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A Method of Computing Absolute Perturbations.

By *G. W. Hill*, Office of the American Ephemeris, Washington U. S.

The object of this article is to call the attention of astronomers to the notable abbreviations which are produced in some parts of the formulae for perturbations by the introduction of the true anomaly as the variable according to which the integrations are to be executed. Prof. *Hansen* in his later disquisitions has substituted the eccentric anomaly as the independent variable in place of the mean anomaly, or what is the same thing, the time; and he regards this step as constituting a

remarkable amelioration of the method. The method here explained will, as far as coordinates are concerned, be the same as that of *Laplace*, but the same use will be made of the true anomaly in the elliptic orbit, as independent variable, as that which *Hansen* has made of the eccentric anomaly.

The following notation and equations are so familiar that they seem to need no explanation:

$$R = m' \left(\frac{1}{\Delta} - \frac{r \cos \psi}{r'^2} \right) + m'' \left(\frac{1}{\Delta_1} - \frac{r \cos \psi_1}{r''^2} \right) + \dots,$$
$$(1) \dots \dots \begin{cases} \frac{d^2 x}{dt^2} + \frac{\mu}{r^3} x = \frac{dR}{dx}, \\ \frac{d^2 y}{dt^2} + \frac{\mu}{r^3} y = \frac{dR}{dy}, \\ \frac{d^2 z}{dt^2} + \frac{\mu}{r^3} z = \frac{dR}{dz}. \end{cases}$$

Let us now suppose that each coordinate of the disturbed planet is separated into two parts, such that

$$\begin{aligned} x &= x_0 + \delta x, \\ y &= y_0 + \delta y, \\ z &= z_0 + \delta z, \end{aligned}$$

the first of which, x_0, y_0, z_0 , satisfy the differential equations

$$\begin{aligned} \frac{d^2 x_0}{dt^2} + \frac{\mu}{r_0^3} x_0 &= 0, \\ \frac{d^2 y_0}{dt^2} + \frac{\mu}{r_0^3} y_0 &= 0, \\ \frac{d^2 z_0}{dt^2} + \frac{\mu}{r_0^3} z_0 &= 0, \end{aligned}$$

where

$$r_0^2 = x_0^2 + y_0^2 + z_0^2,$$

and the second, $\delta x, \delta y, \delta z$, are of the order of the disturbing forces.

It is evident that this separation is, to a certain extent, arbitrary, as certain functions of t might be added to x_0, y_0, z_0 , without their ceasing to satisfy the differential equations determining them, and then $\delta x, \delta y, \delta z$ would necessarily be diminished by the same functions. This indetermination is eliminated in different ways according to the circumstances attending the computation of the perturbations.

If x_0, y_0, z_0 are derived from elements osculating for a certain epoch, it is plain that $\delta x, \delta y, \delta z$ ought to vanish at this epoch, as also their first differential coefficients with respect to the time. This will be accomplished by taking all the integrations, which $\delta x, \delta y, \delta z$ involve, between the limits $t=0$ and $t=t$. If the perturbations are computed from so-called mean elements, the six arbitrary constants, which $\delta x, \delta y, \delta z$ involve, must be determined in accordance with the suppositions upon which the mean elements have been derived.

We shall now write

$$r = r_0 + \delta r,$$

$$dR = \frac{dR}{dx} dx + \frac{dR}{dy} dy + \frac{dR}{dz} dz,$$

$$r \frac{dR}{dr} = x \frac{dR}{dx} + y \frac{dR}{dy} + z \frac{dR}{dz}.$$

dR is then the differential of R when the coordinates of the disturbed planet alone vary. The last equation is evidently correct, when in the first member we suppose R to be expressed in terms of r and two other coordinates which make $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$ independent of r .

By multiplying the equations which determine x , y , z , severally by $2dx$, $2dy$, $2dz$, adding the products and integrating,

$$(2) \dots \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2\mu}{r} + \frac{\mu}{a} = 2\int dR,$$

$$\frac{d^2(r_0 \delta r)}{dt^2} + \frac{\mu}{r_0^3} r_0 \delta r = 2\int dR + r \frac{dR}{dr} - \frac{1}{2} \frac{d^2(\delta r)^2}{dt^2} + \frac{\mu(\delta r)^2}{rr_0^2}.$$

In like manner equations (1) can be transformed into

$$\frac{d^2 \delta x}{dt^2} + \frac{\mu}{r_0^3} \delta x = \frac{dR}{dx} + \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right) \mu x,$$

$$\frac{d^2 \delta y}{dt^2} + \frac{\mu}{r_0^3} \delta y = \frac{dR}{dy} + \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right) \mu y,$$

$$\frac{d^2 \delta z}{dt^2} + \frac{\mu}{r_0^3} \delta z = \frac{dR}{dz} + \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right) \mu z.$$

For brevity's sake now put

$$(3) \dots \begin{cases} Qr = 2\int dR + r \frac{dR}{dr} = \frac{1}{2} \frac{d^2(\delta r)^2}{dt^2} + \frac{\mu(\delta r)^2}{rr_0^2}, \\ Qx = \frac{dR}{dx} + \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right) \mu x, \\ Qy = \frac{dR}{dy} + \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right) \mu y, \\ Qz = \frac{dR}{dz} + \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right) \mu z. \end{cases}$$

