Waves in Canals and on a Sloping Bank. By H. M. MACDONALD.

Read June 11th, 1896. Received in revised form December 3rd, 1896.

1. In his treatise on *Hydrodynamics* (1895), Ch. ix., Professor Lamb has alluded to a former paper on the above subject in the following terms:—"There is some divergence here, and elsewhere in the text, from the views maintained by Macdonald in the paper cited" (Footnote, p. 435). The first statement in my paper alluded to, occurs on p. 109, Vol. xxv. of the *Proceedings*, where it is stated that a long wave with infinite velocity of propagation would not be generated; this statement was due to the idea that such a system required to be set up all along its length simultaneously, and would need an infinity of energy to do it. There is nothing in this statement which conflicts with Professor Lamb's interpretation of the motion (with which I entirely agree), imagining it to be set up. The other statement alluded to occurs on p. 111, where it is stated that straight-crested waves in a canal of triangular cross section are only possible in certain cases.

Professor Lamb, § 239, assuming the existence of free oscillations of the liquid contained between two transverse partitions in a canal, deduces the existence of straight-crested waves in a canal of unlimited length. There is a serious objection to the use of a physical assumption as the basis of such a proof, that it assumes that the mathematical theory of the physical phenomena is correct; in the present case it assumes the existence of an irrotational fluid motion, and that the boundary conditions are those satisfied in nature.

What is really proved in my former paper is that the cases there mentioned are the only ones in which the velocity potential can be expanded in a Fourier series of the type

$$\Sigma R_n \cos \frac{s\pi\theta}{\alpha} \cos (mx - nt),$$

where r, θ, x are cylindrical coordinates of origin in the lowest line of the canal, θ the angle between the vector r and a side of the canal. What was assumed in the final statement was that the solution of the equation

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} - m^2 f = 0,$$

subject to $\frac{\partial f}{\partial \theta} = 0, \quad \theta = 0 \text{ or } a,$

can always be expressed in the form

$$\sum R_s \cos \frac{s\pi\theta}{a}.$$

This assumption is usually made, but, as an assumption, it is not legitimate; the possibility of such an expansion depends on the character of the other boundary conditions. The only case where the expansion can easily be shown to be possible is when the region within which the function is required can be extended, so that the boundary conditions, where $\theta = 0$ or a , are of the same character for all values of r . In this case it is evidently true, and the conditions here stated are satisfied in all the problems solved by such expansions which I have examined. It will be shown below that these conditions are satisfied for symmetric waves both along and across canals of uniform isosceles triangular cross section, vertical angles $\pi/2$ and $2\pi/3$, and for asymmetric waves when the vertical angle is $\pi/2$.

2. Taking the case when the vertical angle is $\pi/2$, if a velocity potential exists in the region AOB satisfying the required conditions,

FIG. 1.

then functions exist in the spaces DOA , OOD , BOC , and the function can be continued throughout the space $ABOD$, $\frac{\partial f}{\partial \theta}$ being zero along AO and BD . The function can then be continued throughout all space as the squares can be repeated so as to fill all space, and further

$\frac{\partial f}{\partial \theta} = 0$ all along the line $COAO$, produced to infinity, and also along DOB produced to infinity. Hence the velocity potential, both for symmetric and asymmetric waves along or across this canal, can be expressed in the form

$$\sum R_n \cos \frac{s\pi\theta}{\alpha}.$$

When the vertical angle is $2\pi/3$, if a velocity potential exists in the space AOB (Fig. 2) satisfying the required conditions, then functions exist in the spaces AOC , BOC , and the function can be continued throughout the space ABC , $\frac{\partial f}{\partial \theta}$ being zero along OA , OB , and OC .

The function can then be continued throughout all space, as the equilateral triangles can be repeated so as to fill all spaces. When the wave motion is symmetric, $\frac{\partial f}{\partial \theta}$ is zero along OD , OE , and OG ; therefore $\frac{\partial f}{\partial \theta}$ vanishes all along the lines OA and OB produced indefinitely.

