

4 Sir W. Thomson on the *Thermoelastic, Thermomagnetic,*

sphere, will keep the top of the tube sufficiently moist for an immense time.

The only possible changes which can go on in this cell are in the zinc and the solution in immediate contact with it. This solution can at any time be drawn off with a pipette and replaced by fresh, without greatly affecting the liquid in the bottle (if the cork be air-tight); and the zinc can still more easily be taken out and replaced by a new piece.

I have described the cell as at present made; but if there were any chance of its coming into use as a standard, a few modifications might be introduced. Thus the zinc might be a short rod with an india-rubber collar fitting the tube and with a short copper wire attached to it, which should project above the cork instead of the zinc, the joint being a little way down the tube and protected by a coat of varnish from damp air. A set of experiments would have to be made to determine the dependence of electromotive force on temperature; and then a thermometer with a short scale might be fixed in each cork.

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II. *On the Thermoelastic, Thermomagnetic, and Pyroelectric Properties of Matter.* By WILLIAM THOMSON, M.A., late Fellow of St. Peter's College, Cambridge, Professor of Natural Philosophy in the University of Glasgow\*.

1. **A** BODY which is either emitting heat, or altering its dimensions against resisting forces, is doing work upon matter external to it. The mechanical effect of this work in

\* [This paper is in the main a reprint from an article which appeared under the title "On the Thermoelastic and Thermomagnetic Properties of Matter, Part I," in April 1855, in the first number of the 'Quarterly Journal of Mathematics,' but which was confined to the thermoelastic part of the subject. The continuation, in which it was intended to make a similar application of thermodynamic principles to magnetic induction, was never published or written; but the results which it should have contained were sufficiently indicated in a short article on "Thermomagnetism," which I wrote at the request of my friend and colleague the late Professor J. P. Nichol for the second edition of his 'Cyclopædia,' published in 1860, and which I include in the present reprint. The addition of "Pyro-Electricity," which I now make to the title of the former article, is justified by another short quotation from the second edition of Nichol's 'Cyclopædia' (article "Thermo-Electricity, Division I.--Pyro-Electricity, or Thermo-Electricity of Nonconducting Crystals"), and a short addition, now written and published for the first time, in which the same thermodynamic principles are applied to this form of thermoelectric action.

Several additions both in the shape of text and footnote are appended in the course of the reprint. These are all distinguished by being enclosed in brackets, [ ].]

one case is the excitation of thermal motions, and in the other the overcoming of resistances. The body must itself be altering in its circumstances, so as to contain a less store of energy within it, by an amount precisely equal to the aggregate value of the mechanical effects produced; and conversely, the aggregate value of the mechanical effects produced must depend solely on the initial and final states of the body, and is therefore the same, whatever be the intermediate states through which the body passes, provided the *initial* and *final* states be the same.

2. The *total intrinsic energy* of a body might be defined as the mechanical value of all the effect it would produce, in heat emitted and in resistances overcome, if it were cooled to the utmost, and allowed to contract indefinitely or to expand indefinitely according as the forces between its particles are attractive or repulsive, when the thermal motions within it are all stopped; but in our present state of ignorance regarding perfect cold, and the nature of molecular forces, we cannot determine this "total intrinsic energy" for any portion of matter; nor even can we be sure that it is not infinitely great for a finite portion of matter. Hence it is convenient to choose a certain state as standard for the body under consideration, and to use the unqualified term *intrinsic energy* with reference to this standard state; so that the "intrinsic energy of a body in a given state" will denote the mechanical value of the effects the body would produce in passing from the state in which it is given, to the standard state—or, which is the same, the mechanical value of the whole agency that would be required to bring the body from the standard state to the state in which it is given.

3. In Part V.\* of a series of papers on the Dynamical Theory of Heat, communicated to the Royal Society of Edinburgh, a system of formulæ founded on propositions established in Part I.† of the same series of papers, and expressing, for a given fluid mass, relations between its pressure, its thermal capacities, its intrinsic energy (all considered as functions of its temperature and volume), and Carnot's function of the temperature, were brought forward for the purpose of pointing out the importance of making the *intrinsic energy* of a fluid in different states an object of research along with the other elements which have hitherto been considered, and partially investigated in some cases. In the present communication a similar mode of treatment, extended to include solid bodies, unmagnetic [and unelectricified], or magnetized [or electricified]

\* Trans. Roy. Soc. Edinb. December 15, 1851.

† Ibid. March 17, 1851.

in any way, is shown to lead to the most general possible theory of elasticity, whether of solids or fluids, and to point out various thermodynamic properties of solids and various thermal effects of magnetism [and of electricity] not hitherto discovered.

SECTION I.—*Elasticity of Solids or Fluids not subjected to Magnetic Force.*

4. Let  $x, y, z, \xi, \eta, \zeta$  be six independent variables expressing the mechanical condition of a homogeneous solid mass, homogeneously strained in any way\*, and let  $t$  be its temperature; and (in accordance with the preceding explanations) let  $e$  denote its intrinsic energy, reckoned from a certain "standard state" defined by particular values,  $x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t_0$ , on which its physical condition depends. Thus, if  $\phi$  denotes a certain function depending on the nature of the substance, and vanishing for the values  $x_0, y_0, \dots, t_0$  of the independent variables, we have

$$e = \phi(x, y, z, \xi, \eta, \zeta, t); \dots \dots (1)$$

and a knowledge of the function  $\phi$  [with besides a knowledge of  $w$  for one particular temperature†] comprehends all the thermoelastic qualities of the solid.

5. Now let us suppose the body to be strained so as to pass from the mechanical state  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0)$  to  $(x, y, z, \xi, \eta, \zeta)$  while it is constantly kept at the temperature  $t$ ; and let  $H$  denote the quantity of heat that must be supplied to it during this process to prevent its temperature from being lowered (a quantity which of course is zero, or negative, for such strains as cause no thermal effects, or which cause positive evolutions of heat). Let the body be brought back to its mechanical condition  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0)$  through the same or any other of all the infinitely varied successions of states by which it may be made to pass from one to the other of the two which have been named, its temperature being kept always at  $t$ . Then, by the second Fundamental Law of the Dynamical Theory of Heat (see *Trans. Roy. Soc. Edinb.* May 1, 1854, p. 126), we must have

$$\frac{H}{t} + \frac{H'}{t} = 0,$$

and therefore  $H' = -H$ .

\* The terms *a strain*, or *to strain*, are used simply with reference to alterations of dimensions or form in a solid—the forces by which "a strain" is produced being called the *straining tensions or pressures*, or sometimes merely *the tensions or pressures*, to which the solid is subjected. This distinction of terms is adopted in accordance with the expressions used by Mr. Rankine in his paper on the Elasticity of Solids (*Cambridge and Dublin Mathematical Journal*, February 1851).

† [See equations (10), (11) of § 7 below.]