Then our differential equations take the form

$$(4) \dots \begin{cases} \frac{d^2(r_0 \delta r)}{dt^2} + \frac{\mu}{r_0^3} r_0 \delta r = Qr, \\ \frac{d^2 \delta x}{dt^2} + \frac{\mu}{r_0^3} \delta x = Qx, \\ \frac{d^2 \delta y}{dt^2} + \frac{\mu}{r_0^3} \delta y = Qy, \\ \frac{d^2 \delta z}{dt^2} + \frac{\mu}{r_0^3} \delta z = Qz. \end{cases}$$

The problem of elliptic motion being supposed

where $\frac{\mu}{a}$ is the constant added to complete the integral, and we shall suppose that it is such that the equation

$$\frac{dx_0^2 + dy_0^2 + dz_0^2}{dt^2} + \frac{2\mu}{r_0} + \frac{\mu}{a} = 0,$$

is satisfied; if there is any residual constant part, it must be supposed contained in the term $2\int dR$. By multiplying the differential equations determining x , y , z , severally by these quantities and adding the products to equation (2), we get

$$\frac{1}{2} \frac{d^2 r^2}{dt^2} - \frac{\mu}{r} + \frac{\mu}{a} = 2\int dR + r \frac{dR}{dr}.$$

By using the equation $r = r_0 + \delta r$, this can be readily transformed into

completely solved, $\frac{\mu}{r_0^3}$ is a known function of t , and

$$(5) \dots \frac{d^2 q}{dt^2} + \frac{\mu}{r_0^3} q = 0$$

is a linear differential equation. According to the theory of this class of differential equations, the value of q has the form

$$q = K_1 q_1 + K_2 q_2,$$

K_1 and K_2 being the arbitrary constants, and q_1 and q_2 any two particular solutions which are independent of other. Then there must necessarily exist the two equations

$$(6) \dots \begin{cases} \frac{d^2 q_1}{dt^2} + \frac{\mu}{r_0^3} q_1 = 0, \\ \frac{d^2 q_2}{dt^2} + \frac{\mu}{r_0^3} q_2 = 0. \end{cases}$$

By the elimination of $\frac{\mu}{r_0^3}$ from these, is obtained

$$\frac{q_1 d^2 q_2 - q_2 d^2 q_1}{dt^2} = 0.$$

This is an exact differential, and integrating

$$(7) \dots \frac{q_1 dq_2 - q_2 dq_1}{dt} = \text{a constant.}$$

This constant is arbitrary and may be taken at will; we shall, for the sake of simplicity, assume it equal to unity.

Taking now the more general equation

$$\frac{d^2 q}{dt^2} + \frac{\mu}{r_0^3} q = Q,$$

let us eliminate $\frac{\mu}{r_0^3}$ from this and equations (6).

We get

$$\frac{q_1 d^2 q - q d^2 q_1}{dt^2} = Qq_1,$$

$$\frac{q_2 d^2 q - q d^2 q_1}{dt^2} = Qq_2,$$

and taking the integrals,

$$\frac{q_1 dq - q dq_1}{dt} = K_2 + \int q_1 Q dt,$$

$$\frac{q_2 dq - q dq_2}{dt} = -K_1 + \int q_2 Q dt.$$

Whence is obtained, regard being had to equation (7),

$$q = K_1 q_1 + K_2 q_2 + q_2 \int q_1 Q dt - q_1 \int q_2 Q dt$$

The constants K_1 and K_2 may be regarded as contained in the integrals $\int Q q_2 dt$ and $\int Q q_1 dt$. Applying these results to equations (4), there result

$$\frac{dx_0}{dt} \frac{d\delta x}{dt} + \frac{dy_0}{dt} \frac{d\delta y}{dt} + \frac{dz_0}{dt} \frac{d\delta z}{dt} + \frac{\mu}{rr_0^2} r_0 \delta r = \int dR - \frac{1}{2} \left[\left(\frac{d\delta x}{dt} \right)^2 + \left(\frac{d\delta y}{dt} \right)^2 + \left(\frac{d\delta z}{dt} \right)^2 \right].$$

In the case, mentioned above, of osculating elements, all the constants are determined by making each integral expression vanish with t .

There is a remarkable procedure for reducing the right members of equations (8) to contain a single integral expression, which is due to Prof. *Hansen*. The factors q_1 , q_2 , outside the signs of integration, may evidently be removed within the signs, if it is agreed to regard the t , they contain, as constant in the integration. As it is necessary to keep this t distinct from the t of the quantities already under the sign of integration, we may write τ for it, and to denote that any quantity, which is a function of t , has its t changed into τ , we will write $(-)$ above it. Thus making

$$(10) \dots \dots N = \bar{q}_2 q_1 - \bar{q}_1 q_2,$$

we have the very simple expressions

$$(11) \dots \dots \begin{cases} r_0 \delta r = \int N \bar{Q}_r dt, \\ \delta x = \int N \bar{Q}_x dt, \\ \delta y = \int N \bar{Q}_y dt, \\ \delta z = \int N \bar{Q}_z dt. \end{cases}$$