Hence the velocity potential of a symmetric wave motion, either along

FIG. 2.

or across a canal vertical angle $2\pi/3$, can be expressed in the form]

$$\sum R_n \cos \frac{s\pi\theta}{\alpha}.$$

When the wave motion is asymmetric, the condition $\frac{\partial f}{\partial \theta}$ zero is no longer satisfied all along OA and OB , and it follows from the analysis of the

previous paper that the velocity potential is not expressible in this form. When the vertical angle is $\pi/3$, all space can be partitioned by means of hexagons, but parts of the fixed boundaries when produced are free surfaces in other partitions, the condition $\frac{\partial f}{\partial \theta}$ zero all

along them is satisfied neither for symmetric nor asymmetric waves, and in this case also the velocity potential cannot be expressed in the form

$$\Sigma R, \cos \frac{s\pi\theta}{a}.$$

3. When, in the results obtained for the wave motion in these canals, the depth of the canal is made indefinitely great, velocity potentials for waves standing against a sloping bank or along it are obtained, when the bank makes angles $\pi/4$ or $\pi/6$ with the horizon. The velocity potential in these cases takes the form of a sum of exponentials, and it suggests the inquiry for what angles of slope is there a solution of this form. In the case of waves standing against a sloping bank, a solution of this form has been given by Kirchhoff, *Gesam. Abhand.*, Bd. II., p. 428, when the angle with the horizon is of the form $(2m+1)\pi/2n$, m and n being integers. It will be shown below that the solution in the form of a sum of exponentials makes the velocity finite at a great distance from the edge only when the angle of slope is of the form $\pi/2n$. Further, it is shown below that there is a solution of this form for waves parallel to the bank when the angle of slope is of the form $\pi/(4n+2)$, and that in this case the velocity is finite at a great distance from the edge. In all these cases where the condition of finite velocity at a great distance from the edge is satisfied, the velocity of propagation of the waves is that of deep sea waves.

4. Taking the axis of x along the intersection of the free surface and the bank, that of y horizontally and perpendicular to it, and that of z vertically upwards, the velocity potential satisfies the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (1)$$

throughout the fluid, subject to

$$\frac{\partial \phi}{\partial z} \cos \alpha + \frac{\partial \phi}{\partial y} \sin \alpha = 0, \quad (2)$$

when $z \cos \alpha + y \sin \alpha = 0$, α being the angle which the bank makes with the horizon, and

$$l \frac{\partial \phi}{\partial z} - \phi = 0, \quad (3)$$

when $z = 0$, where $l = g/n^2$, $2\pi/n$ being the period of the motion. Further, the velocity at a great distance from the edge must be finite.

Considering first the case of waves standing against the bank, ϕ takes the form $f(y, z) e^{i\omega t}$, where f may be written

$$f = \left\{ \cos \left(z \frac{\partial}{\partial y} \right) + \frac{\sin \left(z \frac{\partial}{\partial y} \right)}{l \frac{\partial}{\partial y}} \right\} f_0, \quad (4)$$

for the above expression for f satisfies (1) and (3), f_0 being a function of y only. It is proposed to investigate in what cases f_0 is of the form $\Sigma A_\nu e^{\nu y}$. When f_0 is of this form it may be written

$$f_0 = \int e^{-i\omega y t} \chi(t) dt,$$

where the path of integration is any closed curve enclosing all the poles of $\chi(t)$ which are simple.*

$$\text{Then } f = \frac{1}{2} \int \left[\left(1 + \frac{1}{mlt} \right) e^{-i\omega(y+z)t} + \left(1 - \frac{1}{mlt} \right) e^{-i\omega(y-z)t} \right] \chi(t) dt.$$

To satisfy (2) change to polars; then $y + iz = re^{i\theta}$, and (2) becomes

$$\int \left[\left(1 + \frac{1}{mlt} \right) mte^{-i\alpha} e^{-imre^{-i\alpha}t} + \left(1 - \frac{1}{mlt} \right) mte^{i\alpha} e^{-imre^{i\alpha}t} \right] \chi(t) dt = 0.$$

This is equivalent to

$$\begin{aligned} e^{-2i\alpha} \Sigma (t_s + \lambda) e^{-imre^{-i\alpha}t_s} [\chi(t)(t - t_s)]_{t=t_s} \\ = \Sigma (t_s - \lambda) e^{-imre^{i\alpha}t_s} [\chi(t)(t - t_s)]_{t=t_s} \end{aligned}$$

for all values of r , where $\lambda = \frac{1}{ml}$.

