formula (9), n the total number of observations, and k the number of constants. Hence, in order that the two values of the mean error may not be inconsistent with each other, or that our adjustment may be satisfactory, we must have approximately

$$n = k + \sum w_x (D_x - D'_x)^2 \quad . \quad . \quad . \quad (15)$$

Example.

x	X	D	$-\text{Log} \frac{(X-D)}{X}$	x	X	D	$-\text{Log} \frac{(X-D)}{X}$
5	1953	32	0 00717	45	1447	22	0 0067
6	2079	28	588	46	1413	27	83
7	2082	13	272	47	1383	18	57
8	2063	5	105	48	1369	16	51
9	2037	12	256	49	1345	20	64
10	1969	10	0 00222	50	1307	17	0 0057
11	1900	15	344	51	1262	21	73
12	1868	3	070	52	1195	22	81
13	1784	9	219	53	1143	23	88
14	1568	7	195	54	1091	21	84
15	1365	8	0 00255	55	1051	27	0 0117
16	1121	7	272	56	928	32	152
17	890	3	15	57	880	19	095
18	691	4	25	58	829	16	085
19	530	3	25	59	793	29	162
20	393	3	0 0033	60	739	30	0 0182
21	290	1	15	61	676	18	117
22	172	3	76	62	634	27	189
23	84	1	52	63	574	24	185
24	91	0	0 0000	64	514	21	182
25	112	0	00	65	474	22	0 0466
26	166	3	79	66	405	23	568
27	213	0	00	67	342	25	730
28	287	2	31	68	305	20	657
29	405	2	22	69	270	16	594
30	506	2	0 0018	70	241	14	0 0581
31	627	4	28	71	203	8	395
32	849	8	48	72	175	16	913
33	931	6	31	73	148	12	812
34	1037	8	37	74	113	10	866
35	1138	10	0 0042	75	86	15	0 1743
36	1207	13	49	76	57	7	1228
37	1284	11	40	77	44	5	1137
38	1322	13	45	78	34	8	2352
39	1386	12	40	79	23	2	0 870
40	1452	13	0 0041	80	17	2	0 1176
41	1461	16	48	81	13	2	1538
42	1456	20	69	82	9	3	3333
43	1452	16	48	83	5	2	4000
44	1447	15	0 0045	84	3	3	1 0000

The series of observations above set out is the same as the Danish Life Insurance Company of 1871 has adopted for the basis of its calculations in the case where the life of a male is assured, the table having been graduated by Professor Oppermann. The fourth column above is calculated from the second and third; and it is to be noticed that for the 20 highest ages the probability of dying at each year of age is tabulated instead of the negative logarithm of the probability of surviving a year.

In order to find the seven constants of my formula answering to the above observations, I have made use of Professor Oppermann's graduated table as the basis for forming the five first normal places for the higher ages, while I have preferred a graphic delineation of the mortality as the basis for obtaining the two of the first set of normal places which related to the youngest ages. With these normal places the seven constants are determined. Then the values following from them are compared with the observations, the weights calculated, a new set of normal places formed, and with these the whole of the above described process was repeated and gave as a result the following set of

(1) $\text{Log } \frac{l(7)}{l(8)}$	=0.00350	observations for the ages	5- 9
(2) $\text{Log } \frac{l(13)}{l(14)}$	=0.00210	„ „	10-18
(3) $\text{Log } \frac{l(25)}{l(26)}$	=0.00285	„ „	19-32
(4) $\text{Log } \frac{l(36)}{l(37)}$	=0.00412	„ „	32-39
(5) $\text{Log } \frac{l(45)}{l(46)}$	=0.00554	„ „	40-51
(6) $\text{Log } \frac{l(56)}{l(57)}$	=0.01097	„ „	52-64
(7) $\text{Log } \frac{l(70)}{l(71)}$	=0.03278	„ „	65-84