6. We conclude that the quantity of heat absorbed by the body in being strained from one state to another at the same temperature is quite independent of the particular succession of states through which it is made to pass, provided it has throughout the same temperature. Hence we must have

$$H = \psi(x, y, z, \xi, \eta, \zeta, t) - \psi(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t), \quad (2)$$

where  $\psi$  denotes a function of the variables. Now the mechanical value of the heat taken in by the body while it passes from one condition to the other, together with the work spent in compelling it to do so, constitutes the whole augmentation of mechanical energy which it experiences; so that if  $\epsilon$  denote this augmentation—that is, if

$$\epsilon = \phi(x, y, z, \xi, \eta, \zeta, t) - \phi(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t), \quad (3)$$

and if  $w$  denote the work done by the applied forces and  $J$  the mechanical equivalent of the thermal unit, we have

$$\epsilon = w + JH. \quad (4)$$

From this we conclude that the work required to strain the body from one to another of two given mechanical states, keeping it always at the same temperature, is independent of the particular succession of mechanical states through which it is made to pass, and is always the same when the initial and final states are the same. This theorem was, I believe, first given by Green (as a consequence of the most general conceivable hypothesis that could be framed to explain the mutual actions of the different parts of a body on which its elasticity depends), who inferred from it that there cannot be 36, but only 21, independent coefficients [or “modulus”] of elasticity, with reference to axes chosen arbitrarily in any solid whatever. It is now demonstrated as a particular consequence of the Second General Thermodynamic Law. It might at first sight be regarded as simply a consequence of the general principle of mechanical effect; but this would be a mistake, fallen into from forgetting that heat is in general evolved or absorbed when a solid is strained in any way; and the only absurdity to which a denial of the proposition could lead would be the possibility of a self-acting machine going on continually drawing heat from a body surrounded by others at a higher temperature, without the assistance of any at a lower temperature, and performing an equivalent of mechanical work.

7. The full expression of the Second Thermodynamic Law for the circumstances of elastic force is, as is shown in the passage referred to above (Trans. Roy. Soc. Edinb. May 1, 1854, p. 126), that if  $H$ ,  $H'$ , &c. denote the quantities of heat emitted

from a body when at temperatures\*  $t, t'$  respectively, during operations changing its physical state in any way, the sum  $\Sigma \frac{H_t}{t}$  must vanish for any cycle of changes, if each is of a perfectly reversible character, and if at the end of all the body is brought back to its primitive state in every respect. Let us consider, for instance, the following cycle, which obviously fulfils both conditions.

(I.) Let the body, initially in the state  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t)$ , be raised in temperature from  $t$  to  $t'$ , its form and dimensions being maintained constant.

(II.) Let it be strained from the state  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0)$  to the state  $(x, y, z, \xi, \eta, \zeta)$ , while its temperature is kept always at  $t'$ .

(III.) Let it be lowered in temperature from  $t'$  to  $t$ , its form and dimensions being retained.

(IV.) Let it be brought back to the mechanical state  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0)$ , while its temperature is kept constantly at  $t$ .

The quantities of heat taken in by the body in these successive operations are respectively :—

$$(I.) \quad \frac{1}{J} \{ \phi(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t') - \phi(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t) \},$$

because the difference of the whole mechanical energies is simply the mechanical value of the heat taken in or emitted in all cases in which no work is either done on the body or received by it in virtue of the action of applied forces ;

(II.)  $\psi(x, y, z, \xi, \eta, \zeta, t') - \psi(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t')$ ,  
according to the notation expressed by equation (2) above ;

$$(III.) \quad - \frac{1}{J} \{ \phi(x, y, z, \xi, \eta, \zeta, t') - \phi(x, y, z, \xi, \eta, \zeta, t) \},$$

and

$$(IV.) \quad - \{ \psi(x, y, z, \xi, \eta, \zeta, t) - \psi(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t) \}.$$

\* Reckoned on the absolute thermodynamic scale, according to which "temperature" is defined as the mechanical equivalent of the thermal unit divided by "Carnot's function." In a paper "On the Thermal Effects of Fluids in Motion," by Mr. Joule and myself, communicated to the Royal Society last June [1854], and since published in the *Philosophical Transactions*, it is shown that temperature on the absolute thermodynamic scale does not differ sensibly from temperature on the ordinary scale of the air-thermometer, except by the addition of a constant number, which we find to be about 273·7 for the Centigrade scale. Thus, on the system now adopted, the temperature of melting ice is 273·7, that of boiling water is 373·7, and differences of temperature are sensibly the same as on an ordinary standard Centigrade thermometer.

If we suppose  $t' - t$  to be infinitely small, these expressions become respectively, in accordance with the previous notation:—

$$(I.) \quad \frac{1}{J} \frac{de_0}{dt} (t' - t),$$

where  $e_0$  denotes the value of  $e$  for  $(x_0, y_0, z_0, \xi_0, \eta, \zeta_0, t)$ ;

$$(II.) \quad H + \frac{dH}{dt} (t' - t);$$

$$(III.) \quad -\frac{1}{J} \frac{de}{dt} (t' - t);$$

$$(IV.) \quad -H.$$

Hence we have

$$\begin{aligned} \Sigma \frac{H_t}{t} &= \frac{\frac{1}{J} \frac{de_0}{dt} (t' - t)}{\frac{1}{2}(t + t')} + \frac{H + \frac{dH}{dt} (t' - t)}{t'} \\ &\quad + \frac{-\frac{1}{J} \frac{de}{dt} (t' - t)}{\frac{1}{2}(t + t')} + \frac{-H}{t}; \end{aligned}$$

or, since  $e - e_0$  is what we have denoted by  $\epsilon$ ,

$$\Sigma \frac{H_t}{t} = (t' - t) \left\{ \frac{d\left(\frac{H}{t}\right)}{dt} - \frac{1}{J} \frac{d\epsilon}{tdt} \right\};$$

and the expression of the Second Thermodynamic Law becomes

$$\frac{d\left(\frac{H}{t}\right)}{dt} - \frac{1}{J} \frac{d\epsilon}{tdt} = 0. \quad \dots \dots \dots (5)$$

Eliminating  $\epsilon$  from this by (4), we have

$$H = -\frac{t}{J} \frac{dw}{dt}; \quad \dots \dots \dots (6)$$

and, eliminating  $H$ ,

$$\epsilon = w - t \frac{dw}{dt}. \quad \dots \dots \dots (7)$$

This is equivalent to

$$e = e_0 + w - t \frac{dw}{dt}; \quad \dots \dots \dots (8)$$

or, if  $N_0$  denote the specific heat of the mass at any temperature  $t$ , when kept constantly in the mechanical state  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0)$ ,

$$e = J \int_{t_0}^t N_0 dt + w - t \frac{dw}{dt}, \quad \dots \dots \dots (9)$$

an expression which shows how the “intrinsic energy” of the body may be determined from observations giving  $w$  as a function of the seven independent variables, and  $N_0$  as a function of the temperature, for a particular set of values of the geometrical elements. Conversely, by (5) we have

$$H = \int \frac{t}{J} \frac{d\epsilon}{dt} \cdot dt; \quad \dots \dots \dots (10)$$

and by (6) and (7), or simply by (4),

$$w = \epsilon - JH; \quad \dots \dots \dots (11)$$

which show how  $H$  and  $w$  may be determined for all temperatures from a knowledge of the intrinsic energy of the body, and of [one of] those functions themselves for a particular temperature.