Therefore, if t_0 is a pole of $\chi(t)$, $t_0 e^{-2i\alpha}$ is one and also $t_0 e^{-4i\alpha}$, &c.

* It may be shown that this form of f_0 , when the poles of $\chi(t)$ are not simple, does not lead to a solution of the problem in any of the cases where the above form fails.

Hence $t_0 e^{-2qa} = t_0$, which requires $a = p\pi/q$, where p and q are integers. Writing $\gamma = e^{-2ia}$,

$$\chi(t) = \frac{\chi_1(t)}{(t-t_0)(t-t_0\gamma)\dots(t-t_0\gamma^{q-1})},$$

then

$$\frac{\gamma(t_0+\lambda)\chi_1(t_0)}{t_0^{q-1}(1-\gamma)(1-\gamma^2)\dots(1-\gamma^{q-1})} = \frac{(t_0\gamma-\lambda)\chi_1(t_0\gamma)}{t_0^{q-1}(\gamma-1)(\gamma-\gamma^2)\dots(\gamma-\gamma^{q-1})}, \text{ \&c.};$$

therefore

$$\chi_1(t_0)(t_0+\lambda) = \chi_1(t_0\gamma)(t_0\gamma-\lambda),$$

$$\chi_1(t_0\gamma)(t_0\gamma+\lambda) = \chi_1(t_0\gamma^2)(t_0\gamma^2-\lambda),$$

$$\text{\&c.} = \text{\&c.},$$

$$\chi_1(t_0\gamma^{q-1})(t_0\gamma^{q-1}+\lambda) = \chi_1(t_0)(t_0-\lambda);$$

that these equations may be consistent, q must be an even integer.

Let $q = 2k$; then

$$\chi_1(t_0\gamma^s) = \frac{(t_0\gamma^{s-1}+\lambda)(t_0\gamma^{s-2}+\lambda)\dots(t_0+\lambda)}{(t_0\gamma^s-\lambda)(t_0\gamma^{s-1}-\lambda)\dots(t_0\gamma-\lambda)} \chi_1(t_0),$$

$$\text{that is, } \chi_1(t_0\gamma^s) = \frac{(t_0\gamma^{s+1}-\lambda)(t_0\gamma^{s+2}-\lambda)\dots(t_0\gamma^{k+s-1}-\lambda)\gamma^{ks}}{(t_0\gamma-\lambda)(t_0\gamma^2-\lambda)\dots(t_0\gamma^{k-1}-\lambda)} \chi_1(t_0);$$

$$\text{therefore } \chi_1(t) = A(t\gamma-\lambda)(t\gamma^2-\lambda)\dots(t\gamma^{k-1}-\lambda)t^k,$$

$$\text{and } f_0(y) = A \int e^{-imy} \frac{(t\gamma-\lambda)(t\gamma^2-\lambda)\dots(t\gamma^{k-1}-\lambda)t^k dt}{t^{2k}-1}.$$

That the velocity may be finite everywhere, it will be sufficient to make $f_0(y)$ finite when y is infinite; that this may be so, e^{-imy} must be finite when y is infinite for all the values of s which can occur, and these are therefore the integers given by