After the integration is finished, τ will be replaced by t . Since τ is regarded as constant, an arbitrary function of τ must be added to each of these expressions, which, after τ is changed into t , becomes an arbitrary function of t . These must, in each case, be so determined that the expressions (11) may coincide with (8). Any consideration of these arbitrary functions will be rendered unnecessary, by agreeing to take the inte-

$$(8) \dots \dots \begin{cases} r_0 \delta r = q_2 \int q_1 Q_r dt - q_1 \int q_2 Q_r dt, \\ \delta x = q_2 \int q_1 Q_x dt - q_1 \int q_2 Q_x dt, \\ \delta y = q_2 \int q_1 Q_y dt - q_1 \int q_2 Q_y dt, \\ \delta z = q_2 \int q_1 Q_z dt - q_1 \int q_2 Q_z dt. \end{cases}$$

These equations must satisfy the relation

$$(9) \dots r_0 \delta r = x_0 \delta x + y_0 \delta y + z_0 \delta z + \frac{1}{2} [\delta x^2 + \delta y^2 + \delta z^2 - \delta r^2].$$

It is however necessary to employ all four of equations (8) since in proceeding by successive approximations, as we are obliged to, we cannot get the values of Q_x , Q_y , Q_z until δr is known. These equations contain, in the general case, nine arbitrary constants, viz., the one added to the term $2 \int dR$ in Q_r , and the eight introduced by the eight integrals of equations (8). But the last will be reduced to six, independent of each other, by the condition (9), and the constant annexed to $\int dR$ will be determined in function of these six, by the condition

$$K_1 q_1 + K_2 q_2,$$

must be added to each equation. In the case of osculating elements, mentioned above, if the lower limit is taken at zero, this arbitrary expression vanishes.

Equations (11) may be exhibited in the form of definite integrals, thus

$$r_0 \delta r = - \int_0^t N \bar{Q}_r dt,$$

$$\delta x = - \int_0^t N \bar{Q}_x dt,$$

$$\delta y = - \int_0^t N \bar{Q}_y dt,$$

$$\delta z = - \int_0^t N \bar{Q}_z dt.$$

N may be regarded as an integrating factor whose value is virtually zero, but a part of the time, involved in its expression, is regarded as constant in the integration.

The values of q_1 and q_2 must now be determined. If

$$n = \sqrt{\frac{\mu}{a^3}},$$

$$nt + c = \mathcal{Z} = u - e \sin u,$$

where c and e are constants, and $\mathcal{Z}$ and u respectively

the mean and eccentric anomalies in the elliptic orbit of the disturbed planet, then

$$\frac{r_0}{a} = 1 - e \cos u,$$

$$d\mathcal{C} = \frac{r_0}{a} du.$$

Equation (5) becomes then

$$\frac{d^2 q}{d\mathcal{C}^2} + \frac{a^3}{r_0^3} q = 0.$$

If u is made the independent variable, it becomes

$$(12) \dots (1-e \cos u) \frac{d^2 q}{du^2} - e \sin u \frac{dq}{du} + q = 0.$$

Differentiating this and removing the useless factor $1 - e \cos u$, we get,

$$\frac{d^3 q}{du^3} + \frac{dq}{du} = 0,$$

the integral of which is

$$q = K_1 \cos u + K_2 \sin u + K_3.$$

Determining K_3 so that equation (12) may be satisfied,

$$K_3 = -K_1 e.$$

Hence the complete integral of (12) is

$$q = K_1 (\cos u - e) + K_2 \sin u.$$

It is evident now that we may take

$$q_1 = k (\cos u - e),$$

$$q_2 = k \sin u.$$

If these values are substituted in equation (7) it is found that $k = \frac{1}{n}$, thus

$$q_1 = \sqrt{\frac{an}{\mu}} (\cos u - e) = \sqrt{\frac{an}{\mu}} r_0 \cos v,$$

$$q_2 = \sqrt{\frac{a^3 n}{\mu}} \sin u = \sqrt{\frac{an}{\mu(1-e^2)}} r_0 \sin v,$$

if v is the true anomaly of the disturbed planet in its elliptic orbit. Thus

$$N = \frac{a^3 n}{\mu} [\sin(\bar{u} - u) - e \sin \bar{u} + e \sin u]$$

$$= \frac{an}{\mu \sqrt{1-e^2}} \bar{r}_0 r_0 \sin(\bar{v} - v).$$

We shall now change the independent variable t for the variable v . We have

$$dt = \frac{r_0^2 dv}{a^2 n \sqrt{1-e^2}}.$$

$$\delta \lambda = \int \left[\int \frac{an}{\mu} Q_\lambda dt - \frac{1}{1-e^2} \cos i \frac{(r+r_0)\delta r - (z+z_0)\delta z}{r_0^2 - z_0^2} \right] \frac{a^2 n dt}{r^2 - z^2}.$$

Or if v is made the independent variable, and for brevity we put $p = a(1-e^2)$, the expressions for the perturbations are

Whence

$$N dt = \frac{1}{\mu a (1-e^2)} \bar{r}_0 r_0^3 \sin(\bar{v} - v) dv.$$

Thus the expressions for the perturbations become

$$(13) \dots \dots \begin{cases} \delta r = \frac{1}{\mu a (1-e^2)} \int Q_r r_0^3 \sin(\bar{v} - v) dv, \\ \delta x = \frac{r_0}{\mu a (1-e^2)} \int Q_x r_0^3 \sin(\bar{v} - v) dv, \\ \delta y = \frac{r_0}{\mu a (1-e^2)} \int Q_y r_0^3 \sin(\bar{v} - v) dv, \\ \delta z = \frac{r_0}{\mu a (1-e^2)} \int Q_z r_0^3 \sin(\bar{v} - v) dv. \end{cases}$$

It will be perceived that, by this transformation, we have been enabled to get rid of the factor r_0 before δr , with a simplification of the right member of the equation.