8. Let  $K$  denote the specific heat of the body at any temperature  $t$ , when it is allowed or compelled to vary in form and dimensions with the temperature, according to any fixed law—that is, when each of the variables  $x, y, z, \xi, \eta, \zeta$  is a given function of  $t$ ; and let  $N$  denote what this becomes in the particular case of each of these elements being maintained at a constant value; or, which is the same, let  $N$  be the specific heat of the body at any temperature when maintained at constant dimensions ( $x, y, z, \xi, \eta, \zeta$ ). We have

$$JN = \frac{de}{dt}, \quad \dots \dots \dots (12)$$

$$JK = \frac{de}{dt} + \frac{d(JH)}{dx} \frac{dx}{dt} + \frac{d(JH)}{dy} \frac{dy}{dt} + \frac{d(JH)}{dz} \frac{dz}{dt} + \frac{d(JH)}{d\xi} \frac{d\xi}{dt} + \frac{d(JH)}{d\eta} \frac{d\eta}{dt} + \frac{d(JH)}{d\zeta} \frac{d\zeta}{dt}. \quad (13)$$

Since  $JH$  is equal to  $e - w$ , this expression may be modified as follows:—If  $D$  denote the total differential of  $e$ ,

$$JK = \frac{De}{dt} - \left( \frac{dw}{dx} \frac{dx}{dt} + \frac{dw}{dy} \frac{dy}{dt} + \frac{dw}{dz} \frac{dz}{dt} + \frac{dw}{d\xi} \frac{d\xi}{dt} + \frac{dw}{d\eta} \frac{d\eta}{dt} + \frac{dw}{d\zeta} \frac{d\zeta}{dt} \right). \quad (13 \text{ bis})$$

9. These equations may be applied to any kind of matter; and they express all the information that can be derived, from the general thermodynamic principles, regarding the relations between thermal and mechanical effects produced by condensations, rarefactions, or distortions of any kind. For the case of a fluid they become reduced at once to the forms investigated specially for fluids in my previous communications. Thus, if the mass considered be one pound of any kind of fluid, we may

take one of the six variables  $x, y, z, \xi, \eta, \zeta$  as the volume  $v$ , which it is made to occupy in any particular condition, and the remaining five will not affect its physical properties, and will therefore disappear from all the preceding equations; and the state of the fluid will be completely defined by the values of the two independent variables  $v, t$ . Then, if  $p$  denote the pressure, we must have

$$\frac{dw}{dv} = -p;$$

since  $-p dv$  is the work done upon a fluid in compressing it under pressure  $p$  from a volume  $v$  to a volume  $v + dv$ . But from (9) we have

$$\frac{de}{dv} = \frac{dw}{dv} - t \frac{d}{dt} \frac{dw}{dv},$$

and, therefore,

$$\frac{de}{dv} = t \frac{dp}{dt} - p, \quad . . . . . (14)$$

and

$$\frac{d(\text{JH})}{dv} = t \frac{dp}{dt}. \quad . . . . . (14 \text{ bis})$$

Hence (13) becomes

$$\text{JK} = \frac{de}{dt} + t \frac{dp}{dt} \left( \frac{dv}{dt} \right),$$

where  $\left( \frac{dv}{dt} \right)$  expresses the assumed relation between the naturally independent variables  $v, t$ . If this be such that the pressure is constant, we have

$$\frac{dv}{dt} = \frac{\frac{dp}{dt}}{-\frac{dp}{dv}};$$

and,  $\text{K}$  being now the specific heat under constant pressure, we have finally

$$\text{JK} = \frac{de}{dt} + \frac{t \left( \frac{dp}{dt} \right)^2}{-\frac{dp}{dv}}. \quad . . . . . (15)$$

10. These equations (14) and (15), together with the unmodified equation (12), which retains the same form in all cases, express the general thermodynamic relations between the intrinsic energy, the pressure, and the specific heats of a fluid. If we eliminate  $e$ , we have



$$JK - JN = \frac{t \left( \frac{dp}{dt} \right)^2}{-\frac{dv}{dt}} \dots \dots \dots (16)$$

and  $\frac{d(JN)}{dv} = t \frac{d^2p}{dt^2}, \dots \dots \dots (17)$

which are the equations used to express those relations in a recent paper by Mr. Joule and myself, "On the Thermal Effects of Fluids in Motion" \*.

If, instead of  $\frac{Jdt}{t}$ , we substitute  $\mu dt$ , considering  $\mu$  as a function of Carnot's function of the temperature, they become identical with the two fundamental equations (14) and (16) given in Part III. of my first communication "On the Dynamical Theory of Heat" †.

11. To apply the preceding equations to a body possessing rigidity, it is necessary to take the form as well as the bulk into account, and therefore to retain, besides the temperature, six independent variables to express those elements. There is, of course, an infinite variety of ways in which the form and bulk of a homogeneously strained body may be expressed by means of six independent variables. Thus the lengths (three variables) and the mutual inclinations (three variables) of the edges of a parallelepiped enclosing always the same portion of the solid in all states of strain (which of course always remains a parallelepiped, provided the strain is homogeneous throughout the solid), may be chosen for the independent variables; or we may choose the six elements of an ellipsoid enclosing always the same portion of the solid (which will always remain an ellipsoid however the solid be strained, provided it is strained homogeneously). Thus, let us actually take for  $x, y, z$  the lengths of three conterminous edges OX, OY, OZ of a certain parallelepiped of the solid, and for  $\xi, \eta, \zeta$  the angles between the planes meeting in these edges respectively, the parallelepiped being so chosen that it becomes strained into a cube of unit dimensions, when the solid is in the particular state at which we wish to investigate its thermoelastic properties.

12. If then we take

$$x_0 = 1, \quad y_0 = 1, \quad z_0 = 1, \\ \xi_0 = \frac{1}{2}\pi, \quad \eta_0 = \frac{1}{2}\pi, \quad \zeta_0 = \frac{1}{2}\pi,$$

and if we suppose  $x, y, z, \xi, \eta, \zeta$  to differ infinitely little from  $x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0$  respectively, the actual state  $(x, y, z, \xi, \eta, \zeta)$  will be one in which the body is strained from the state

\* Transactions of the Royal Society, June 15, 1854.

† Transactions of the Royal Society of Edinburgh, March 17, 1851.

$(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0)$  by the edges of the cube being elongated by  $x-x_0, y-y_0, z-z_0$ , and the angles meeting in three conterminous edges receiving augmentations of  $\xi-\xi_0, \eta-\eta_0, \zeta-\zeta_0$ . It is clear, since the altered angles differ each infinitely little from a right angle, that the strains represented by  $\xi-\xi_0, \eta-\eta_0, \zeta-\zeta_0$  involve no change of volume, and are simple deformations, each of a perfectly definite kind, in the planes YOZ, ZOX, XOY respectively, and that the change of volume due to the six coexistent strains is actually an infinitely small augmentation amounting to

$$x-x_0+y-y_0+z-z_0.$$

13. Considering still  $x-x_0$  &c. as each very small, we have the following development by Maclaurin's theorem, the zero suffixes to the differential coefficients being used for brevity to denote the values of the different coefficients at  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0)$ .