$$\cos\left(\frac{\pi}{2} + 2sa\right) \leq 0,$$

which requires

$$k \geq (2p+1)s \geq 0, \quad 3k \geq (2p+1)s \leq 2k, \text{ \&c.},$$

where

$$a = \frac{2p+1}{2k} \pi.$$

When $p = 0$, $a = \pi/2k$, s has values 0, 1 to k ; and

$$(t\gamma-\lambda)(t\gamma^2-\lambda)\dots(t\gamma^{k-1}-\lambda)t^k$$

must have

$$(t-\gamma^{k+1})(t-\gamma^{k+2})\dots(t-\gamma^{2k-1})$$

for a factor; this requires that $\lambda = 1$, that is, $ml = 1$, and then

$$f_0(y) = A \int e^{-myt} \frac{t^k dt}{(t-1)(t-\gamma)\dots(t-\gamma^k)}.$$

When p has any other value than zero, the condition of finite velocity everywhere cannot be satisfied, as is easily seen from the form of $f_0(y)$, it being necessary that λ should be real. In the case where $\alpha = \pi/2k$, f is given by

$$f = A \int \left(\cosh mzt + \frac{\sinh mzt}{mlt} \right) e^{-myt} \frac{t^k dt}{(t-1)(t-\gamma)\dots(t-\gamma^k)}.$$

The velocity potential can also be expressed in the form of a series as follows:—

$$f = A' \left\{ \cos \left(z \frac{\partial}{\partial y} \right) + \frac{\sin \left(z \frac{\partial}{\partial y} \right)}{l \frac{\partial}{\partial y}} \right\} \sum_{s=0}^{s=k} e^{-\frac{1}{2}(s)(k-1)-s(s-1)\pi} e^{-im\gamma^s} \cos(s-1)\alpha \dots \cos(s-k+1)\alpha.$$

There are two cases, according as k is an odd or an even integer.

(1) k odd $= 2k' + 1$; then

$$f = C \sum_{s=0}^{s=k'} \cos(s-1)\alpha \cos(s-2)\alpha \dots \cos(s-2k')\alpha \\ \times \left[e^{-im(y \sin 2sa - z \cos 2sa)} \cos sa \cos \left\{ m(z \sin 2sa + y \cos 2sa) + (k'-s) \frac{\pi}{2} \right\} \right. \\ \left. + e^{-m(y \sin 2sa + z \cos 2sa)} \sin sa \sin \left\{ m(z \sin 2sa - y \cos 2sa) - (k'-s) \frac{\pi}{2} \right\} \right].$$

(2) k even $= 2k'$; then

$$f = C \sum_{s=0}^{s=k'-1} \cos(s-1)\alpha \dots \cos(s-2k'+1)\alpha \\ \times \left[e^{-im(y \sin 2sa - z \cos 2sa)} \cos sa \cos \left\{ m(z \sin 2sa + y \cos 2sa) + k'-s-\frac{1}{2} \right\} \frac{\pi}{2} \right\} \\ + e^{-m(y \sin 2sa + z \cos 2sa)} \sin sa \sin \left\{ m(z \sin 2sa - y \cos 2sa) - (k'-s-\frac{1}{2}) \frac{\pi}{2} \right\} \right] \\ + \frac{C}{2} e^{-my} (\cos mz + \sin mz).$$

Particular cases—

$$\alpha = \pi/4, \quad f = \frac{1}{2}C \{ e^{mz} (\cos my - \sin my) + e^{-my} (\cos mz + \sin mz) \};$$

$$\alpha = \pi/6, \quad f = C \left\{ e^{-m \frac{1}{2}(y \sqrt{3} + z)} \sin m \frac{z \sqrt{3} - y}{2} \right. \\ \left. + \sqrt{3} e^{-m \frac{1}{2}(y \sqrt{3} - z)} \cos m \frac{z \sqrt{3} + y}{2} - e^{mz} \sin my \right\}.$$

In all these cases the wave motion is that due to the reflection of deep sea waves at the bank, as appears from the relation $ml = 1$.