These equations, although very symmetrical, present the inconvenience of being one too many. Hence for the second and third we shall substitute a single one. From the differential equations of motion,

$$\frac{xdy - ydx}{dt} = h + \int \left[x \frac{dR}{dy} - y \frac{dR}{dx} \right] dt,$$

where h is such a constant that

$$\frac{x_0 dy_0 - y_0 dx_0}{dt} = h,$$

and i denoting the inclination of the plane of the elliptic orbit to the plane of xy ,

$$h = \sqrt{\mu a (1-e^2)} \cos i.$$

Denoting by λ the longitude measured in the plane of xy , so that

$$\tan \lambda = \frac{y}{x}$$

and putting

$$Q_\lambda = x \frac{dR}{dy} - y \frac{dR}{dx} = \frac{dR}{d\lambda},$$

we shall have

$$(r^2 - z^2) \frac{d\lambda}{dt} = h + \int Q_\lambda dt.$$

Supposing $\lambda = \lambda_0 + \delta \lambda$, where

$$\tan \lambda_0 = \frac{y_0}{x_0},$$

the following equation is obtained for the determination of $\delta \lambda$,

$$(14) \dots\dots\dots \left\{ \begin{array}{l} \delta r = \frac{1}{\mu p} \int Q_r r_0^3 \sin(\bar{v}-v) dv, \\ \delta z = \frac{r_0}{\mu p} \int Q_z r_0^3 \sin(\bar{v}-v) dv, \\ \delta \lambda = \int \left[\int \frac{r_0^2}{\mu p} Q_\lambda dv - \cos i \frac{(r+r_0)\delta r - (z+z_0)\delta z}{r_0^2 - z_0^2} \right] \frac{r_0^2 dv}{r^2 - z^2}. \end{array} \right.$$

These formulae are absolutely rigorous, since no terms have been neglected, and also perfectly general, as no restriction has been put upon the position of the plane of xy from which the coordinate z is measured. By adopting the plane of the elliptic orbit of the

disturbed planet as the plane of xy , the last equation is somewhat simplified. For then $i=0$, and $z_0=0$, and $z=\delta z$; thus

$$(15) \dots \delta \lambda = \int \left[\int \frac{r_0^2}{\mu p} Q_\lambda dv - \frac{(r+r_0)\delta r - \delta z^2}{r_0^2} \right] \frac{r_0^2 dv}{r^2 - \delta z^2}.$$

Perturbations of the first order with respect to the disturbing forces.

Since in this case, elliptic values are to be substituted for the coordinates in the functions Q_r , Q_z , Q_λ , there is no need any further of making a distinction between r_0 and r , hence we shall omit the $(_0)$ from the former. If we put

$$\begin{aligned} T &= \frac{r^3}{\mu p} Q_r, \\ &= \frac{r^3}{\mu p} \left[2 \int dR + r \frac{dR}{dr} \right], \\ Z &= \frac{r^3}{\mu p} \frac{dR}{dz}, \\ Y &= \frac{r^2}{\mu p} Q_\lambda, \end{aligned}$$

and $\delta\beta$ is the latitude of the disturbed planet measured from the plane of its elliptic orbit, and $\delta\lambda$ the perturbation of the longitude measured in this plane, our formulae in this case reduce to

$$\begin{aligned} \delta r &= \int T \sin(\bar{v}-v) dv, \\ \delta \beta &= \int Z \sin(\bar{v}-v) dv, \\ \delta \lambda &= \int \left[\int Y dv - 2 \frac{\delta r}{r} \right] dv. \end{aligned}$$

Put now

$$X = \frac{r^4}{\mu p} \frac{dR}{dr},$$

then it will easily be found that

$$\frac{1}{\mu p} dR = r^{-2} \left[\frac{e \sin v}{p} X + Y \right] dv.$$

Thus the shape, in which we shall employ our equations, is

$$\begin{aligned} \delta r &= \int \left[X + 2r^3 r^{-2} \left(\frac{e \sin v}{p} X + Y \right) dv \right] \sin(\bar{v}-v) dv, \\ \delta \lambda &= \int \left[\int Y dv - 2 \frac{\delta r}{r} \right] dv, \\ \delta \beta &= \int Z \sin(\bar{v}-v) dv. \end{aligned}$$

The chief thing now to be done is to expand X , Y and Z into periodic series as functions of v . The

elliptic values of the coordinates of the disturbed planet are readily expressed in terms of this variable, but the coordinates of the disturbing bodies will naturally be expressed in terms of their mean anomalies $\mathcal{Z}'$, $\mathcal{Z}''$ etc. These last variables must be eliminated by means of the identical equations

$$\begin{aligned} \mathcal{Z}' &= \frac{n'}{n} v + c' - \frac{n'}{n} c - \frac{n'}{n} (v - \mathcal{Z}), \\ \mathcal{Z}'' &= \frac{n''}{n} v + c'' - \frac{n''}{n} c - \frac{n''}{n} (v - \mathcal{Z}), \\ &\dots\dots\dots \end{aligned}$$