The observation for the age 32 was used with half weight in both the third and fourth normal place. From these normal places I found the following values of the constants:—

$$\begin{array}{lll} \text{Log } a_1 = 7.99475 & \text{Log } a_2 = 7.32162 & \text{Log } a_3 = 6.08650 \\ \beta_1 = .07295 & \beta_2 = .0014735 & \beta_3 = 0.03470 \\ & c' = 31.209. \end{array}$$

Hence the formula for the force of mortality becomes

$$\mu(x) = 0.02474e^{-x \times 1.7023} + 0.00483e^{-\frac{1}{2}\left(\frac{x-31.709}{12.135}\right)^2} + 0.0002699e^{x \times 7.990}$$

The following table of mortality was calculated according to the formula (7b) with the above values of the constants in such a way that $\text{Log } l(5) = 0$, as in the Danish Life Insurance Company's table. The calculation of new values for the normal places had given the following consistent values:—

- (1) $\text{Log } \frac{l(7)}{l(8)} = 0.00353 \pm 0.00039$
- (2) $\text{Log } \frac{l(13)}{l(14)} = 0.00210 \pm 0.00024$
- (3) $\text{Log } \frac{l(25)}{l(26)} = 0.00286 \pm 0.00054$
- (4) $\text{Log } \frac{l(36)}{l(37)} = 0.00413 \pm 0.00047$
- (5) $\text{Log } \frac{l(45)}{l(46)} = 0.00552 \pm 0.00038$
- (6) $\text{Log } \frac{l(56)}{l(57)} = 0.01096 \pm 0.00066$
- (7) $\text{Log } \frac{l(70)}{l(71)} = 0.03287 \pm 0.00200$

The calculation therefore did not require to be carried any further.

Graduated Mortality Table.

x	$\text{Log } l(x)$	$\text{Log } \frac{l(x)}{l(x+1)}$	$\text{Log } \frac{1}{w_x}$	D'	$D - D'$	$w_x(D - D')^2$
0	0.03749	0.01008				
1	2741	0856				
2	1885	0729				
3	1156	0622				
4	0534	0534				
5	0.00000	0.00460	1.310	20.61	+ 11.39	6.37
6	9.99540	400	.277	19.07	+ 8.39	4.22
7	9140	350	.219	16.67	- 3.67	.81
8	8790	310	.165	14.70	- 9.70	6.43
9	8480	278	.110	12.97	- 0.97	.07
10	9.98202	0.00253	1.055	11.42	- 1.42	.18
11	7950	234	.005	10.17	+ 4.83	2.31
12	7716	220	0.973	9.43	- 6.43	4.40
13	7496	211	.934	8.63	+ 0.37	.02
14	7286	205	.867	7.40	- 0.40	.02
15	9.97080	0.00203	0.801	6.35	+ 1.65	.43
16	6877	204	.719	5.26	+ 1.74	.58
17	6673	208	.627	4.26	- 1.26	.38
18	6465	214	.529	3.40	+ 0.60	.11
19	6251	221	.428	2.69	+ 0.31	.04
20	9.96030	0.00230	.314	2.07	+ 0.93	.42

Graduated Mortality Table—(continued).