5. For waves parallel to the bank the velocity potential ϕ may be written

$$f(y, z) \cos(mx - nt);$$

then, as before, putting

$$f = \left\{ \cos z \sqrt{\frac{\partial^2}{\partial y^2} - m^2} + \frac{\sin z \sqrt{\frac{\partial^2}{\partial y^2} - m^2}}{l \sqrt{\frac{\partial^2}{\partial y^2} - m^2}} \right\} f_0(y),$$

the equations (1) and (3) are satisfied, and it is proposed to investigate when the remaining conditions can be satisfied by the assumption

$$f_0(y) = \Sigma A_s e^{my_s}.$$

This assumption being made, f is given by

$$f = \Sigma A_s \left\{ \cosh(mz \sin \psi_s) + \frac{\sinh(mz \sin \psi_s)}{ml \sin \psi_s} \right\} e^{my_s \cos \psi_s},$$

where the constants are determined by the conditions

$$\frac{\partial f}{\partial z} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha = 0,$$

when $z \cos \alpha + y \sin \alpha = 0$ and $f_0(y)$ is finite when y is infinite. Hence, from the first of these,

$$\Sigma A_s \left[\left\{ m \sin \psi_s \sinh(mz \sin \psi_s) + \frac{1}{l} \cosh(mz \sin \psi_s) \right\} \cos \alpha \right. \\ \left. + \left\{ \cosh(mz \sin \psi_s) + \frac{\sinh(mz \sin \psi_s)}{ml \sin \psi_s} \right\} m \cos \psi_s \sin \alpha \right] e^{my_s \cos \psi_s} = 0,$$

when $z \cos \alpha + y \sin \alpha = 0$;

that is, transforming to polars,

$$\Sigma A_s \left[\left(m + \frac{1}{l \sin \psi_s} \right) \sin(\psi_s + \alpha) e^{my_s \cos(\psi_s + \alpha)} \right. \\ \left. - \left(m - \frac{1}{l \sin \psi_s} \right) \sin(\psi_s - \alpha) e^{my_s \cos(\psi_s - \alpha)} \right] = 0,$$

for all positive values of r . Therefore

$$\psi_{s+1} = \psi_s + 2\alpha \quad \text{and} \quad \alpha = p\pi/q,$$

where p and q are integers, and

$$A_0 \left(m + \frac{1}{l} \operatorname{cosec} \psi_0 \right) = A_1 \left(m - \frac{1}{l} \operatorname{cosec} \psi_1 \right),$$

$$A_1 \left(m + \frac{1}{l} \operatorname{cosec} \psi_1 \right) = A_2 \left(m - \frac{1}{l} \operatorname{cosec} \psi_2 \right),$$

&c. = &c.,

$$A_{q-1} \left(m + \frac{1}{l} \operatorname{cosec} \psi_{q-1} \right) = A_0 \left(m - \frac{1}{l} \operatorname{cosec} \psi_0 \right);$$

that these equations may be consistent q must be an even integer. Let $q = 2k$; then

$$A_s \operatorname{cosec} \psi_s = \frac{ml \sin \psi_{s-1} + 1}{ml \sin \psi_s - 1} A_{s-1} \operatorname{cosec} \psi_{s-1},$$

that is,

$$A_s \operatorname{cosec} \psi_s = \frac{(ml \sin \psi_{s-1} + 1)(ml \sin \psi_{s-2} + 1) \dots (ml \sin \psi_0 + 1)}{(ml \sin \psi_s - 1)(ml \sin \psi_{s-1} - 1) \dots (ml \sin \psi_1 - 1)} A_0 \operatorname{cosec} \psi_0,$$

whence, $2ka$ being an odd multiple of π ,

$$A_s \operatorname{cosec} \psi_s = \frac{(-)^s (ml \sin \psi_{s+1} - 1)(ml \sin \psi_{s+2} - 1) \dots (ml \sin \psi_{k+s-1} - 1)}{(ml \sin \psi_1 - 1)(ml \sin \psi_2 - 1) \dots (ml \sin \psi_{k-1} - 1)} \times A_0 \operatorname{cosec} \psi_0.$$

Now that $f_0(y)$ may be finite for all positive values of y , $\cos \psi_s < 0$ for all values of s which can occur; taking the case

$$a = \pi/2k, \quad 3\pi/2 \geq \psi_0 + 2sa \geq \pi/2,$$

let

$$\psi_0 = \pi/2 + \beta,$$

where β lies between 0 and $2a$ (this includes all possibilities); then

$$2k \geq \frac{2\beta k}{\pi} + 2s \geq 0,$$

and s can have values 0 to $k-1$ if β is not zero, 0 to k if β is zero.

