Any other change makes the planes of conjugate foci disappear, and the system becomes completely general. (I.)

12. Again, consider the relations between the positions of conjugate focal lines. All systems of the type IV., and onwards, are, to put it briefly, resolvable into two symmetrical systems acting in two planes at right angles, both as regards position of conjugate lines and magnification; systems of the types I., II., III. are not so resolvable, and the simplest expressions for the relations between conjugate focal lines or for the magnification are involved with the directions of those lines. Nevertheless, the change of form is not very great, and we can even see that, were the systems not related optically at all, it would not be very much greater.

13. This classification of optical systems is probably very imperfect, for the few properties which I have examined and used as a base for it must have omitted many notable cases; but they seem sufficient to show that no analytical obstacle remains against examining every corner of the problem, whether in its broadest or its finest features.

On the Poncelet Polygons of a Limaçon. By F. MORLEY.

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1. *The Limaçon.*

If we take n elements, points or lines, in a determinate order, say they are numbered 1, 2, ..., n , 1, and also the elements, lines or points 12, 23, ..., $n1$, we have the polygon. The problem of determining the condition under which the points are elements of one conic and the lines are elements of a second conic is a famous one, and it is sufficient to make the single reference to Halphen's *Fonctions Elliptiques*, t. II. The two conics constitute a curve of degree and class 4, any element of which determines two elements of the other kind, for a line of the one conic determines two points of the other,

and reciprocally. But there are proper quartics of degree and class 4, for the equations (Salmon, *Curves*, p. 65)

$$2\delta + 3\kappa = 8 = 2\tau + 3\iota$$

are satisfied not only when $\delta = 4$, $\kappa = 0$, the case of two conics, but also when

$$\delta = 1 = \tau, \quad \kappa = 2 = \iota.$$

In this case the possibility of there being polygons at once inscribed and circumscribed, that is, Poncelet polygons of the curve, has been discussed in two papers by R. A. Roberts (*Proc. Lond. Math. Soc.*, Vol. xvi., p. 53, and Vol. xxiii., p. 202). My object is to make some additions to the list given by Roberts. Roberts's method, which reduces the present problem directly to that for two conics, permits us to apply directly tables given by Halphen; but it is to be remarked that this analytical solution is not complete till the questions of reality have been considered. I have preferred to use a method where the algebra keeps in touch with the geometry, and where there is no question as to what factors of the result apply to the problem in hand. As in other cases Roberts's list appears complete, I take here only the one where the cusps may, by a real transformation, be placed at the circular points; the curve is now the limaçon.

The limaçon is described by the vertex x of a parallelogram Ox_1xx_2 , of which the vertex O is fixed, while the vertices x_1 and x_2 describe circles about O with angular velocities whose ratio is two. Thus it is the map on the x plane of the circle $|t| = 1$, where

$$x = 2\alpha t + \beta t^2;$$

and, since we are concerned with shape only, not with size, we put $\beta = 1$. Thus the equations of the curve in circular coordinates are

$$x = 2\alpha t + t^2, \quad y = 2\alpha/t + 1/t^2. \quad (1)$$

Since a change of sign in α merely turns the curve through π , we take $\alpha > 0$. A line $ax + by + c = 0$ meets the curve in four points whose parameters are given by

$$a(2\alpha t + t^2) + b(2\alpha/t + 1/t^2) + c = 0,$$

so that four points t_1, t_2, t_3, t_4 of the curve are collinear when

$$\Sigma t = -2\alpha = \Sigma 1/t. \quad (2)$$

Some special points and lines of the curve are collected for reference.

(i.) *The node, t_1 .*—Here two values of t give the same value of x , say, t_1, t_2 . Then, from (1),

$$t_1 + t_2 = -2a, \quad t_1 t_2 = -x.$$

The first equation shows that the points are conjugate points of the t -plane, and, since they are to be on the unit circle,

$$t_1 t_2 = 1.$$

Hence

$$x = -1, \quad t^2 + 2at + 1 = 0.$$

The node is a point, or rather two points, of the curve only when the points t_i are on the unit circle, that is, when $a \leq 1$.

(ii.) *The outer vertex.*—Here $t = 1$, $x = 1 + 2a$.