x	$\text{Log } l(x)$	$\text{Log } \frac{l(x)}{l(x+1)}$	$\text{Log } \frac{1}{w_x}$	D'	$D - D'$	$w_x(D - D')^2$
21	5800	240	·201	1·60	- 0·60	·16
22	5560	251	9·995	0·99	+ 2·01	4·07
23	5309	263	·703	0·51	+ 0·49	·48
24	5045	275	·757	0·57	- 0·57	·57
25	9·94770	0·00288	9·866	0·74	- 0·74	·75
26	4482	301	0·056	1·15	+ 1·85	3·01
27	4181	313	·182	1·53	- 1·53	1·54
28	3868	325	·327	2·14	- 0·14	·00
29	3543	337	·492	3·13	- 1·13	·41
30	9·93206	0·00349	0·603	4·04	- 2·04	1·04
31	2857	360	·710	5·17	- 1·17	·27
32	2497	371	·794	6·27	+ 1·73	·48
33	2126	382	·867	7·42	- 1·42	·27
34	1745	392	·919	8·37	- 0·37	·02
35	9·91353	0·00402	0·477	9·57	+ 0·43	·02
36	0951	413	1·028	10·77	+ 2·23	·47
37	0538	424	·066	11·73	- 0·73	·05
38	0114	435	·103	12·81	+ 0·19	·00
39	9·89679	447	·128	13·55	- 1·55	·18
40	9·89232	0·00461	1·161	14·66	- 1·66	·19
41	8771	475	·193	15·76	+ 0·24	·00
42	8296	492	·212	16·48	+ 3·52	·76
43	7804	511	·226	17·00	- 1·00	·06
44	7294	532	·242	17·67	- 2·67	·41
45	9·86762	0·00555	1·276	18·39	+ 3·61	·69
46	6207	582	·268	18·79	+ 8·21	3·64
47	5625	612	·281	19·36	- 1·36	·10
48	5013	646	·299	20·20	- 4·20	·89
49	4367	684	·316	21·04	- 1·04	·06
50	9·83683	0·00727	1·328	21·65	- 4·65	1·02
51	2957	774	·341	22·31	- 1·31	·08
52	2183	826	·344	22·49	- 0·49	·01
53	1357	885	·353	23·04	- 0·04	·00
54	0472	949	·363	23·58	- 2·58	·29
55	9·79523	0·01019	1·363	23·58	+ 3·42	·51
56	8504	1097	·354	23·17	+ 8·83	3·45
57	7407	1182	·362	23·63	- 4·63	·93
58	6225	1275	·368	24·01	- 8·01	2·75
59	4950	1376	·378	24·69	+ 4·31	·18
60	9·73574	0·01487	1·381	24·89	+ 5·11	1·09
61	2087	1607	·374	24·54	- 6·54	1·81
62	0481	1738	·378	24·86	+ 2·14	·19
63	9·68743	1880	·367	24·32	- 0·32	·00
64	6863	2035	·351	23·52	- 0·52	·28
65	9·64828	0·02202	1·348	23·44	- 1·44	·09
66	2626	2384	·312	21·65	+ 1·35	·09
67	0242	2581	·269	19·72	+ 5·28	1·50
68	9·57660	2795	·150	18·99	+ 1·01	·06
69	4865	3027	·228	18·16	- 2·16	·28
70	9·51838	0·03279	1·211	17·52	- 3·52	·76
71	·48559	3551	·168	15·97	- 7·97	4·32
72	·45008	3846	·133	14·82	+ 1·18	·10
73	·41163	4166	·090	13·54	- 1·54	·19
74	·36997	4512	·003	11·16	- 1·16	·13
75	9·32486	0·04887	0·913	9·16	+ 5·84	4·17
76	·27599	5293	·763	6·54	+ 0·46	·04

Graduated Mortality Table—(continued).

x	$\text{Log } l(x)$	$\text{Log } \frac{l(x)}{l(x+1)}$	$\text{Log } \frac{1}{w_x}$	D'	$D - D'$	$w_x(D - D')^2$
77	·22306	5733	·679	5·45	- 0·54	·04
78	·16572	6210	·595	4·53	+ 3·47	2·07
79	·10362	6727	·451	3·30	- 1·30	·60
80	9·03635	0·07286	0·346	2 62	- 0·62	·17
81	8·96349	7892	·256	2·16	- 0·16	·01
82	·88457	8549	·121	1·61	+ 1·39	1·46
83	·79908	9260	9·890	0·96	+ 1·04	1·40
84	·70648	0·10030	9·691	0·62	+ 2·38	11·55
85	8·60618	0·10864				
86	·49754	·11768				
87	·37986	·12747				
88	·25239	·13807				
89	·11432	·14956				
90	7·96477	0·16199				
91	·80277	·17547				
92	·62730	·19006				
93	·43724	·20587				
94	·23137	·22300				
95	7·00837	0·24155				
96	6·76682	·26164				
97	6·50519	·28340				
98	6·22179	·30697				
99	5 91481	·33251				
100	5·58231	0·36016				

D' is the number of the deaths according to the graduated table of mortality.