A_s by the above can only vanish for $k-1$ values of ψ_s ; therefore $k+1$ values of s must occur, and hence β is zero and the values of s from 0 to k occur. It will be observed that each exponential now appears twice for

$$\sin 2sa = \sin 2(k-s)a,$$

and it is therefore necessary to rewrite the expression for $f_0(y)$; there are two cases, k odd and k even. When k is odd $= 2k'+1$,

$$f_0(y) = \sum_{k'+1}^{3k'+1} A_s e^{-my \sin 2sa}.$$

and, when k is even $= 2k'$,

$$f_0(y) = \sum_{k'}^{3k'} A_s e^{-my \sin 2sa}.$$

Then, (1) k odd,

$$f = \sum_{k'+1}^{3k'+1} A_s \left[\cosh(mz \cos 2sa) + \frac{\sinh(mz \cos 2sa)}{ml \cos 2sa} \right] e^{-my \sin 2sa},$$

where $A_{s+1} [ml - \sec 2(s+1)\alpha] = A_s (ml + \sec 2sa)$.

Writing f in the form

$$f = \frac{1}{2} \sum_{k'+1}^{3k'+1} A_s \left[e^{m(z \cos 2sa - y \sin 2sa)} \left(1 + \frac{1}{ml \cos 2sa} \right) + e^{-m(2 \cos 2sa + y \sin 2sa)} \left(1 - \frac{1}{ml \cos 2sa} \right) \right],$$

it appears that the velocity will be finite for all values of y and z which occur, if $ml = 1$, for then

$$A_{2k'+2} = 0,$$

and the other constants A , up to $A_{3k'+1}$, and further the coefficient

$$1 + \frac{1}{ml \cos 2(2k'+1)\alpha}$$

of $e^{m[z \cos 2(2k'+1)\alpha - y \sin 2(2k'+1)\alpha]}$ vanishes.

The other constants are given by the relation

$$A_s \sec 2sa = \frac{[\cos 2(s-1)\alpha + 1][\cos 2(s-2)\alpha + 1] \dots [\cos 2(k'-1)\alpha + 1]}{(\cos 2sa - 1)[\cos 2(s-1)\alpha - 1] \dots [\cos 2(k'+2)\alpha - 1]} \\ \times A_{k'+1} \sec 2(k'+1)\alpha,$$

that is,

$$A_s \sec 2sa = \frac{[\cos 2(s+1)\alpha - 1][\cos 2(s+2)\alpha - 1] \dots \cos [2(s+2k')\alpha - 1]}{[\cos 2(k'+2)\alpha - 1][\cos 2(k'+3)\alpha - 1] \dots [\cos 2(3k'+1)\alpha - 1]} \\ \times (-)^{s-k'-1} A_{k'+1} \sec 2(k'+1)\alpha;$$

hence

$$A_s = (-)^s B \cos 2sa \sin^2(s+1)\alpha \sin^2(s+2)\alpha \dots \sin^2(s+2k')\alpha,$$

for values of s from $k'+1$ to $2k'+1$, and is zero for other values.

It follows that

$$f = B \sum_{s=k'+1}^{s=2k'+1} (-)^s \sin^2(s+1) \alpha \sin^2(s+2) \alpha \dots \sin^2(s+2k') \alpha \\ \times \{ \cosh(mz \cos 2sa) \cos 2sa + \sinh(mz \cos 2sa) \} e^{-my \sin 2sa}.$$