(iii.) *The inner vertex.*—Here $t = -1$, $x = 1 - 2a$.

(iv.) *The double tangent.*—From equations (2),

$$t + t' = -a = 1/t + 1/t'.$$

Hence the points of contact are given by

$$t^2 + 1 + at = 0,$$

and the line itself, in circular coordinates, by

$$\begin{aligned} x + y &= 2a(t + 1/t) + t^2 + 1/t^2 \\ &= -2a^2 + a^2 - 2 \\ &= -(2 + a^2). \end{aligned}$$

(v.) *The inflexions.*—In equations (2) let three values of t be t_1 .

Then $3t_1 + t = -2a = 3/t_1 + 1/t_1$,

so that $3a(t_1^3 + 1) + 2(a^3 + 2)t_1 = 0$.

The points are on the curve only when

$$(a^3 + 2)^2 < 9a^2,$$

i.e., when

$$1 \leq a \leq 2.$$

The line of inflexions is

$$\begin{aligned} x + y &= 2a(t + 1/t) + t^2 + 1/t^2 \\ &= 2(8 - 13a^2 - 4a^4)/9a^3. \end{aligned}$$

The equation of any tangent is

$$x(at + 1) + yt^3(a + t) = 2t^3(2a^2 + 1) + 3at(1 + t^2);$$

and this line meets the real axis where

$$x \{ \alpha (t+1/t) + t^2 + 1/t^2 \} = 2(2\alpha^2 + 1) + 3\alpha(t+1/t).$$

Hence the inflexional tangents meet the axis at

$$x, \{ -3\alpha^2(\alpha^2 + 2) + 2(\alpha^2 + 2)^2 - 9\alpha^2 \} = 9\alpha^2(\alpha^2 - 1),$$

or

$$x, = -9\alpha^2/(8 + \alpha^2).$$

Among the various practical constructions of the limaçon the following deserves emphasis from its practical convenience. In a sheet of hard cardboard cut two perpendicular slits, bisecting each other. Let each slit be guided by a fixed pin; then a pencil held in a hole drilled through any point of the sheet will describe a limaçon as the sheet moves. This is, in fact, the elliptic compasses; we have two planes *A* and *B*, and two points of *A* move on two lines of *B*. With reference to *B* any point of *A* describes an ellipse, but with reference to *A* any point of *B* describes a limaçon.

The reciprocal of a limaçon, being a curve with the same Plücker numbers, is also a limaçon if we so choose the base conic that the flex-tangents are the reciprocals of the cusps; this requires that the point *x*, just found is the centre of the conic. *The two reciprocal limaçons have the same shape*, that is, *a* is the same for both. For consider on the axis the points

0 where the cusp-tangents meet,

∞ on the line of cusps,

-1 the node,

$-(1 + \alpha^2/2)$ on the double tangent;

and their reciprocal lines, which are vertical and meet the axis at the points

$(8 - 13\alpha^2 - 4\alpha^4)/9\alpha^2$ on the lines of flexes,

$-9\alpha^2/(8 + \alpha^2)$ where the flex tangents meet,

$-(1 + \alpha^2/2)$ on the double tangent,

-1 the node.

Corresponding double-ratios of these two fours are equal, and, in fact, it is verified at once that the equation

$$(8 + \alpha^2)xx' + 9\alpha^2(x + x') = 8 - 13\alpha^2 - 4\alpha^4$$

sends the one four into the other, and also sends one vertex into the other.*

2. The Correspondence of Cotangential Points.

The curve depends on the single positive constant a ; the critical values of a are $a = 1$ (the cardioidal case) and $a = 2$ (the undulational case). When $a < 1$, the flexes are imaginary, but the curve has two loops; with this we begin. When $1 < a < 2$ the curve is an indented oval (§ 4), and when $a > 2$ the curve is an unindented oval, which obviously does not admit a real polygon.