$$\begin{aligned} w = & \left(\frac{dw}{dx}\right)_0 (x-x_0) + \left(\frac{dw}{dy}\right)_0 (y-y_0) + \left(\frac{dw}{dz}\right)_0 (z-z_0) \\ & + \left(\frac{dw}{d\xi}\right)_0 (\xi-\xi_0) + \left(\frac{dw}{d\eta}\right)_0 (\eta-\eta_0) + \left(\frac{dw}{d\zeta}\right)_0 (\zeta-\zeta_0) \\ & + \frac{1}{2} \left\{ \left(\frac{d^2w}{dx^2}\right)_0 (x-x_0)^2 + \left(\frac{d^2w}{dy^2}\right)_0 (y-y_0)^2 + \left(\frac{d^2w}{dz^2}\right)_0 (z-z_0)^2 \right. \\ & + \left(\frac{d^2w}{d\xi^2}\right)_0 (\xi-\xi_0)^2 + \left(\frac{d^2w}{d\eta^2}\right)_0 (\eta-\eta_0)^2 + \left(\frac{d^2w}{d\zeta^2}\right)_0 (\zeta-\zeta_0)^2 \\ & + 2\left(\frac{d^2w}{dydz}\right)_0 (y-y_0)(z-z_0) + 2\left(\frac{d^2w}{dzdx}\right)_0 (z-z_0)(x-x_0) \\ & \quad + 2\left(\frac{d^2w}{dxdy}\right)_0 (x-x_0)(y-y_0) \\ & + 2\left(\frac{d^2w}{d\eta d\zeta}\right)_0 (\eta-\eta_0)(\zeta-\zeta_0) + 2\left(\frac{d^2w}{d\zeta d\xi}\right)_0 (\zeta-\zeta_0)(\xi-\xi_0) \\ & \quad + 2\left(\frac{d^2w}{d\xi d\eta}\right)_0 (\xi-\xi_0)(\eta-\eta_0) \\ & + 2\left(\frac{d^2w}{dxd\xi}\right)_0 (x-x_0)(\xi-\xi_0) + 2\left(\frac{d^2w}{dyd\eta}\right)_0 (y-y_0)(\eta-\eta_0) \\ & \quad + 2\left(\frac{d^2w}{dzd\zeta}\right)_0 (z-z_0)(\zeta-\zeta_0) \\ & + 2\left(\frac{d^2w}{dxd\eta}\right)_0 (x-x_0)(\eta-\eta_0) + 2\left(\frac{d^2w}{dxd\zeta}\right)_0 (x-x_0)(\zeta-\zeta_0) \\ & + 2\left(\frac{d^2w}{dyd\zeta}\right)_0 (y-y_0)(\zeta-\zeta_0) + 2\left(\frac{d^2w}{dyd\xi}\right)_0 (y-y_0)(\xi-\xi_0) \\ & + 2\left(\frac{d^2w}{dzd\xi}\right)_0 (z-z_0)(\xi-\xi_0) + 2\left(\frac{d^2w}{dzd\eta}\right)_0 (z-z_0)(\eta-\eta_0) + \&c. (18) \end{aligned}$$

14. According to the system of variables \* which we have adopted, as set forth in § 12, when  $x-x_0$  &c. are each infinitely small,  $x$  increasing corresponds to a motion of all the particles in a plane at a distance unity from YOZ, in directions perpendicular to this plane, through a space numerically equal to the increment of  $x$ ;  $\xi$  increasing corresponds to a motion of all the particles at a distance unity from XOY, in directions parallel to YO, through a space equal to the increment of  $\xi$ , or to a motion of all the particles at a distance unity from XOZ, in directions parallel to ZO, through a space equal to the increment of  $\xi$ , or to two such motions superimposed, through any spaces respectively, amounting together to a quantity equal to the increment of  $\xi$ . Similar statements apply to the effects of variations of the other four variables. Hence, if P, Q, R denote the normal components of the superficial tensions experienced respectively by the three pairs of opposite faces of the unit cube of the solid in the state of strain in which we are considering it, and if S, T, U be the components, along the planes of the faces, of the actual tensions, taken in order of symmetry, so that S denotes the component, perpendicular to the edge opposite to OX, of the superficial tension in either of the faces meeting in that edge (which are equal for these two faces, or else the cube would not be in equilibrium, but would experience the effect of a couple in a plane perpendicular to OX), and T and U denote components, perpendicular respectively to OY and OZ, of the superficial tensions of the pairs of faces meeting in those edges, the work done on the parallelepiped during an infinitely small strain in which the variables become augmented by  $dx, dy, \&c.$  respectively will be

$$Pdx + Qdy + Rdz + Sd\xi + Td\eta + Ud\xi.$$

Hence, if the portion of matter of which the intrinsic energy is denoted by  $e$ , and to which the notation  $\epsilon, w, \&c.$  applies, be the matter within the parallelepiped referred to, we have

$$\left. \begin{aligned} \frac{dw}{dx} = P, \quad \frac{dw}{dy} = Q, \quad \frac{dw}{dz} = R, \\ \frac{dw}{d\xi} = S, \quad \frac{dw}{d\eta} = T, \quad \frac{dw}{d\xi} = U. \end{aligned} \right\} \dots (19)$$

15. Using the development of  $w$  expressed by (18), we derive from these equations the following expressions for the six component tensions :—

\* [A method of generalized stress and strain components is fully developed in "Elements of a Mathematical Theory of Elasticity," first published in the Transactions of the Royal Society for April 1856, and embodied in an article on "Elasticity," about to be published in the *Encyclopædia Britannica*.]

$$\begin{aligned}
 P &= \left( \frac{dw}{dx} \right)_0 \\
 &+ \left( \frac{d^2w}{dx^2} \right)_0 (x-x_0) + \left( \frac{d^2w}{dx dy} \right)_0 (y-y_0) + \left( \frac{d^2w}{dx dz} \right)_0 (z-z_0) \\
 &+ \left( \frac{d^2w}{dx d\xi} \right)_0 (\xi-\xi_0) + \left( \frac{d^2w}{dx d\eta} \right)_0 (\eta-\eta_0) + \left( \frac{d^2w}{dx d\zeta} \right)_0 (\zeta-\zeta_0); \\
 Q &= \left( \frac{dw}{dy} \right)_0 \\
 &+ \left( \frac{d^2w}{dx dy} \right)_0 (x-x_0) + \left( \frac{d^2w}{dy^2} \right)_0 (y-y_0) + \left( \frac{d^2w}{dy dz} \right)_0 (z-z_0) \\
 &+ \left( \frac{d^2w}{dy d\xi} \right)_0 (\xi-\xi_0) + \left( \frac{d^2w}{dy d\eta} \right)_0 (\eta-\eta_0) + \left( \frac{d^2w}{dy d\zeta} \right)_0 (\zeta-\zeta_0); \\
 R &= \left( \frac{dw}{dz} \right)_0 \\
 &+ \left( \frac{d^2w}{dz dx} \right)_0 (x-x_0) + \left( \frac{d^2w}{dz dy} \right)_0 (y-y_0) + \left( \frac{d^2w}{dz^2} \right)_0 (z-z_0) \\
 &+ \left( \frac{d^2w}{dz d\xi} \right)_0 (\xi-\xi_0) + \left( \frac{d^2w}{dz d\eta} \right)_0 (\eta-\eta_0) + \left( \frac{d^2w}{dz d\zeta} \right)_0 (\zeta-\zeta_0); \\
 S &= \left( \frac{dw}{d\xi} \right)_0 \\
 &+ \left( \frac{d^2w}{d\xi dx} \right)_0 (x-x_0) + \left( \frac{d^2w}{d\xi dy} \right)_0 (y-y_0) + \left( \frac{d^2w}{d\xi dz} \right)_0 (z-z_0) \\
 &+ \left( \frac{d^2w}{d\xi^2} \right)_0 (\xi-\xi_0) + \left( \frac{d^2w}{d\xi d\eta} \right)_0 (\eta-\eta_0) + \left( \frac{d^2w}{d\xi d\zeta} \right)_0 (\zeta-\zeta_0); \\
 T &= \left( \frac{dw}{d\eta} \right)_0 \\
 &+ \left( \frac{d^2w}{d\eta dx} \right)_0 (x-x_0) + \left( \frac{d^2w}{d\eta dy} \right)_0 (y-y_0) + \left( \frac{d^2w}{d\eta dz} \right)_0 (z-z_0) \\
 &+ \left( \frac{d^2w}{d\xi d\eta} \right)_0 (\xi-\xi_0) + \left( \frac{d^2w}{d\eta^2} \right)_0 (\eta-\eta_0) + \left( \frac{d^2w}{d\zeta d\eta} \right)_0 (\zeta-\zeta_0); \\
 U &= \left( \frac{dw}{d\zeta} \right)_0 \\
 &+ \left( \frac{d^2w}{d\zeta dx} \right)_0 (x-x_0) + \left( \frac{d^2w}{d\zeta dy} \right)_0 (y-y_0) + \left( \frac{d^2w}{d\zeta dz} \right)_0 (z-z_0) \\
 &+ \left( \frac{d^2w}{d\zeta d\xi} \right)_0 (\xi-\xi_0) + \left( \frac{d^2w}{d\zeta d\eta} \right)_0 (\eta-\eta_0) + \left( \frac{d^2w}{d\zeta^2} \right)_0 (\zeta-\zeta_0).
 \end{aligned}
 \tag{20}$$
  