The question now arises whether the formula represents the observations with the necessary precision. As far as can be seen by a simple inspection of the number of signs, 35 being positive and 45 negative, with 38 changes from one to the other, there is nothing to prevent a completely favourable judgment; but a severer test is, as already mentioned, afforded by equation (15). Thus we have

$$\Sigma w_x(D - D')^2 = 90\cdot6$$

If to this we add 7, the number of the constants, there results 97·6, the number of observations being only 80. It might therefore have been expected that the adjustment should have given the sum of the squares of errors less by 17·6 than really is the case. This difference is not very small, but it is however no proof of the adjustment being either wholly or partly at variance with the observations. If in this case we examine the deviations more in detail, we can with good reason maintain that the sum of the squares must be assumed to become a little larger at each adjustment than is usually the case, because the observations for the highest age, 84, exhibit the unusually large product of weight and

square of error = 11.55, notwithstanding that the circumstances are such that that observation can have scarcely any influence at all on the value of the normal places; the weight of the probability is infinitely small. Nevertheless I believe we ought to presume the presence of a real deviation; for if we examine the observations from the youngest ages, the column $w_x(D-D')^2$ shows for those ages many very large numbers; and besides there is a certain approach to regularity in the magnitudes and signs of $D-D'$, and besides this proceeds in such a direction as must be expected from the provisional character of the term $\mu_1(x) = a_1 e^{b_1 x}$, which, it must be admitted, gives rather too low a mortality at the youngest ages. To be sure, I have not in other experiences found this deviation show an influence so far from birth as here. The observations here are however very far indeed from being adequate to give any important contribution towards the improvement of the form of the function $\mu_1(x)$. I therefore rely upon the result we have obtained; but if anybody reproaches me for the deviation from observation, I can only ask for a complete acquittal as regards the ages above 10. For the remaining 75 observations, the sum of the squares $+7 = 79.7$.

It will be of interest to compare this graduation with Professor Oppermann's. The following table gives for every fifth age $\text{Log } l(x)$ and $\text{Log } \frac{l(x)}{l(x+1)}$ according to Professor Oppermann, and the deviation of the latter from my adjustment.

x	$\text{Log } l(x)$	$\text{Log } \frac{l(x)}{l(x+1)}$	$T-O$
5	0.00000	0.00280	+0.00180
10	9.98639	263	- 10
15	9.97349	253	- 50
20	9.96090	252	- 22
25	9.94812	264	+ 24
30	9.93446	292	+ 57
35	9.91895	344	+ 58
40	9.90024	429	+0.00032
45	9.87637	563	- 8
50	9.84448	768	- 41
55	9.80039	1077	- 58
60	9.73799	1542	- 55
65	9.64822	2231	- 19
70	9.51779	3256	+ 23
75	9.32704	4773	+0.00115
80	9.04699	7019	+ 267
85	8.63481	10340	+ 524
90	8.02726	15250	+ 949
95	7.13092	22507	+ 1648

A tolerably trustworthy scale for the magnitude of this deviation is had in the mean errors for my normal places, which are for

$$\begin{array}{lll} x=7, \pm 00039 & x=36, \pm 00047 & \\ x=13, \pm 00024 & x=45, \pm 00038 & x=70, \pm 00200 \\ x=25, \pm 00054 & x=56, \pm 00066 & \end{array}$$