From equations (2), we see that the tangent at t_3 meets the curve at t_1, t_2 , when

$$t_1 + t_2 + 2a = -2t_3,$$

$$1/t_1 + 1/t_2 + 2a = -2/t_3,$$

$$\text{i.e., when} \quad (t_1 + t_2 + 2a)(1/t_1 + 1/t_2 + 2a) = 4. \quad (3)$$

The equation defines a symmetric two-to-two correspondence. The values of t_2 will coincide when

$$[a(t_1^2 + 1) + (2a^2 - 1)t_1]^2 = (2at_1 + 1)(2at_1 + t_1^2),$$

$$\text{that is} \quad (t_1^2 + 1 + 2at_1)[a(t_1^2 + 1) + 2(a^2 - 2)t_1] = 0;$$

the former factor gives the nodal points, the latter the points where the flex tangents meet the curve again.

We now introduce elliptic functions; the notation and especially the necessary σ -formulæ are explained in the treatise on the *Theory of Functions* by Harkness and myself (Ch. VII.).

$$\text{The substitution} \quad \frac{1-t}{1+t} = iz/\kappa, \quad (4)$$

where κ is a real constant, and therefore, for

$$|t| = 1,$$

z is real, brings the quartic last written into the even form

$$Qz = \left(z^2 - \kappa^2 \frac{1+a}{1-a}\right) \left(z^2 - \kappa^2 \frac{1+\gamma}{1-\gamma}\right),$$

* It is therefore possible to so choose a base conic that the limaçon is its own reciprocal; the axes of such a conic are given by

$$a^2 = 4(1-\alpha^2)(4-\alpha^2)^2/(8+\alpha^2)^2, \quad b^2 = \pm 4(1-\alpha^2)\sqrt{1-\alpha^2} \cdot 4-\alpha^2/(8+\alpha^2).$$

where

$$\gamma = a - 2/a,$$

so that

$$\frac{1+\gamma}{1-\gamma} = -\frac{(1-a)(2+a)}{(1+a)(2-a)}.$$

Therefore (3) can be written

$$\frac{dz_1}{\sqrt{Qz_1}} \pm \frac{dz_2}{\sqrt{Qz_2}} = 0, \quad (5)$$

whence, if we put

$$z = \sigma_2 u / \sigma u = \sqrt{pu - e_2}, \quad (6)$$

with

$$e_1 - e_2 = \kappa^2 \frac{1+a}{1-a},$$

$$e_2 - e_3 = -\kappa^2 \frac{(1-a)(2+a)}{(1+a)(2-a)},$$

$$e_1 + e_2 + e_3 = 0;$$

then

$$u_2 \pm u_1 = \text{a constant } c,$$

where u_1 and u_2 correspond to cotangential points.

We choose, in the case $0 < a < 1$,

$$\kappa^2 = (1-a^2)(2-a),$$

and have

$$e_1 - e_2 = (1+a)^2(2-a) = 2+3a-a^3,$$

$$e_2 - e_3 = (1-a)^2(2+a) = 2-3a+a^3,$$

$$e_3 - e_1 = -4,$$

so that e_1, e_2, e_3 are known real numbers.

We have then

$$\left. \begin{aligned} x &= 2at + t^2, \\ \frac{1-t}{1+t} &= i\tau/\kappa = i\sigma_2 u / \kappa \sigma u, \end{aligned} \right\} \quad (7)$$

equations which map the limaçon into (1) a unit circle, (2) the real axis in the τ -plane, (3) a broken line in the u -plane. To determine this broken line, let $2\omega_1$ be the real, $2\omega_2$ the imaginary, period, and note that when

$$t = -1, \quad \tau = \infty, \quad u = 0,$$

$$t = 1, \quad \tau = 0, \quad u = \omega_1 + \omega_2 = \omega_3, \quad \text{say};$$

$$t^2 + 2at + 1 = 0, \quad \tau^2 = e_1 - e_2, \quad pu = e_1, \quad u = \omega_1 \text{ or } \omega_2 + \omega_3.$$

Also observe that the mapping is conform for the points in question, except for the node, near which, since, if $u = \omega_1 + u'$,

$$\sigma_3 u / \sigma u = \sqrt{e_1 - e_3} \sigma_3 u' / \sigma_1 u' = P_3 u',^*$$

we have

$$x + 1 = P_1(t - t_1) = P_3(u - \omega_1),$$

so that angles at the node are double of angles at the corresponding u -points.

Thus, since, for real values of τ , pu is real and $> e_3$, we may take the u -path as in Fig. 1.

FIG. 1.