$$\begin{aligned}
 &+ \left( \frac{d^2w}{d\eta dx} \right)_0 (x-x_0) + \left( \frac{d^2w}{d\eta dy} \right)_0 (y-y_0) + \left( \frac{d^2w}{d\eta dz} \right)_0 (z-z_0) \\
 &+ \left( \frac{d^2w}{d\xi d\eta} \right)_0 (\xi-\xi_0) + \left( \frac{d^2w}{d\eta^2} \right)_0 (\eta-\eta_0) + \left( \frac{d^2w}{d\zeta d\eta} \right)_0 (\zeta-\zeta_0); \\
 &+ \left( \frac{d^2w}{d\zeta dx} \right)_0 (x-x_0) + \left( \frac{d^2w}{d\zeta dy} \right)_0 (y-y_0) + \left( \frac{d^2w}{d\zeta dz} \right)_0 (z-z_0) \\
 &+ \left( \frac{d^2w}{d\zeta d\xi} \right)_0 (\xi-\xi_0) + \left( \frac{d^2w}{d\zeta d\eta} \right)_0 (\eta-\eta_0) + \left( \frac{d^2w}{d\zeta^2} \right)_0 (\zeta-\zeta_0).
 \end{aligned}
 \tag{21}$$

16. These equations express in the most general possible man-

ner the conditions of equilibrium of a solid in any state of strain whatever at a constant temperature. They show how the straining forces are altered with any infinitely small alteration of the strain. If we denote by  $P_0$  &c. the values of  $P$  &c. for the state  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0, t)$ , the values of  $P - P_0$ ,  $Q - Q_0$ ,  $R - R_0$ ,  $S - S_0$ ,  $T - T_0$ ,  $U - U_0$  given by these equations as linear functions of the strains  $(x - x_0)$ ,  $(y - y_0)$ ,  $(z - z_0)$ ,  $(\xi - \xi_0)$ ,  $(\eta - \eta_0)$ ,  $(\zeta - \zeta_0)$ , with twenty-one coefficients, express the whole tensions required to apply these strains to the cube, if the condition of the solid when the parallelepiped is exactly cubical is a condition of no strain, and in this case become (if single letters are substituted for the coefficients  $\left(\frac{d^2w}{dx^2}\right)_0$  &c.) identical

with the equations of equilibrium of an elastic solid subjected to infinitely small strains, which have been given by Green, Cauchy, Haughton, and other writers. Many mathematicians and experimenters have endeavoured to show that in actual solids there are certain essential relations between these twenty-one coefficients [or moduluses] of elasticity. Whether or not it may be true that such relations do hold for natural crystals, it is quite certain that an arrangement of actual pieces of matter may be made, constituting a homogeneous whole when considered on a large scale (being, in fact, as homogeneous as writers adopting the atomic theory in any form consider a natural crystal to be), which shall have an arbitrarily prescribed value for each one of these twenty-one coefficients. No one can legitimately deny for all natural crystals, known and unknown, any property of elasticity, or any other mechanical or physical property, which a solid composed of natural bodies artificially put together may have in reality. To do so is to assume that the infinitely inconceivable structure of the particles of a crystal is essentially restricted by arbitrary conditions imposed by mathematicians for the sake of shortening the equations by which their properties are expressed. It is true experiment might, and does, show particular values for the coefficients for particular bodies; but I believe even the collation of recorded experimental investigations is enough to show bodies violating every relation that has been imposed; and I have not a doubt that an experiment on a natural crystal, magnetized if necessary, might be made to show each supposed relation violated. Thus it has been shown, first I believe by Mr. Stokes, that the relation which the earlier writers supposed to exist between rigidity and resistance to compression is not verified, because experiments on the torsion of wires of various metals, rods of india-rubber, &c. indicate, on the whole, less rigidity than would be expected, according to that relation, from their resistance to

compression, and less in different proportions for different metals. It is quite certain that india-rubber, jelly of any kind (ever so stiff), and gutta-percha are all of them enormously less rigid in proportion to their resistance to compression than glass or the metals; and they are all certainly substances which may be prepared so as to be at least as homogeneous as rods, wires, bars, or tubes of metals. From some experiments communicated to me by Mr. Clerk Maxwell, which he has made on iron wire by flexure and torsion, it appears highly probable that iron is more rigid in proportion to its resistance to compression than M. Wertheim's experiments on brass and glass show these bodies to be.

[17. Since the publication of this paper, the same conclusion as to the relative qualities of iron and brass has been arrived at by Everett (Transactions of the Royal Society, 1865 and 1866) as a result of fresh experiments made by himself on these substances—but an opposite conclusion with reference to two specimens of flint glass upon which he experimented, and which both showed greater rigidity in proportion to compressibility than either his own experiments or those of others had shown for iron or any other substance accurately experimented on. Far beyond these specimens of glass, with respect to greatness of rigidity in proportion to compressibility, is cork; which though not hitherto accurately experimented on, and though no doubt very variable in its elastic quality, shows obviously a very remarkable property, on which its use for corking bottles depends, viz. that a column of it compressed endwise does not swell out sidewise to any sensible degree, if at all. It is easy to construct a model elastic solid, on the plan suggested above, which shall actually show lateral shrinking when compressed longitudinally, and lateral swelling when pulled out longitudinally. The false theory, referred to above as having been first proved to be at fault by Stokes, gives for every kind of solid  $\frac{1}{4}$  as the ratio of the lateral shrinking to the longitudinal elongation when a rod is pulled out lengthwise. The following Table (p. 18) shows how different are the values of that ratio determined by experiment on several real solids.]

18. The known fact that [many] gelatinous bodies, and the nearly certain fact that most bodies of all kinds, when their temperatures are raised, become less rigid to a much more marked extent than that of any effect on their compressibilities, are enough to show that neither the relation first supposed to exist, nor any other constant relation between compressibility and rigidity, can hold even for one body at different temperatures.

Substances.	Authority.	Ratio of lateral shrinking or swelling to longitudinal extension or shortening under the influence of push or pull on ends of a column of the substance.
Cork .....	General experience and some accurate measurements of diameter of a cork under various degrees of end-pressure, producing shortenings from small amounts up to as much as $\frac{1}{8}$ of the original length .....	0
Specimens of "crystal" glass.....	} Wertheim .....	.33
A specimen of flint glass.	} Everett (1865) .....	.26
Another specimen of flint glass .....	} Everett (1866) .....	.23
A specimen of brass .....	} Wertheim .....	.34
Drawn brass rod .....	} Everett (1866) .....	.47
Copper .....	} W. Thomson .....	from .40 to .23
Iron .....	} Clerk Maxwell .....	.27
Steel .....	} Kirchhoff .....	.29
Cast steel .....	} Everett (1866) .....	.31
Vulcanized india-rubber.	} Joule .....	{ Less than .5 by an exceedingly small amount.