The velocity of propagation of the waves is given by

$$V^2 = g/m,$$

that is,

$$V^2 = g\lambda/2\pi,$$

where λ is the wave-length, and the velocity potential by

$$\phi = B \sum_{s=k'+1}^{s=2k'+1} (-)^s \sin^2(s+1) \alpha \sin^2(s+2) \alpha \dots \sin^2(s+2k') \alpha \\ \times \{ \cos^2 sa e^{-m(y \sin 2sa - z \cos 2sa)} - \sin^2 sa e^{-m(y \sin 2sa + z \cos 2sa)} \} \cos(mx - nt),$$

where

$$\alpha = \pi/(4k' + 2).$$

When k is even $= 2k'$,

$$f = A_{k'} e^{-my} \left(1 + \frac{z}{l} \right) \\ + \sum_{k'=1}^{3k'-1} A_s \left\{ \cosh(mz \cos 2sa) + \frac{\sinh(mz \cos 2sa)}{ml \cos 2sa} \right\} e^{-my \sin 2sa} \\ + A^{3k'} e^{my} \left(1 + \frac{z}{l} \right) + B y e^{mz};$$

but the constants cannot be determined to satisfy the conditions that the velocity is everywhere finite, and that the motion is the same at a great distance from the bank, as in the case of deep sea waves. It should be observed that Sir George Stokes' solution is included in the general form.

If α is of the form $\frac{2p+1}{2k} \pi$, where p is other than zero, the conditions cannot be satisfied, and it follows that the only cases where the velocity potential can be expressed as a sum of exponentials, the velocity being sensible at a distance from the edge, is when the angle of the bank is of the form $\pi/(4k+2)$, and that then the velocity of propagation of the waves is that of deep sea waves.

Particular cases :—

$$(1) \alpha = \frac{\pi}{2}, \quad \phi = C e^{mz} \cos(mx - nt);$$

$$(2) \alpha = \frac{\pi}{6}, \quad \phi = C \{ e^{mz} + e^{-m \frac{1}{2}(y\sqrt{3}+z)} - 3e^{-m \frac{1}{2}(y\sqrt{3}-z)} \} \cos(mx - nt);$$

and so on.

The following presents to the Library were received during the Recess :—

"Proceedings of the Royal Society," Vol. *LIX.*, Nos. 357, 358; Vol. *LX.*, Nos. 359, 360.

"Journal of the Institute of Actuaries," Vol. *XXXII.*, Pt. 6, July, 1896.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. *XX.*, St. 6-8; Leipzig, 1896.

"Nyt Tidsskrift for Matematik," A. Aargang 7, Nos. 3, 4; B. Aargang 7, No. 2; Copenhagen, 1896.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," *II.*, *III.*; 1896.

"Archives Néerlandaises," Tome *XXX.*, Liv. 2; Harlem, 1896.

"Monatshefte für Mathematik und Physik," Hefte 7-9; Wien, 1896.

"Russian Mathematical Society—Proceedings," Tom. *XVII.*; Odessa, 1895.

"Reale Istituto Lombardo—Rendiconti," Serie 2, Vol. *XXVIII.*; Milano, 1895.

"Prace Matematyczno-Fizyczne," Tom *VII.*; Warsaw, 1896.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. *V.*, Faso. 11, 12; Sem. 2, Vol. *V.*, Fasc. 1-5; Roma, 1896.

"Educational Times," July-Oct., 1896.

"Indian Engineering," Vol. *XIX.*, Nos. 21-26, Vol. *XX.*, Nos. 1-11, May 23-Sept. 12, 1896.

"Atti del R. Istituto Veneto," Tomo *LIII.*, Dispense 4-10, 1894-5; Tomo *LIV.*, Dispense 1-4, 1895-6.

Martello, T.—"L'Imposta Progressiva in Teoria e in Pratica," 8vo; Venezia, 1894. (Appendice al Tomo *VI.*)

"Annales de la Faculté des Sciences de Toulouse," Tome *X.*, Fasc. 1, 2; Paris, 1896.

"Journal de l'Ecole Polytechnique," *II.* Serie, Cahier 1; Paris, 1895.

"Annales de la Faculté des Sciences de Marseille," Tome *VII.*; Paris, 1896.

"Annali di Matematica," Tomo *XXIV.*, Fasc. 3; Milano, June, 1896.