Again, if u_1 and u_2 correspond to cotangential points, we have, as already said

$$u_2 \pm u_1 = c, \quad (8)$$

and the sign is to be determined.

* By P_n is meant a Taylor series beginning with the n^{th} power.

Evidently we are concerned only with points on the outer loop, *i.e.*, with points on the vertical part of the u -path, so that c is imaginary. We must distinguish *inner* tangents which touch the inner loop, for which the points u_1, u_2 include the point of contact, from outer tangents which touch the outer loop, for which the point of contact lies beyond u_1 or u_2 . For the former, u_2 and u_1 increase together, and we have

$$u_2 - u_1 = c.$$

For the latter, u_1 and u_2 move oppositely, and we have

$$u_2 + u_1 = c',$$

where the relation of c' and c is found, by considering the transition case of a nodal tangent, to be for the upper part of the figure

$$c' + c = 2(\omega_2 + \omega_3),$$

and for the lower half $c'' - c = 2\omega_1$.

Accordingly, starting from u_1 , we construct u_2 for an inner tangent in the u -plane by going a vertical distance c , along the vertical part of the u -path; but for an outer tangent this brings us to the end of the vertical part, and we merely turn back and complete the distance c . Thus, in any case, regarding the vertical part as an endless chain, we measure along it an absolute distance $|c|$ in a definite direction, to pass from u_1 to u_2 . And, if, going on in this way, we ever get back to u_1 , we have a Poncelet polygon. Thus, since the length of the chain is $|4\omega_2|$, we must have

$$nc = 4m\omega_2, \quad (9)$$

where n and m are integers.

3. Poncelet Polygons when $\alpha < 1$.

In the preceding equation let m be 1, so that one trip round the chain brings u to the starting point. We have then, when $n = 2h$,

$$\sigma_2 hc/2 = 0;$$

when n is odd,

$$\sigma nc/2 = 0.$$

To determine c we take a special case, *e.g.*, the double tangent. From (8),

$$\kappa^2 \left(\frac{1-t}{1+t} \right)^2 = -\sigma_2^2 u / \sigma^2 u = -(pu - e_2),$$

and for the double tangent we may take

$$u = c''/2 = c/2 + \omega_1,$$

$$t + 1/t = -a.$$

Hence
$$\kappa^2 \left(1 - \frac{4}{2-a}\right) = - \left\{ p \left(\frac{c}{2} + \omega_1 \right) - e_2 \right\},$$

or
$$(1-a^2)(2+a) = e_1 - e_2 + p \left(\frac{c}{2} + \omega_1 \right) - e_1,$$

or
$$-2a(1+a) = \frac{e_1 - e_2 \cdot e_1 - e_2}{p c/2 - e_1},$$

or
$$p c/2 - e_1 = -2(1+a)(2-a)/a,$$

so that
$$p c/2 - e_2 = (1+a)(2-a)(1+a-2/a),$$

$$= -(1-a^2)(4-a^2)/a,$$

$$p c/2 - e_3 = -2(1-a)(2+a)/a.$$

Therefore $\sigma_2^2 c/2 : \sigma_3 c/2 : \sigma_2^2 c/2 : \sigma_1^2 c/2 : \sigma_1^2 c/2 : \sigma_2^2 c/2$

$$= (1-a)(2+a) : 2 : (1+a)(2-a),$$

and (H. and M., p. 316), if we write σ_λ for $\sigma_\lambda c$,

$$\sigma_1 : \sigma_2 : \sigma_3 = 1+a : 1-a^2 : 1-a,$$

$$\sigma_2 \sigma_3 : \sigma_3 \sigma_1 : \sigma_1 \sigma_2 = 1-a : 1 : 1+a. \quad (10)$$

Hence, if τ_λ stand for $\sigma_\lambda 2c$, we have (*loc. cit.*)

$$\tau_1 : \tau_2 : \tau_3 = 1+4a : 1+2a^2 : 1-4a.$$

If, now, the limaçon admits a $2h$ -gon, we have

$$\sigma_2 h c/2 = 0.$$

If $h = 3$,

$$\sigma_2 3c/2 = 0,$$

$$\sigma_1 \sigma_2 = \sigma_3 (\sigma_1 + \sigma_2),$$

$$1+a = 2-a,$$

$$a = 1/2.$$

If $h = 4$,

$$\sigma_2 2c = 0,$$

i.e.,

$$\tau_2 = 0,$$

$$a = 1/4.$$

If $h = 5$, $\sigma_3 5c/2 = 0$.

