IS THE TREATMENT OF THE THEORY OF LIMITS
IN OUR ELEMENTARY GEOMETRIES LOGICAL?*

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Our early American geometries, which were largely translations from the French, followed the plan of Legendre and *assumed*, without any proof, that the circle was the limit of inscribed, or of circumscribed, regular polygons when the number of sides is indefinitely increased.

The modern treatment of the theory of limits was introduced into our later geometries to supply a proof for this assumption. The only excuse then for the introduction of the doctrine of limits into elementary geometry is that it was to give a satisfactory proof for the proposition quoted above, and for the analogous proposition in solid geometry relating to cylinder, cone and sphere.

The proofs of the propositions of plane geometry in which the theory of limits is used in our American geometries is fairly logical, but what is given as proofs in the analogous propositions of solid geometry are no proofs at all, but assumptions pure and simple, as were those of Legendre.

Before discussing these proofs, let me say that I showed a list of the geometries that I had examined on this subject to an official of a prominent text book publishing company, and asked him what proportion of the geometries used in the United States was represented there.

He said: "You have here all the *prominent* geometries published in this country. There are a number of geometries which do not appear on your list, but they are what we call *local books*—books of small circulation, used only in restricted localities. Your list represents ninety per cent of the geometries sold in this country."

In view of this statement I will affirm that ninety per cent of the boys and girls of this country who are studying geometry are, in their solid geometry, studying demonstrations whose logic is vile. I am referring to those propositions of solid geometry in which the theory of limits is used.

*From the opening address to the Association of Ohio Teachers of Mathematics and Science at their meeting in Columbus, Dec. 29, 1904.

Take the following as an example: "If a prism whose base is a regular polygon is inscribed in a circular cylinder, and if the number of the sides of the base is indefinitely increased, the volume of the prism approaches the volume of the cylinder as its limit, and the lateral surface of the prism approaches the lateral surface of the cylinder as its limit."

The proof given (stripped of its verbiage) is: "Since the circumference of the base of the cylinder is the limit of the perimeter of the base of the prism, and the area of the base of the cylinder is the limit of the area of the base of the prism, *therefore* the volume of the cylinder is the limit of the volume of the prism, and the lateral surface of the cylinder is the limit of the lateral surface of the prism."

This is, of course, the baldest kind of an assumption. When it was proved in plane geometry that the circumference was the limit of the perimeter of an inscribed polygon, nothing was proved about *position* or *shape* of the perimeter. The term limit as used in mathematics has nothing to do with position or shape. The definition states that the limit of a variable is a constant from which the variable can be made to differ by less than any assignable quantity. The perimeter cannot differ from the circumference in *position* or *shape* by less than any assignable quantity. All that has been proved in that proposition is that the *length* of the broken line forming the perimeter can be made to differ from the *length* of the curved line forming the circumference by less than any assignable quantity.

To state without proof in solid geometry that the volume of that irregular solid lying between the lateral surfaces of the cylinder and inscribed prism, a solid for whose volume we have no expression up to this point—to state, I repeat, that the volume of this solid can be made less than any assignable quantity because its base can be made less than any assignable quantity, *is an assumption*.

To state without proof that the lateral surface of the cylinder, a surface for whose area there is no expression up to this point, is the limit of the lateral surface of the inscribed prism, because the perimeter of the base of the prism can be made to differ from the circumference of the base of the cylinder by less than any assignable quantity, is another assumption as bad as the former.

The only assumptions admissible in a logical geometry are axioms and postulates. The foregoing assumptions being capable of proof are not axiomatic.

In only two geometries of my list was there even an attempt made to prove this assumption.

In one of these the proof given for the proposition relating to the cylinder is wrong. The author fails to note that a right section of an oblique circular cylinder is an ellipse, and that proposition relating to circles cannot be used as authorities here. The corresponding proposition relating to cone and sphere are also wrong.

In the other, after stating the propositions in the same manner that all the others do, the author adds a foot note after each one: "For a vigorous proof of this proposition, see my appendix."

But unfortunately several of the "*rigorous proofs*" in the appendix contain the same vicious assumption that spoiled the demonstrations of all the books of the list.

The fact that ninety per cent of the geometries used in this country give no proof for the propositions quoted relating to cylinder, cone and sphere, is a matter worthy of the serious consideration of the teachers of geometry.

GRAPH TRACING.

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Teachers unfamiliar with elementary analytical geometry may be glad to know a more organized way of dealing with graph tracing than mere plotting of points, and teachers who are mathematicians to learn that such methods may be used without difficulty in ordinary algebraical teaching for young pupils. My experience is that boys can be led to trace curves with more certainty and with an understanding of the aims of graph tracing.

In the case of an *equation of the first degree* it might be demonstrated:

(1) That if the equation is put in the form $y = mx + c$, m determines its direction and c its actual position. Thus, in the case of $5y = 3x + c$, if any value of x is increased by 5, the corresponding value of y is increased by 3, and so, if a series of points, A, B, C, D, etc., be found in this way, AB, BC, CD, etc.,