19. Again, some of the relations which have been supposed to exist lead to three principal axes of elasticity. Many natural crystals do certainly exhibit perfect symmetry of form with reference to three rectangular axes, and therefore probably possess all their physical properties symmetrically with reference to those axes; but as certainly many, and among them some of the best-known, of natural crystals do not exhibit symmetry of form with reference to rectangular axes, and possess the mechanical property of resisting fracture differently in different directions, without symmetry about any three rectangular axes—for instance, Iceland spar, which has three planes of greatest brittleness ("cleavage-planes"), inclined at equal angles to one another and to a common axis (the "optic axis" of the crystal). If, as probably must be the case, the elastic properties within the limits of elasticity have correspondence with the mechanical properties on which the brittleness in different directions depends, the last-mentioned class of crystals cannot have three principal axes of elasticity at right angles to one another. It will be an interesting inquiry to examine thoroughly the various directional properties of an elastic solid represented by the different coefficients (of which the entire number may of course be reduced from twenty-one to eighteen by a choice of axes), or by various combinations of them.

20. The general thermodynamic principles expressed above in the equations (6), (8), (12), and (13) enable us to determine the relations between the evolution of heat or cold by strains of any kind effected on an elastic solid, the variation of its elastic forces with temperature, and the differences and variations of its specific heats. Thus (6) gives at once, when the development of  $w$  expressed by (18) is used, and for  $\left(\frac{dw}{dx}\right)_0$  &c. are substituted P &c., which are infinitely nearly equal to them, the following expression for the heat absorbed by an infinitely small straining, namely from  $(x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0)$  to  $(x, y, z, \xi, \eta, \zeta)$  :—

$$H = \frac{t}{J} \left\{ \frac{-dP}{dt} (x-x_0) + \frac{-dQ}{dt} (y-y_0) + \frac{-dR}{dt} (z-z_0) - \frac{dS}{dt} (\xi-\xi_0) - \frac{dT}{dt} (\eta-\eta_0) - \frac{dU}{dt} (\zeta-\zeta_0) \right\} . \quad (22)$$

21. We conclude that cold is produced whenever a solid is strained by opposing, and heat when it is strained by yielding to, any elastic force of its own, the strength of which would diminish if the temperature were raised—but that, on the contrary, heat is produced when a solid is strained against, and cold when it is strained by yielding to, any elastic force of its own, the strength of which would increase if the temperature were raised. When the stress is a pressure, uniform in all directions, fluids may be included in the statement. Thus we may conclude as certain :—

(1) That a cubical compression of any elastic fluid or solid in an ordinary condition would cause an evolution of heat ; but that, on the contrary, a cubical compression would produce cold in any substance, solid or fluid, in such an abnormal state that it would contract if heated, while kept under constant pressure.

(2) That if a wire already twisted be suddenly twisted further, always, however, within its limits of elasticity, cold will be produced ; and that if it be allowed suddenly to untwist, heat will be evolved from itself (besides heat generated externally by any work allowed to be wasted, which it does in untwisting). For I suppose it is certain that the torsive rigidity of every wire is diminished by an elevation of temperature.

(3) That a spiral spring suddenly drawn out will become lower in temperature, and will rise in temperature when suddenly allowed to draw in. [This result has since been experimentally verified by Joule ("Thermodynamic Properties of Solids," Trans. Roy. Soc. 1858), and the amount of the effect



found to agree with that calculated, according to the preceding thermodynamic theory, from the amount of the weakening of the spring which he found by experiment.]

(4) That a bar or rod or wire of any substance with or without a weight hung on it, or experiencing any degree of end thrust, to begin with, becomes cooled if suddenly elongated by end pull or by diminution of end thrust, and warmed if suddenly shortened by end thrust or by diminution of end pull: except abnormal cases, in which, with constant end pull or end thrust, elevation of temperature produces shortening; in every such case pull or diminished thrust produces elevation of temperature, thrust or diminished pull lowering of temperature.

(4') That an india-rubber band suddenly drawn out (within its limits of elasticity) produces cold, and that, on the contrary, when allowed to contract, heat will be evolved from it. For it is certain that an india-rubber band with a weight suspended by it will expand in length if the temperature be raised. [Alas for overconfident assertion! This is not true—at all events not true *in general* for either natural or vulcanized india-rubber, but only true for india-rubber in somewhat exceptional circumstances. It was founded on the supposition that india-rubber becomes less rigid when raised in temperature, which, besides, seeming to be expectable for solids generally, seemed to be experimentally proved for india-rubber by the familiar stiffness of common india-rubber in very cold weather. My original supposition is in fact correct for india-rubber which has become rigid by being kept at rest at a low temperature for some time. In this condition india-rubber was found by Joule to be cooled when suddenly stretched, and heated when the stretching weight was removed; and therefore, when in this condition, it is certain, from the thermodynamic principle, that a band of the substance bearing a weight will expand in length if the temperature is raised, and shrink when the temperature is lowered. But the very piece of india-rubber in which Joule found a cooling effect by pull when its temperature was  $5^{\circ}$  C., gave him a heating effect by pull, and a cooling effect on withdrawal of pull, when the temperature was  $15^{\circ}$  C. Joule experimented also on vulcanized india-rubber, and with it always found a heating effect when the substance was pulled out, and a cooling effect when it was allowed to shrink back. I pointed out to him that therefore, by thermodynamic theory, a vulcanized india-rubber band, when stretched by a constant weight of sufficient amount hung on it, must, when heated, pull up the weight, and when cooled, allow the weight to descend. This is an experiment which any one can make with the greatest ease by hanging a

few pounds weight on a common india-rubber band, and taking a red-hot coal, in a pair of tongs, or a red-hot poker, and moving it up and down close to the band. The way in which the weight rises when the red-hot body is near, and falls when it is removed, is quite startling. Joule experimented on the amount of shrinking per degree of elevation of temperature, with different weights hung on a band of vulcanized india-rubber, and found that they closely agreed with the amounts calculated by my theory from the heating effects of pull, and cooling effects of ceasing to pull, which he had observed in the same piece of india-rubber. Joule's experiments leave the statements of the following paragraph (5) true for common india-rubber at  $5^{\circ}$  C., but reverse it for common india-rubber at higher temperatures and for vulcanized india-rubber—that is to say, leave it applicable to these substances with "pull" substituted for "push" throughout.]

(5) We may conclude as highly probable, that pushing a column of india-rubber together longitudinally while leaving it free at its sides will cause the evolution of heat, when the force by which its ends are pushed together falls short of a certain limit; but that, on the contrary, if this force exceeds a certain limit, cold will be produced by suddenly increasing the force a very little, so as to contract the column further. For I suppose it is certain that a column of india-rubber with no weight, or only a small weight on its top, will expand longitudinally when its temperature is raised; but it appears to me highly probable that if the weight on the top of the column exceed a certain limit, the diminished rigidity of the column will allow it to descend when the temperature is raised. [This second change we now know to be contrary to the true state of the case; for we have seen that the rigidity of india-rubber is augmented by elevation of temperature.]

22. The specific heat of an elastic solid homogeneously strained under given pressures or tensions will be obtained by finding the differential coefficients of  $x, y, z, \xi, \eta, \zeta$  with reference to  $t$ , so as to make P, Q, R, S, T, U each remain constant or vary in a given manner—that is to say, by finding the coefficients of expansion in various dimensions for the body with an infinitely small change of its temperature, and using these in (3) above.