Here we are delayed a minute by not having the appropriate σ -formula in readiness.

We have, from (60), p. 315, putting $u = 2u$, $v = u/2$,

$$\sigma_\lambda 5u/2 \sigma 3u/2 = \sigma_\lambda 2u \sigma 2u \sigma_\mu \mu/2 \sigma_\nu u/2 - \sigma_\lambda u/2 \sigma_\mu u/2 \sigma_\nu 2u \sigma_\nu 2u;$$

or, writing σ_λ for $\sigma_\lambda u$, so that

$$\sigma 2u = 2\sigma \sigma_1 \sigma_2 \sigma_3,$$

$$\frac{\sigma_\lambda 5u/2 \sigma 3u/2}{\sigma_\lambda u/2 \sigma u/2} = 4\sigma_1 \sigma_2 \sigma_3 \sigma_\lambda 2u \sigma_\mu^2 u/2 \sigma_\nu^2 u/2 - \sigma_\mu 2u \sigma_\nu 2u,$$

$$= 2\sigma_1 \sigma_2 \sigma_3 (\sigma_\mu + \sigma_\nu) \sigma_\lambda 2u - \sigma_\mu 2u \sigma_\nu 2u;$$

whence, substituting for $\sigma_\lambda 2u$ its value in terms of σ (p. 316), and removing the factor

$$\frac{\sigma 3u/2}{\sigma u/2} \quad \text{or} \quad \sigma_2 \sigma_3 + \sigma_3 \sigma_1 + \sigma_1 \sigma_2,$$

we have
$$\frac{\sigma_\lambda 5u/2}{\sigma_\lambda u/2} = 2\sigma_\mu \sigma_\nu (-\sigma_\mu^2 \sigma^2 + \sigma_\nu^2 \sigma_\lambda + \sigma_\lambda^2 \sigma_\mu^2)$$

$$-(-\sigma_2 \sigma_3 + \sigma_3 \sigma_1 + \sigma_1 \sigma_2)(\sigma_2 \sigma_3 - \sigma_3 \sigma_1 + \sigma_1 \sigma_2)(\sigma_2 \sigma_3 + \sigma_3 \sigma_1 - \sigma_1 \sigma_2),$$

a result which may be written

$$\frac{\sigma_\lambda 5u/2}{\sigma_\lambda u/2} = 2\sigma_\lambda 2u \sigma_\mu \sigma_\nu u - \frac{\sigma 3u \sigma u/2}{\sigma 3u/2 \sigma u}.$$

Hence in our case the equation $\sigma_3 5c/2 = 0$ entails, from (10),

$$2(1+a)(1-4a) - (1+2a)(1-2a) = 0,$$

or
$$1 - 6a - 4a^2 = 0,$$

or
$$a = \frac{-3 + \sqrt{13}}{4}.*$$

For greater values of h , a is a number of a higher irrationality. For example, if $h = 6$, we have

$$\sigma_3 3c = 0,$$

$$\tau_1 \tau_2 = r_3 (\tau_1 + \tau_2),$$

$$16a^3 + 8a^2 + 8a = 1.$$

* We observe incidentally that now $/5$ is a number of the form $a + b\sqrt{13}$; we have, in fact, when

$$\begin{aligned} e_1 &= 2 + \sqrt{13}/24, & e_2 &= -\sqrt{13}/12, & e_3 &= -2 + \sqrt{13}/24, \\ p2\omega_3/5 &= -9/4 - 29\sqrt{13}/24. \end{aligned}$$