Let us put then

$$\begin{aligned} S' &= \frac{n'}{n} v + c' - \frac{n'}{n} c, \\ S'' &= \frac{n''}{n} v + c'' - \frac{n''}{n} c, \\ &\dots\dots\dots \end{aligned}$$

so that

$$\frac{dS'}{dv} = \frac{n'}{n}, \quad \frac{dS''}{dv} = \frac{n''}{n}, \quad \dots\dots\dots$$

Then $\mathcal{Z}'$, $\mathcal{Z}''$ etc. will be replaced by the following values

$$\mathcal{Z}' = S' - \frac{n'}{n} (v - \mathcal{Z}), \quad \mathcal{Z}'' = S'' - \frac{n''}{n} (v - \mathcal{Z}), \quad \dots\dots$$

In the development of X , Y and Z in periodic series from particular values of these quantities, it will be better to make the differences of S' , S'' etc. ..., from v , the variables to be employed. Thus we shall put

$$w' = S' - v, \quad w'' = S'' - v, \quad \dots\dots\dots$$

The formulae, to be written now, will be confined to the case of the action of one planet. The expressions for X , Y and Z are

$$\begin{aligned} X &= \frac{m'}{h^2} r^4 \left[\frac{1}{\Delta^3} - \frac{1}{r^3} \right] r' \cos \beta' \cos(\lambda' - \lambda) - \frac{m' r^5}{h^2 \Delta^3}, \\ Y &= \frac{m'}{h^2} r^3 \left[\frac{1}{\Delta^3} - \frac{1}{r^3} \right] r' \cos \beta' \sin(\lambda' - \lambda), \\ Z &= \frac{m'}{h^2} r^3 \left[\frac{1}{\Delta^3} - \frac{1}{r^3} \right] r' \sin \beta', \end{aligned}$$

where $h^2 = \mu a(1-e^2)$, and

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos \beta' \cos (\lambda' - \lambda).$$

If the inclinations of the orbits of the two planets to some fixed plane, as the ecliptic, are denoted by i, i' , and the longitudes of their ascending nodes by Ω, Ω' , and the longitudes of their perihelia by π, π' , we compute I, Θ, Θ', Π and Π' from

$$\begin{aligned} \cos I &= \cos i \cos i' + \sin i \sin i' \cos (\Omega' - \Omega), \\ \sin I \cos (\Theta - \Omega) &= -\sin i \cos i' + \cos i \sin i' \cos (\Omega' - \Omega), \\ \sin I \sin (\Theta - \Omega) &= \sin i' \sin (\Omega' - \Omega), \\ \sin I \cos (\Theta' - \Omega') &= \cos i \sin i' - \sin i \cos i' \cos (\Omega' - \Omega), \\ \sin I \sin (\Theta' - \Omega') &= \sin i \sin (\Omega' - \Omega), \\ \Pi &= \pi - \Theta, \\ \Pi' &= \pi' - \Theta'. \end{aligned}$$

The circumference being divided into k equal parts with reference to v , compute for each of the k values of $v, 0, \frac{2}{k}\pi, \frac{4}{k}\pi, \dots, \frac{2(k-1)}{k}\pi$, the following quantities:

$$\tan \frac{u}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{v}{2},$$

$$\mathcal{Z} = u - e \sin u,$$

$$V = r - \frac{u'}{n} (v - \mathcal{Z}),$$

$$r = \frac{p}{1 + e \cos v},$$

$$K \cos (\Pi' - A) = \cos (v + \Pi),$$

$$K \sin (\Pi' - A) = \cos I \sin (v + \Pi),$$

$$K' \cos (\Pi' - A') = \cos I \cos (v + \Pi),$$

$$K' \sin (\Pi' - A') = \sin (v + \Pi),$$

$$G = 2Kr,$$

$$G' = \frac{m'}{h^2} r^4 K,$$

$$G'' = \frac{m'}{h^2} r^3,$$

$$G''' = \frac{m'}{h^2} r^3 K',$$

$$G'''' = \frac{m'}{h^2} \sin I. r^3.$$

Several of these quantities as $u, \mathcal{Z}, V, r$ will need to be computed only $\frac{k}{2}$ times, if k is a multiple of 2;

and K, K', A, A' only $\frac{k}{4}$ times in the same case.