23. The elastic properties of such a crystal as is frequently found in natural specimens of garnet—a regular rhombic dodecahedron—must, if they correspond to the crystalline form, be symmetrical with reference to six axes in the substance perpendicular to the six pairs of opposite faces of the dodecahedron, or to the six edges of a regular tetrahedron related to

the dodecahedron in a determinate manner (having for its corners four of the eight trihedral corners of the dodecahedron); and yet they may differ, and in all probability they do differ, in different directions through the crystal. The relations among the coefficients of elasticity, according to the system of independent variables used in the preceding paper, which are required to express such circumstances, may be investigated by choosing for the normal cube a cube with faces perpendicular to the lines joining the three pairs of opposite tetrahedral corners of the dodecahedron. This choice of the normal cube makes all the coefficients vanish except nine, and makes these nine related one to another as follows:—

$$\left(\frac{d^2w}{dx^2}\right)_0 = \left(\frac{d^2w}{dy^2}\right)_0 = \left(\frac{d^2w}{dz^2}\right)_0 = \lambda + 2\mu,$$

$$\left(\frac{d^2w}{d\xi^2}\right)_0 = \left(\frac{d^2w}{d\eta^2}\right)_0 = \left(\frac{d^2w}{d\zeta^2}\right)_0 = \mu + \kappa,$$

and

$$\left(\frac{d^2w}{dy\,dz}\right)_0 = \left(\frac{d^2w}{dz\,dx}\right)_0 = \left(\frac{d^2w}{dx\,dy}\right)_0 = \lambda;$$

where  $\lambda$ ,  $\mu$ ,  $\kappa$  are three independent coefficients, introduced merely for the sake of comparison with M. Lamé's notation. In different natural crystals of the cubical system, such as fluor-spar, garnet, &c., it is probable that the three coefficients here left have different relations with one another. The body would, as is known, be, in its elastic qualities, perfectly isotropic if, and not so unless, the further relation

$$\left(\frac{d^2w}{dx^2}\right)_0 = \left(\frac{d^2w}{dy\,dz}\right)_0 + 2\left(\frac{d^2w}{d\xi^2}\right)_0$$

were fulfilled. Hence the quantity  $\kappa$  in the preceding formulæ expresses the crystalline quality which I suppose to exist in the elasticity of a crystal of the cubic class.

24. The fact of there being six axes of symmetry in cubic crystals (diagonals of sides of the cube), has suggested to me a system of independent variables, symmetrical with respect to those axes, which I believe may be found extremely convenient in the treatment of a mechanical theory of crystallography, and which, so far as I know, has not hitherto been introduced for the expression of a state of strain in an elastic solid. It is simply the *six edges of a tetrahedron enclosing always the same part of the solid*, a system of variables which might be used in all expressions connected with the theory of the elasticity of solids. To apply it to express the elastic properties of a crystal of the cubical class, let the tetrahedron be chosen with its edges parallel

to the six lines which are lines of symmetry when the solid is unstrained. In any state of strain let  $x, y, z$  be the lengths of three edges lying in one plane, and  $\xi, \eta, \zeta$  those of the three others (which meet in a point). Let  $x_0, y_0, z_0, \xi_0, \eta_0, \zeta_0$  denote the values (equal among themselves) of these variables for the unstrained state, and let  $w$  be the work required to bring any portion of the solid (whether the tetrahedron itself or not is of no consequence) from the unstrained state to the state  $(x, y, z, \xi, \eta, \zeta)$  while kept at a constant temperature. The relations among the coefficients of elasticity according to this system of variables, to express perfect symmetry with reference to the six axes, will clearly be:—

$$\begin{aligned} \left(\frac{d^2w}{dx^2}\right)_0 &= \left(\frac{d^2w}{dy^2}\right)_0 = \left(\frac{d^2w}{dz^2}\right)_0 = \left(\frac{d^2w}{d\xi^2}\right)_0 = \left(\frac{d^2w}{d\eta^2}\right)_0 = \left(\frac{d^2w}{d\zeta^2}\right)_0 = \varpi; \\ \left(\frac{d^2w}{dy\,dz}\right)_0 &= \left(\frac{d^2w}{dz\,dx}\right)_0 = \left(\frac{d^2w}{dx\,dy}\right)_0 = \left(\frac{d^2w}{d\eta\,d\xi}\right)_0 = \left(\frac{d^2w}{d\xi\,d\zeta}\right)_0 \\ &= \left(\frac{d^2w}{d\xi\,d\eta}\right)_0 = \left(\frac{d^2w}{dx\,d\eta}\right)_0 = \left(\frac{d^2w}{dx\,d\zeta}\right)_0 = \left(\frac{d^2w}{dy\,d\xi}\right)_0 \\ &= \left(\frac{d^2w}{dy\,d\xi}\right)_0 = \left(\frac{d^2w}{dz\,d\xi}\right)_0 = \left(\frac{d^2w}{d\xi\,d\eta}\right)_0 = \sigma; \\ \left(\frac{d^2w}{dx\,d\xi}\right)_0 &= \left(\frac{d^2w}{dy\,d\eta}\right)_0 = \left(\frac{d^2w}{dz\,d\zeta}\right)_0 = \omega, \text{—} \end{aligned}$$

where  $\varpi, \sigma, \omega$  denote three independent coefficients of elasticity for the substance. The definition of the new system of variables may be given as simply, and in some respects more conveniently, by referring to the dodecahedron, whose faces are perpendicular to the edges of the tetrahedron. Thus the six variables  $x, y, z, \xi, \eta, \zeta$  may be taken to denote respectively the mutual distances of the six pairs of parallel faces of the rhombohedron into which the regular dodecahedron is altered when the solid is strained in any manner. Thus, if the portion of the solid considered be the dodecahedron itself, and of such dimensions that when it is in its normal state the area of each face is unity, the values of  $\frac{dw}{dx}$  &c., denoted, as in the preceding paper,

by P, Q, R, S, T, U, are normal tensions (reckoned, as usual, per unit of area), on surfaces in the solid parallel to the faces of the dodecahedron, which compounded give the actual straining force to which the solid is subjected. The coefficients denoted above by  $\varpi, \sigma, \omega$  are such as to give the following expressions for the component straining tensions in terms of the strains:—

$$P = \varpi(x - x_0) + \omega(\xi - \xi_0) + \sigma(y - y_0 + z - z_0 + \eta - \eta_0 + \zeta - \zeta_0);$$

$$Q = \varpi(y - y_0) + \omega(\eta - \eta_0) + \sigma(z - z_0 + x - x_0 + \zeta - \zeta_0 + \xi - \xi_0);$$

$$R = \varpi(z - z_0) + \omega(\zeta - \zeta_0) + \sigma(x - x_0 + y - y_0 + \xi - \xi_0 + \eta - \eta_0);$$

$$S = \varpi(\xi - \xi_0) + \omega(x - x_0) + \sigma(y - y_0 + z - z_0 + \eta - \eta_0 + \zeta - \zeta_0);$$

$$T = \varpi(\eta - \eta_0) + \omega(y - y_0) + \sigma(z - z_0 + x - x_0 + \zeta - \zeta_0 + \xi - \xi_0);$$

$$U = \varpi(\zeta - \zeta_0) + \omega(z - z_0) + \sigma(x - x_0 + y - y_0 + \xi - \xi_0 + \eta - \eta_0).$$

25. The three quantities,  $\varpi$ ,  $\omega$ ,  $\sigma$ , or the three coefficients of elasticity according to the new system of independent variables, will express, by their different relative values, the elastic properties of all crystals of the cubical class. For a perfectly isotropic body, a particular numerical relation, which I have not yet determined, must hold between  $\varpi$ ,  $\rho$ , and  $\sigma$ ; and two independent coefficients of elasticity will remain. To determine this relation, and to find the formulæ of transformation from one set of variables to another on the new system, or from the new system to the ordinary system (that which was used in the preceding portion of this paper), or *vice versâ*, may be interesting objects of inquiry.

Glasgow College, March 10, 1855.