Again, for odd polygons, we have

$$\sigma n c/2 = 0.$$

From $\sigma 5 v \sigma^3 v = \sigma 4 v \sigma^3 2 v - \sigma v \sigma^3 3 v,$

let $v = c/2;$

then $\sigma 2 v = \sigma,$

$$\sigma 3 v = \sigma v (\sigma_2 \sigma_3 + \sigma_3 \sigma_1 + \sigma_1 \sigma_2),$$

$$\sigma 4 v = 2 \sigma \sigma_1 \sigma_2 \sigma_3;$$

and, since $\sigma_2 + \sigma_3 = 2 \sigma_2^2 v \sigma_3^2 v,$

$$(\sigma_2 + \sigma_3)(\sigma_2 + \sigma_1)(\sigma_1 + \sigma_2) = 8 (\sigma_1 v \sigma_2 v \sigma_3 v)^4 = \sigma^4/2 \sigma^4 v,$$

or $\sigma^4 v = \frac{1}{2} \sigma^4 / (\sigma_2 + \sigma_3)(\sigma_2 + \sigma_1)(\sigma_1 + \sigma_2).$

Hence $\sigma 5 c/2 \sigma^3 c/2 = 2 \sigma^4 \sigma_1 \sigma_2 \sigma_3 - \frac{1}{2} \frac{\sigma^4 (\sigma_2 \sigma_3 + \sigma_2 \sigma_1 + \sigma_1 \sigma_2)^3}{(\sigma_2 + \sigma_3)(\sigma_2 + \sigma_1)(\sigma_1 + \sigma_2)}$
 $= \sigma^4 \sigma_1 \sigma_2 \sigma_3 (2 - P^3/2Q),$

where $P = \sigma_2 \sigma_3 + \sigma_2 \sigma_1 + \sigma_1 \sigma_2,$

$$Q = \sigma_1 (\sigma_2 + \sigma_3) \sigma_2 (\sigma_2 + \sigma_1) \sigma_3 (\sigma_1 + \sigma_2).$$

In our case $P = 3, \quad Q = 2(4 - a^2);$

and therefore $\sigma 5 c/2 = 0,$

when $8(4 - a^2) = 27, \quad \text{or} \quad a = \sqrt{10}/4,$

the condition for a pentagon.

Similarly $\sigma 7 v \sigma^5 v = \sigma 5 v \sigma^3 3 v - \sigma 2 v \sigma^5 4 v$

$$= \sigma^2 v \sigma 5 v (\sigma_2 \sigma_3 + \sigma_2 \sigma_1 + \sigma_1 \sigma_2) - 8 \sigma^4 \sigma_1^3 \sigma_2^3 \sigma_3^3$$

$$= \sigma^4 \sigma_1 \sigma_2 \sigma_3 \{2P^3 - P6/2Q - 8\sigma_1^3 \sigma_2^3 \sigma_3^3\}.$$

This is zero when

$$2 \cdot 3^3 - 3^3/4 (4 - a^2) - 8(1 - a^2) = 0,$$

i.e., when

$$32a^4 + 56a^2 - 7 = 0,$$

$$a = \frac{1}{4} \sqrt{-14 + 6\sqrt{7}}.$$

4. Poncelet Polygons when $1 < a < 2.$

Turning now to the case $1 < a < 2$, we observe that $\kappa^2 = (1 - a^2)(2 - a)$, is now negative, and r in equation (4) is a pure imaginary. We have the same equation as before, only now pu is real and $< e_3$.

The points where the flex-tangents meet the curve now take the place of the double point; they are

$$u = \omega_3, \quad u = \omega_1 + \omega_3.$$

The inner tangents touch at points on the indentation, between the inflexions; and the outer tangents at points between the inflexions and the points of contact of the double tangent. The mapping is indicated in Fig. 2.

FIG. 2.

For inner tangents
and for outer tangents
where

$$\begin{aligned} u_3 - u_1 &= c, \\ u_3 + u_1 &= c' \text{ or } c'', \\ c' - c &= 2(\omega_1 + \omega_3), \\ c'' + c &= 2\omega_3. \end{aligned}$$

The lower point of contact of the double tangent is $c''/2$ or $c/2 + \omega_3$.