The deviations may therefore be described as large for ages under about 20 and as not unimportant for the rest of life. For a correct judgment hereupon it must not however be overlooked that the learned Professor's graduation advances still less claim to complete agreement with the whole series of observations. Professor Oppermann has verbally requested me to bear in mind in making this comparison, that it was resolved by the Direction of the Life Insurance Company to use this adjustment, notwithstanding the undoubted deviations which the two observations for the youngest ages exhibit. Thus in Professor Oppermann's adjustment the sum of the squares is 129·4, and adding to this the number of the constants, 5, we get 134·4 instead of 80; but the unquestionable deviation here seen falls almost entirely upon the first ages. Omitting, as in my adjustment, the five youngest ages, the sum of the squares becomes 77·7, and $77\cdot7 + 5 = 82\cdot7$, which is not too large in 75 observations. It will be remembered that my corresponding number was 79·7, or 3· less. Omitting in both graduations the ten first ages, the sum of the squares + the number of constants, becomes for Professor Oppermann, 74·5, for me 72·8. My graduation appears thus throughout to agree a little more closely with the experience than Professor Oppermann's.

But excluding the observations on the youngest ages of all, the difference is but very small, and both graduations can be considered as good notwithstanding their mutual deviations. Great as these may be, it must be admitted that they are less, and mostly a good deal less, than the deviations which the nature of the observations makes probable, not to speak of the deviations which the future experience of the Life Insurance Company may possibly exhibit.

It follows as a matter of course that the various premiums of the Life Insurance Company would have been rather different if they had been based upon my graduation instead of Professor Oppermann's; but I have had neither the necessary time nor the necessary information to give an opinion upon the financial result of the difference in the graduations. It will however undoubtedly appear that the difference could not be so very great as may be easily supposed on a comparison of the tables of mortality them-

selves, because in calculating the values of annuities the necessary summations will cause the negative and positive deviations in the different ages partly to balance each other. This can be partly seen by a comparison of the following values of the average lifetime for certain ages according to the two graduations, which I have calculated with this view.

Age.	AVERAGE LIFETIME ACCORDING TO		Difference.
	Oppermann.	Thiele.	
5	52.380	51.924	- 1 percent.
15	45.371	45.246	0 "
25	37.802	37.430	- 1 "
35	30.071	30.067	0 "
45	22.618	22.843	+ 1 "
55	15.912	15.991	0 "
65	10.359	10.271	- 1 "
75	6.203	5.801	- 7 "
85	3.422	3.273	- 4 "
95	1.757	1.658	- 6 "

Hence we can conclude with tolerable certainty that at the younger ages the premiums for the simplest kind of insurances according to the two graduations will also not differ by more than a small percentage. This shows that the observations, few in number as they are, are yet sufficient to use as a basis for calculating insurance premiums. But the magnitude of the premium, as is well known, depends very much upon the rate of interest and the loading. These quantities it is evident cannot be deduced from the mortality of the past. With respect to the surprisingly small addition for safety which the Life Insurance Company is required by law to make, it may be said that even with a future mortality not materially differing from that here investigated, it is not altogether improbable that the loading for safety will prove insufficient. Against the dangers of a real change in the rate of mortality in the country or a possible change in the manner of deciding as to the health of applicants for insurance it furnishes scarcely any security at all.

In conclusion, I must give my thanks to Professor Oppermann, not only because he was the first to strike out the path along which I have endeavoured to proceed, but also because he has favoured me with a series of conversations on the question of mortality which have not been without influence upon this work, notwithstanding that until its conclusion he did not know what

formula I worked with. Every one who knows how important it is to have a problem properly stated, even if the solution is not given, will understand what advantage I have been able to gain from my intercourse with Professor Oppermann.

[Mr. Makeham, having had the opportunity of perusing the foregoing paper while in manuscript, has drawn our attention to a paper on the same subject by the late Mr. Gompertz, and suggested that it would interest our readers to have this paper reprinted, so as to appear with Dr. Thiele's. We have great pleasure in complying with this suggestion, and also in laying before our readers a paper by Mr. Makeham himself, treating of the same questions.—ED. J. I. A.]