26. *Extracted from Nichol's 'Cyclopædia of the Physical Sciences,' second edition, 1860. Thermomagnetism. (1) Experimental Facts.*—Gilbert found that if a piece of soft iron between the poles of a magnet be raised to a bright red heat it loses all its ordinary indications of magnetism, and it only retains (Faraday, 'Exp. Res.' 2344–2347) slight traces of the paramagnetic character. Nickel loses its magnetic inductive capacity very rapidly as its temperature rises about 635° Fahr., and has very little left at the temperature of boiling oil. Cobalt loses its inductive capacity at a far higher temperature than that of either, near the melting-point of copper. Of the three metals, iron remains nearly constant, nickel falls gradually, and cobalt actually rises in inductive capacity as the temperature is raised from 0° to 300° Fahr. (Faraday, 'Exp. Res.' 3428; 'Phil. Trans.' Nov. 1855). Cobalt, of course, must have a maximum inductive capacity at some temperature intermediate between 300° Fahr. and the temperature of melting copper. Crystals, when their temperatures are raised, have their magnetic inductive capacities in different directions of the crystalline substance rendered less unequal, and in general to a very marked degree. Thus Faraday found the difference of inductive capacities in different directions in a crystal of bismuth (a diamagnetic crystal) reduced to less than half when the temperature was raised from 100° to 280°.

In carbonate of iron (a paramagnetic crystal) the difference of inductive capacities in different directions was reduced to one third when the temperature was raised from  $70^{\circ}$  to  $289^{\circ}$  Fahr., and was tripled when the temperature was again brought down to  $70^{\circ}$  (Exp. Res. 3400 and 3411).

(2) *Thermodynamic Relations.*—The theory of the mutual convertibility of heat and mechanical work in reversible operations when applied to these phenomena proves:—1. That a piece of soft iron at a moderate or low red heat, when drawn gently away from a magnet experiences a cooling effect, and when allowed to approach a magnet experiences a heating effect; that nickel at ordinary temperatures, and cobalt at high temperatures, within some definite range below that of melting copper, experience the same kind of effects when subjected to similar magnetic operations. 2. That cobalt at ordinary atmospheric temperatures, and at all temperatures upwards to its temperature of maximum inductive capacity, experiences a cooling effect when allowed to approach a magnet slowly, and a heating effect when drawn away. 3. That a crystal in a magnetic field experiences a cooling effect when its axis of greatest paramagnetic or of least diamagnetic inductive capacity is turned round from a position along to a position across the lines of force, and a heating effect when such a motion is reversed.

[27. Let there be three rectangular axes fixed relatively to the movable body, whether soft iron, or copper, or a crystal in a magnetic field, and, considering the whole magnetic motive\* on the body, reduce it, after the manner of Poinso't, to three component forces along the magnetic axes and three couples round these axes. Let P, Q, R be the force-components, and S, T, U the couple-components thus obtained, which we must suppose to be known functions of  $t$ , the temperature. Equation (22) of § 20 above gives H, the quantity of heat which must be supplied to prevent the body from becoming cooler when it is moved through infinitesimal spaces  $x-x_0$ ,  $y-y_0$ ,  $z-z_0$  in the directions of the three axes, and turned through infinitesimal angles  $\xi-\xi_0$ ,  $\eta-\eta_0$ ,  $\zeta-\zeta_0$  round the same axes. The lowering of temperature which it experiences if heat is neither given to it nor taken from it is equal to  $\frac{H}{C}$ ,

where C denotes the whole capacity for heat of the body, or the product of its mass or bulk by its specific heat per unit of

\* [In dynamics the want is keenly felt of an expression for a system of forces acting on a body: adopting a suggestion of my brother, Professor James Thomson, the word "motive" is used in the text to supply this want.]

mass or per unit of bulk. If the directions of  $x, y, z$  and  $\xi, \eta, \zeta$  are such that  $P, Q, R, S, T, U$  are positive, then for iron and nickel, and for cobalt at temperatures above that of its maximum inductive capacity,

$$-\frac{dP}{dt}, \quad -\frac{dQ}{dt}, \quad -\frac{dR}{dt}, \quad -\frac{dS}{dt}, \quad -\frac{dT}{dt}, \quad -\frac{dU}{dt}$$

are positive, and therefore the substance experiences a cooling effect when it is moved in such a manner as to require work to be done against magnetic force; and the reverse is the case for cobalt at ordinary temperatures.]

28. *Extracted from Nichol's 'Cyclopædia of the Physical Sciences,'* second edition, 1860.—The most probable account that can be given of the pyroelectric quality of dipolar crystals is, that these bodies intrinsically possess the same kind of *bodily electro-polarization* which Faraday, in his 'Experimental Researches,' has clearly proved to be temporarily produced in solid and liquid nonconductors, and that they possess this property to different degrees at different temperatures.

The inductive action exercised by this electro-polar state of the substance, on the matter touching the body all round, induces a superficial electrification which perfectly balances its electric force on all points in the external matter; but when the crystal is broken in two across its electric axis, the two parts exhibit as wholes contrary electrifications, not only by the free electro-polarities on the fractured surfaces discovered by Canton, but by the induced electrification on the old surface, belonging to the old state of electric equilibrium, and gradually lost by slow conduction, while a new superficial distribution of electricity on each fragment is acquired which ultimately masks all external symptoms of electric excitement. When the temperature of the substance is changed, its electro-polarization changes simultaneously, while the masking superficial electrification follows the change only by slow degrees—more or less slow according to the greater or less resistance offered to electric conduction in the substance or along its surface.

[29. If the preceding explanation of pyroelectricity be true, it must follow that a pyroelectric crystal moved about in an electric field will experience cooling effects or heating effects calculable by formula (22) of § 20, with the same notation for the electric subject as that of § 27 for magnetism. Thus the effects will be the same for a crystal at the same temperature whatever be the electrification of its surface. Thus it is remarkable that, in virtue of the wholly latent electric

polarity of a seemingly neutral pyroelectric crystal (that is to say, a crystal at the surface of which there is an electrification neutralizing for external space the force due to its internal electric polarity), the same cooling and heating effects will be produced by moving it in an electric field, as similar motions would produce in a similar crystal which, by having been heated in hot water, dried at the high temperature, and cooled, is in a state of pyroelectric excitement.]

Yacht 'Lalla Rookh,'  
Largs, Sept. 13, 1877.

### III. On *Binocular Vision*

*To the Editors of the Philosophical Magazine and Journal.*

GENTLEMEN,

IN Mr. Thompson's excellent article on the Chromatism of the Eye, in the July Number of your Journal, I notice the following sentence:—"In Wheatstone's classical research of 1838 it was demonstrated how great is the capacity of the brain to combine two slightly differing retinal images." From this sentence I conclude that Wheatstone's theory of binocular relief is still held by many scientists: I am confirmed in this conclusion by finding the expression "*mental fusion*" or "*conscious fusion*" of dissimilar images in nearly all works on the subject, even the latest, viz. Hermann's '*Physiology*.' Now, as I am quite sure that this theory is not only untenable, but positively hurtful to science by discouraging that careful analysis of visual perception so necessary to a true theory and yet so difficult to most persons, I have thought it would not be amiss to state briefly what I conceive to be the present condition of science on this subject.

In all investigations on binocular vision we are met at the very threshold by the difficulty which most persons find in analyzing what I would call *visual judgments*, i. e. judgments which by long habit and inherited tendency seem at first to be direct sensuous perceptions incapable of further analysis. It is difficult to convince *many* persons that they ever see double images at all; and yet they, of course, every day form judgments based on the existence and the unconscious perception of such double images. It is difficult to convince *most* persons, even the thoughtfully observant, that in regarding a stereoscopic scene there is no complete fusion of the two pictures, but that, when the eye is fixed on the foreground, objects in the background are double, and *vice*