The circumference being divided into k' equal parts with reference to the variable w' , compute for each of the kk' values of v and w' the following quantities, w' taking in succession the values $0, \frac{2}{k'}\pi, \frac{4}{k'}\pi, \dots, \frac{2(k'-1)}{k'}\pi$:

$$\mathcal{Z}' = V + w', \quad 1 - r' \cos \frac{v'}{2} = V \sqrt{a'(1-e')} \cos \frac{u'}{2},$$

$$u' - e' \sin u' = \mathcal{Z}', \quad V r' \sin \frac{v'}{2} = V \sqrt{a'(1+e')} \sin \frac{u'}{2}.$$

If we have tables of the disturbing planet giving the true anomaly or the equation or the equation of the centre and the radius vector or its logarithm with the argument mean anomaly, we can derive $\log r'$, and v' by means of their aid, and thus dispense with computing the last three equations. We now compute kk' times

$$\Delta^2 = r^2 + r'^2 - G r' \cos (v' + A),$$

$$X = G' \left[\frac{1}{\Delta^3} - \frac{1}{r'^3} \right] r' \cos (v' + A) - \frac{G''}{\Delta^3},$$

$$Y = G''' \left[\frac{1}{\Delta^3} - \frac{1}{r'^3} \right] r' \sin (v' + A'),$$

$$Z = G'''' \left[\frac{1}{\Delta^3} - \frac{1}{r'^3} \right] r' \sin (v' + \Pi').$$

From these kk' special values of each of the quantities X, Y and Z , we deduce their development in periodic series of the form

$$\sum_{i,i'} \left(K_{i,i'}^{(e)} \cos (iv - i'w') + K_{i,i'}^{(s)} \sin (iv - i'w') \right).$$

This process is so well known that we need not here insert the formulae required for it; they will be found in *Hansen's „Auseinandersetzung“, Part I, p. 159*. A double application of these formulae will be necessary, the first relative to v , the second relative to w' . After these series are obtained, w' can be replaced by $\mathcal{S}' - v$.

The series X is now to be multiplied by $\frac{e}{p} \sin v$, which, for every periodic term in X , will give two periodic terms, which will be added to the series Y . This result is next to be multiplied by

$$\left(\frac{1+e \cos v}{p} \right)^2 = \frac{1+\frac{1}{2}e^2}{p^2} + \frac{2e}{p^2} \cos v + \frac{e^2}{2p^2} \cos 2v.$$

There is now an integration to be effected. A table of logarithms of the integrating factors

$$[i, i'] = \frac{1}{i-i'} \frac{n'}{n}$$

will now be made for all combinations of i and i' which occur in the periodic series. If the last result contains a term

$$K_{\sin}^{\cos} (iv - i' \mathcal{S}'),$$

the corresponding term of the integrated result will be $+ [i, i'] K_{\cos}^{\sin} (iv - i' \mathcal{S}')$.

A multiplication by $2r^3$ is now to be made. We have

$$\frac{r^3}{a^3} = \frac{1}{2} E_0 - E_1 \cos v + E_2 \cos 2v - E_3 \cos 3v + \dots,$$

where the rigorous value of the coefficients is given by the equation

$$E_i = \sqrt{1-e^2} [2+e^2+3i\sqrt{1-e^2}+i^2(1-e^2)] \left(\frac{e}{1+\sqrt{1-e^2}} \right)^i.$$

This multiplication accomplished, the product is to be added to X . If this result has a term

$$K_{\sin}^{\cos} (iv - i' S'),$$

then δr has as its corresponding term

$$-[i-1, i'] [i+1, i'] K_{\sin}^{\cos} (iv - i' S'),$$

Perturbations of the second order with respect to the disturbing forces.

Calling the parts of the perturbations of r , β , λ which are of two dimensions with respect to the planetary masses, $\delta^2 r$, $\delta^2 \beta$, $\delta^2 \lambda$, so that we have with errors of the third order

$$\delta^2 r = \int \frac{r^3}{h^2} \delta Q_r \sin(\bar{v}-v) dv,$$

$$\delta^2 \beta = \int \frac{r^3}{h^2} \delta Q_\beta \sin(\bar{v}-v) dv - \frac{\delta r}{r} \delta \beta,$$

$$\delta^2 \lambda = \int \left[\int \frac{r^2}{h^2} \delta Q_\lambda dv - 2 \frac{\delta^2 r}{r} - \left(\frac{\delta r}{r} \right)^2 + \delta \beta^2 - 2 \frac{\delta r}{r} \frac{d \delta \lambda}{dv} \right] dv.$$

Where as before, there is no need of any distinction between r_0 and r . The following are the expressions for δQ_r and δQ_β ,

$$\frac{r^3}{h^2} \delta Q_r = \frac{r^3}{h^2} \delta \left(r \frac{dR}{dr} \right) + 2 \frac{r^3}{h^2} \int d\delta R - \frac{1}{2r} \frac{d^2(\delta r)^2}{dv^2} + \frac{e}{p} \sin v \frac{d(\delta r)^2}{dv} + \frac{1}{p} (\delta r)^2,$$

$$\frac{r^3}{h^2} \delta Q_\beta = \frac{r^3}{h^2} \delta \left(\frac{dR}{d\beta} \right) + \frac{3}{p} \delta r \delta \beta.$$