Kantor, S.—"Theorie der Transformationen im R , welche sich aus quadratischen zusammensetzen lassen," 4to pamphlet.

"Journal für die reine und angewandte Mathematik," Bd. *CXVI.*, Heft 3; Berlin, 1896.

"Annals of Mathematics," Vol. *X.*, Nos. 4-6; University of Virginia, 1896.

"Proceedings of the Physical Society," Vol. *XIV.*, Pt. 7-9, July-Sept., 1896.

"Bulletin des Sciences Mathématiques," Tome *XX.*, with Index, Juin-Août; Paris, 1896.

"Jahrbuch über die Fortschritte der Mathematik," Jahrgang 1893 und 1894, Bd. *XXV.*, Heft 2; Berlin, 1896.

"Proceedings of the Edinburgh Mathematical Society," Vol. *XIV.*, 1895-6.

"Proceedings of the Royal Society of Edinburgh," Sessions 1893-6, Vol. *XX.*

"Transactions of the Canadian Institute," No. 8, Vol. *IV.*, Pt. 2; Toronto, 1895.

"Rendiconti del Circolo Matematico," Tomo *X.*, Fasc. 4; Palermo, 1896.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 1896, 1-25.

Frey, M. von.—“Untersuchungen über die Sinnesfunctionen der menschlichen Haut,” 8vo; Leipzig, 1896.

Drude, P.—“Zur Theoriestehender elektrischer Drahtwellen,” 8vo; Leipzig, 1896.

“Proceedings of the Royal Irish Academy,” Vol. III., No. 5; Dublin, 1896.

“Transactions of the Royal Irish Academy,” Vol. XXX., Pts. 18–20.

“Mathematical Magazine,” Vol. II., Nos. 9, 10; Washington, 1896.

Murani, O.—“Sperimenti sul Raggi Röntgen,” 4to; Milano, 1896.

Gill, D.—“Report on the Geodetic Survey of S. Africa,” folio; Cape Town, 1896.

“Sitzungsberichte der Physikalisch-Medicinischen Societät in Erlangen,” Heft 27; 1895.

“Revue Semestrielle des Publications Mathématiques,” Tome IV., Partie 2; 1896.

“Wiskundige Opgaven met de Oplossingen,” Deel VII., St. 1; Amsterdam, 1896.

“Bulletin de la Société Mathématique de France,” Tome XXIV., Nos. 5–7.

“The University of Toronto Quarterly,” Vol. II., Nos. 1–4; 1895–6.

“Rendiconto dell’ Accademia delle Scienze Fisiche e Matematiche,” Vol. II., Fasc. 5–7; Napoli, 1896.

“Jornal de Sciencias Mathematicas e Astronomicas,” Vol. XII., No. 5; Coimbra, 1896.

“Bulletin of the American Mathematical Society,” 2nd Series, Vol. II., Nos. 9, 10, and List of Members: New York.

“Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen,” Heft 2, 1896.

Cayley, A.—“Collected Mathematical Papers,” Vol. X., 4to; Cambridge, 1896. (Two copies.)

“On 2 as a 16-ic Residue and Note,” by Lt.-Col. Allan Cunningham, R.E. (Corrected offprint of pp. 85–122 of Vol. XXVII. of the Society’s *Proceedings*.)

ERRATA AND ADDENDA.

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Page 144. The results, in this paper, stated for surfaces of revolution should be deleted, as the differential equations there given are obeyed, not by H and E , as stated, but by $dH/d\rho$ and $dE/d\rho$.

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Page 103, table, last line but one, column of $\frac{1}{2}e$, for “32,” read “16.”

„ 104, result (117), for “ n^4 ,” read “ $16n^4$.”

„ 105, table, last line, for “ $e =$,” read “ $2e =$.”

„ 109, table, for “5,576,681,” read “5,576,881.”

„ 117, line 5 up, for “556,313,” read “556,513.”

„ 118, line of 155,377, column of $\frac{p-1}{k}$, for “8.9.23,” read “8.9.13.”