Hence, repeating the process of the last article, we have

$$(1-a^2)(2+a) = p \left(\frac{c}{2} + \omega_3 \right) - e_3,$$

whence

$$pc/2 - e_3 = 2(1-a),$$

$$pc/2 - e_1 = -2(1+a),$$

$$pc/2 - e_2 = a(1-a^2),$$

$$\sigma_2^2 c/2 \sigma_3^2 c/2 : \sigma_3^2 c/2 \sigma_1^2 c/2 : \sigma_1^2 c/2 \sigma_2^2 c/2 = a^3 - a : 2 : a^2 + a,$$

$$\sigma_1 : \sigma_2 : \sigma_3 = 1+a : a^2-1 : 1-a,$$

and

$$\sigma_2 \sigma_3 : \sigma_3 \sigma_1 : \sigma_1 \sigma_2 = 1-a : -1 : 1+a; \quad (11)$$

and these results may be verified by observing that, if c_+ be the present constant c , and c_- that of the previous article when a was < 1 , while c'' is analytically the same in both, we had

$$c'' = c_+ + 2\omega_3 = c_- + 2\omega_1,$$

so that

$$\sigma_\lambda c_+ = \sigma_\lambda (c_- + 2\omega_1 - 2\omega_3),$$

or (H. and M., p. 302, f. 34)

$$\sigma_1 c_+ : \sigma_2 c_+ : \sigma_3 c_+ = -\sigma_1 c_- : \sigma_2 c_- : -\sigma_3 c_-.$$

The ratios of r_λ are as before.

The condition of closure is, as before,

$$nc = 4m\omega_3.$$

Running through the preceding σ -formulae with the new values (11) instead of (10), we find only two new cases: *first*, when

$$\sigma_3 (5c/2) = 0;$$

in this case $2(1+a)(1-4a) - 3(1+2a)(1-2a) = 0$,

or $4a^2 - 6a - 1 = 0$,

or $a = (3 + \sqrt{13})/4$,

the condition for a decagon.*

* It will be observed that, whereas $c < \omega_3$ when $a < 1$, $c > \omega_3$ when $a > 1$. Thus in the present decagon a figure would show that

$$10c = 12\omega_3;$$

while for the decagon of the preceding article

$$10c = 4\omega_3.$$

Second, we have, when $\sigma 7c = 0$,

$$8a^3 + 1 + 32a^3(1 - a^2) = 0,$$

whence

$$a^3 = (5 \pm 3\sqrt{3})/8.$$

We have then the following table of the simpler Poncelet polygons:—

Number of Sides...	5	6	7	8	10
Value of α	$\sqrt{10}/4$	$1/2$	$\sqrt{-14 + 6\sqrt{7}}/4$ $\sqrt{10 + 6\sqrt{3}}/4$	$1/4$	$(\sqrt{13} + 3)/4$ $(\sqrt{13} - 3)/4$

Four of these, namely those with an even number of sides, are not given by Roberts.

5. The Hexagonal Case.

When $\alpha = 1/2$, the curve is the trisectrix $r = \cos \theta/3$. The intersection of the cusp-tangents, $x = 0$, is now the inner vertex; reciprocally the join of the inflexions is the tangent at the outer vertex.

Referring to Fig. 3, let $t_1, t_2, \dots, t_6$ be the points of the hexagon, and let t_{12} be the point of contact of the line t_1, t_2 . We have, from (3),

$$(t_1 + t_2 + 1)(1/t_1 + 1/t_2 + 1) = 4.$$

The roots of this quadratic in t , are t_1 and t_2 ; hence

$$t_1 t_2 = t_2,$$

$$t_1 + t_2 = -\frac{t_2^2 - t_2 + 1}{t_2 + 1};$$

or, since

$$t_2 = 1/t_6,$$

$$t_1 t_2 t_6 = 1,$$

and

$$t_1 + t_2 = -\frac{1 - t_6 + t_6^2}{t_6 + t_6^2}.$$

Hence

$$t_1 + t_2 = -\frac{t_1 t_2 - 1 + t_6}{1 + t_6},$$

or, for odd suffixes,

$$\sum t_1 + \sum t_1 t_2 = 1,$$

or, for all suffixes, $\sum_1^6 t_\lambda = 1.$

Now, from (2), $2t_{12} = -1 - t_1 - t_3.$

Therefore $t_{12} + t_{34} + t_{56} = -2.$

FIG. 3.

Therefore the points $t_{12}, t_{34}, t_{56}, 1$ lie on a line; that is, the points of contact of alternate sides lie on a line through the outer vertex.

Reciprocally, the tangents at alternate angular points meet on the tangent at the inner vertex.