Bearing in mind that X , Y , Z are homogeneous functions of r and r' , it will be easy to deduce the following equations

$$\frac{r^3}{h^2} \delta \left(r \frac{dR}{dr} \right) = \left(r \frac{dX}{dr} - 3X \right) \frac{\delta r}{r} - \left(r \frac{dX}{dr} - X \right) \frac{\delta r'}{r'} + \frac{dX}{d\lambda} (\delta \lambda - \delta \lambda') + r \left(r \frac{dZ}{dr} - 3Z \right) \delta \beta + \frac{dX}{d\beta'} \delta \beta',$$

$$\frac{r^2}{h^2} \delta Q_\lambda = \frac{1}{r} \frac{dX}{d\lambda} \frac{\delta r}{r} - \left(\frac{1}{r} \frac{dX}{d\lambda} + 2Y \right) \frac{\delta r'}{r'} + \frac{dY}{d\lambda} (\delta \lambda - \delta \lambda') + \frac{dZ}{d\lambda} \delta \beta + \frac{dY}{d\beta'} \delta \beta',$$

$$\frac{r^3}{h^2} \delta \left(\frac{dR}{d\beta} \right) = \left(r \frac{dZ}{dr} - 3Z \right) \frac{dr}{r} - r \frac{dZ}{dr} \frac{\delta r'}{r'} + \frac{dZ}{d\lambda} (\delta \lambda - \delta \lambda') + \frac{dZ}{d\beta} \delta \beta + \frac{dZ}{d\beta'} \delta \beta',$$

$$2 \frac{r^3}{h^2} \int d\delta R = 2r^3 \int r^{-2} \left[r^{-1} X \frac{d \delta r}{dv} + Y \frac{d \delta \lambda}{dv} + Z \frac{d \delta \beta}{dv} + \frac{e}{p} \sin v \frac{r^3}{h^2} \delta \left(r \frac{dR}{dr} \right) + \frac{r^2}{h^2} \delta Q_\lambda \right] dv,$$

where the differential coefficients $\frac{d \delta r}{dv}$, $\frac{d \delta \lambda}{dv}$, $\frac{d \delta \beta}{dv}$

are complete with respect to the independent variable v .

except in the case where $i=1$, $i'=0$, when we have, instead of this,

$$+ \frac{1}{2} K_{\cos}^{\sin} v.$$

Having thus obtained δr , we multiply it by

$$\frac{1}{r} = \frac{1}{p} + \frac{e}{p} \cos v.$$

The result, which is the perturbation of the natural logarithm of r , must be doubled, and then subtracted from $\int Y dv$. Another integration being executed on this result, we have $\delta \lambda$ the perturbation of the longitude measured in the plane of the fixed elliptic orbit.

Finally $\delta \beta$ will be obtained by treating Z to the same kind of integration as that last used in obtaining δr ; that is, in general, each coefficient of Z will be multiplied by the proper value of $-[i-1, i'] [i+1, i']$ which corresponds to it.

$$r = r_0 + \delta r + \delta^2 r,$$

$$\beta = \delta \beta + \delta^2 \beta,$$

$$\lambda = \lambda_0 + \delta \lambda + \delta^2 \lambda,$$

where δr , $\delta \beta$, $\delta \lambda$ are the perturbation which have previously been determined, we shall have

In computing the values of these functions, $\frac{\delta r'}{r'}$,

$\delta \lambda'$ and $\delta \beta'$ must be expressed as functions of v .

Hence if they are at first expressed in terms of t , it must be eliminated by means of the equation

$nt + c = v - E_1 \sin v + \frac{1}{2} E_2 \sin 2v - \frac{1}{3} E_3 \sin 3v + \dots$,
where the rigorous value of E_i is

$$E_i = 2 [1 + i \sqrt{1 - e^2}] \left(\frac{e}{1 + \sqrt{1 - e^2}} \right)^i.$$

We may have only given the perturbation of the orbit longitude and the latitude above the elliptic orbit, of the disturbing planet; in this case, calling the latter $\delta \eta'$, the values of $\delta \lambda'$ and $\delta \beta'$ will be given by the equations

$$\begin{aligned} \delta \lambda' &= \frac{\cos I}{\cos^2 \beta'} \delta v' - \frac{\sin I \cos(v' + \Pi')}{\cos^2 \beta'} d\eta', \\ \delta \beta' &= \frac{\cos I}{\cos \beta'} \delta \eta' + \frac{\sin I \cos(v' + \Pi')}{\cos \beta'} dv'. \end{aligned}$$

When we see that in order to obtain the perturbations of the second order, it will be necessary to have, expressed in periodic series in terms of v , the following nine quantities: $r \frac{dX}{dr}$, $r \frac{dZ}{dr}$, $\frac{dZ}{d\lambda}$,

$$\begin{aligned} \frac{dX}{d\lambda}, \quad \frac{dY}{d\lambda}, \quad \frac{dZ}{d\lambda}, \\ \frac{dX}{d\beta'}, \quad \frac{dY}{d\beta'}, \quad \frac{dZ}{d\beta'}, \end{aligned}$$

For six of these, whose expressions are

$$\begin{aligned} r \frac{dX}{dr} &= 4X - \frac{m'}{h^2} \frac{r^5}{\Delta^3} + 3 \frac{m'}{h^2} r^3 \left(\frac{(r'^2 - r^2)^2}{4\Delta^5} - \frac{r'^2 r^2}{2\Delta^3} + \frac{1}{4\Delta} \right), \\ r \frac{dZ}{dr} &= 3Z + \frac{3m'}{2h^2} r^3 \left(\frac{r'^2 - r^2}{\Delta^5} - \frac{1}{\Delta^3} \right) r' \sin \beta', \\ \frac{dZ}{dr} &= -\frac{m'}{h^2} r^4 \left(\frac{1}{\Delta^3} - \frac{3r'^2 \sin^2 \beta'}{\Delta^5} \right), \\ \frac{dX}{d\beta'} &= \frac{m'}{h^2} r^4 \left(\frac{3}{2} \frac{r^2 - r'^2}{\Delta^5} + \frac{1}{2} \frac{1}{\Delta^3} + \frac{1}{r'^3} \right) r' \sin \beta' \cos(\lambda' - \lambda), \\ \frac{dY}{d\beta'} &= -\frac{m'}{h^2} r^3 \left(\frac{3r^2 + r'^2}{2\Delta^5} - \frac{1}{2} \frac{1}{\Delta^3} - \frac{1}{r'^3} \right) r' \sin \beta' \sin(\lambda' - \lambda), \\ \frac{dZ}{d\beta'} &= \frac{m'}{h^2} r^3 \left[\left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) r' \cos \beta' - \frac{3r r'^2 \sin^2 \beta' \cos(\lambda' - \lambda)}{\Delta^5} \right], \end{aligned}$$

the same method must be used as that which has been given for X , Y , Z . The remaining three, X , Y and Z being considered as functions of the two variables v and S' , can be obtained from the equations

$$\begin{aligned} \frac{dX}{d\lambda} &= \frac{dX}{dv} + \frac{n'}{n} \left(1 - \frac{r^2}{a^2 \sqrt{1 - e^2}} \right) \frac{dX}{dS'} - \frac{c}{p} \sin v \cdot r^2 \frac{dX}{dr}, \\ \frac{dY}{d\lambda} &= \frac{dY}{dv} + \frac{n'}{n} \left(1 - \frac{r^2}{a^2 \sqrt{1 - e^2}} \right) \frac{dY}{dS'} - \frac{c}{p} \sin v \cdot r^2 \frac{dY}{dr}, \\ \frac{dZ}{d\lambda} &= \frac{dZ}{dv} + \frac{n'}{n} \left(1 - \frac{r^2}{a^2 \sqrt{1 - e^2}} \right) \frac{dZ}{dS'} - \frac{c}{p} \sin v \cdot r^2 \frac{dZ}{dr}, \end{aligned}$$

The Factor

$$1 - \frac{r^2}{a^2 \sqrt{1 - e^2}} = E_1 \cos v - E_2 \cos 2v + E_3 \cos 3v - \dots,$$

where $E_i = 2 [1 + i \sqrt{1 - e^2}] \left(\frac{e}{1 + \sqrt{1 - e^2}} \right)^i$.

Moreover we have the relation

$$r^2 \frac{dY}{dr} = \frac{dX}{d\lambda} + rY.$$

The factor r is given by the equation

$$\frac{r}{a} = \frac{1}{2} E_0 - E_1 \cos v + E_2 \cos 2v - E_3 \cos 3v + \dots,$$

where $E_i = 2 \sqrt{1 - e^2} \left(\frac{e}{1 + \sqrt{1 - e^2}} \right)^i$.

The values of $\lambda' - \lambda$ and β' , necessary for the computation of the first six quantities, can be obtained from the equations

$$\begin{aligned} \cos \beta' \cos(\lambda' - \lambda) &= K \cos(v' + A), \\ \cos \beta' \sin(\lambda' - \lambda) &= K' \sin(v' + A'), \\ \sin \beta' &= \sin I \sin(v' + \Pi'). \end{aligned}$$

The terms to be integrated in the second approximation have the general form,

$$(C + C'v) \frac{\sin}{\cos} (iv - i'S' - i''S'').$$

If these terms are integrated with respect to v , we have

$$\begin{aligned} &+ [i, i', i''] (C + C'v) \frac{\cos}{\sin} (iv - i'S' - i''S'') \\ &+ [i, i', i'']^2 C' \frac{\sin}{\cos} (iv - i'S' - i''S''), \end{aligned}$$

where

$$[i, i', i''] = \frac{1}{i - i' \frac{n'}{n} - i'' \frac{n''}{n}}.$$

If they are integrated after having been multiplied by the factor $\sin(\bar{v} - v)$, the result is

$$\begin{aligned} &- [i-1, i', i''] [i+1, i', i''] (C + C'v) \frac{\sin}{\cos} (iv - i'S' - i''S'') \\ &+ \frac{2[i-1, i', i'']^2 [i+1, i', i'']^2}{[i, i', i'']} C' \frac{\cos}{\sin} (iv - i'S' - i''S''), \end{aligned}$$

except in the case where $i=1$, $i'=0$, $i''=0$, when we shall have

$$\begin{aligned} &+ \left(\frac{1}{8} C' - \frac{1}{2} C'v - \frac{1}{4} C'v^2 \right) \frac{\cos}{\sin} v \\ &+ \frac{1}{4} (C + C'v) \frac{\sin}{\cos} v. \end{aligned}$$

The labor of computing perturbations of the second order is, in some sort, measured by the number of multiplications to be made of two periodic series each involving double arguments. In this method, in the case of one disturbing planet, there are 22, or 25 if one thinks that the multiplications involving $\delta \lambda'$ ought to be considered as distinct from those involving $\delta \lambda$. If all the terms involving $\sin I$ as a factor be neglected, the number of these multiplications is diminished by 12.

It is my intention to illustrate this method by applying it to the computation of the perturbations of the first order of Ceres by Jupiter.