III. The impulsive motion of an electrified sphere

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To cite this article: G.F.C. Searle M.A. F.R.S. (1909) III. The impulsive motion of an electrified sphere, Philosophical Magazine Series 6, 17:97, 43-71, DOI: [10.1080/14786440108636580](https://doi.org/10.1080/14786440108636580)

To link to this article: <http://dx.doi.org/10.1080/14786440108636580>

Published online: 21 Apr 2009.

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With $C_1 = 11.87$ mfd. the frequency of the arc note was 578, and of the damped oscillation 678. In these cases also the arc frequencies are considerably less than those of the same circuits without the arc.

With $C_1 = 9.55$ and C_2 about $.000275$ mfd. one note was heard, and the secondary potential reached very high values during the oscillations, as was shown by copious "brushing" at the plates of the secondary condenser. The curve is shown in Pl. I. fig. 11. The amplitude was at times even greater than is there shown, and the zero-line had to be brought below the tuning-fork wave in order to keep the vibrating ray on the rotating mirror. The frequency is 613, and the octave appears to have an amplitude several times as great as that of the fundamental. The value of $L_1 C_1$ is 4.411×10^{-8} , of $L_2 C_2$ about 1.929×10^{-8} ; the frequencies calculated by (4) from the inductances and capacities are approximately 723 and 1300. The high potential at the plates of the secondary condenser in this case appears to be due to approximate agreement in frequency between the octave of the arc note and the higher natural vibration of the secondary coil. Similar evidences of "resonance" of a transitory kind were frequently observed with the other condensers when the arc note (either the high note or the low note) during its fluctuations fell rather low in pitch. All the effects here described may also be conveniently obtained by using a variable condenser with oil dielectric as the secondary capacity.

Bangor, September 1908.

III. *The Impulsive Motion of an Electrified Sphere.* By G. F. C. SEARLE, M.A., F.R.S., *University Lecturer in Experimental Physics, Cambridge*.*

§ 1. **A**T the end of a paper on "The Impulsive Motion of Electrified Systems,"† published in the *Philosophical Magazine* for January 1907, I have calculated the energy and momentum radiated when the velocity of a

* Communicated by the Author.

† I regret that when I wrote this paper I was unaware that Dr. Paul Hertz had obtained some of the results in his *Untersuchungen über unstatige Bewegungen eines Electrons*, Göttingen, 1904. His paper was in my possession at the time, but I was under the impression that it referred to the force required to set a charged sphere in motion and not to the problems which I was considering. Where my work overlaps that of Dr. Hertz, it gives an independent confirmation of his results, as it was done without any reference to his paper.

charged sphere is impulsively changed by an infinitesimal amount, the change of velocity making any angle with the initial velocity. In the present paper I complete the investigation by calculating the energy and momentum which are radiated when the velocity of the sphere is impulsively changed in *any* manner, the only restrictions being that both the initial and the final velocities are less than that of light, and that the sphere has no angular velocity about any axis through its centre. The results in the restricted case, in which the centre of the sphere moves along a single straight line throughout, have been given by Heaviside and by Paul Hertz.

§ 2. The investigation turns mainly upon a certain integral. In the evaluation of this integral I have been greatly aided by Mr. G. T. Bennett, Fellow of Emmanuel College. I had carried out the integration in the manner given in § 10 below, but I was unable to put the result into a symmetrical form. Mr. Bennett then devised the very elegant semi-geometrical process which is given in §§ 6, 7, 8, and, guided by the result, reduced the complicated expression arising in my method of integration to a symmetrical form. I am also indebted to Mr. F. J. W. Whipple, of the Merchant Taylors' School, for verifying the integration by an alternative method.

§ 3. Now let us consider a sphere of radius a , carrying a charge Q on its surface, and let us suppose that its velocity is impulsively changed from $\mathbf{u}_1$ to $\mathbf{u}_2$, and that the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$ is α , **heavy type** denoting vectors. Then it will be seen from § 31 of my earlier paper that, when the pulse generated by the change of velocity has travelled out through a distance r , which is very great compared with the diameter of the sphere, the electric force in the pulse is at right angles to the radius and is given by

$$\mathbf{E} = \frac{Q}{2Kra} \left\{ \frac{\mathbf{R}(\mathbf{u}_2\mathbf{R}) - \mathbf{u}_2}{v - \mathbf{u}_2\mathbf{R}} - \frac{\mathbf{R}(\mathbf{u}_1\mathbf{R}) - \mathbf{u}_1}{v - \mathbf{u}_1\mathbf{R}} \right\} \dots \quad (1)$$

Here $\mathbf{R}$ denotes, for the moment, a unit vector along the radius, and $\mathbf{u}_1\mathbf{R}$ denotes the scalar product of $\mathbf{u}_1$ and $\mathbf{R}$, while K is the specific inductive capacity and v the velocity of light.

The magnetic force in the pulse is at right angles to both $\mathbf{E}$ and $\mathbf{R}$, and is given by

$$\mathbf{H} = vK\mathbf{V}\mathbf{R}\mathbf{E}, \quad \dots \dots \dots (2)$$

where $\mathbf{V}\mathbf{R}\mathbf{E}$ is a vector product.

If θ_1 and θ_2 be the angles between the radius and the directions of $\mathbf{u}_1$ and $\mathbf{u}_2$, we have $\mathbf{u}_1\mathbf{R}=u_1\cos\theta_1$ and $\mathbf{u}_2\mathbf{R}=u_2\cos\theta_2$, and $\mathbf{u}_1\mathbf{u}_2=u_1u_2\cos\alpha$, where α is the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$. Thus we find

$$E^2 = \frac{Q^2}{4K^2r^2a^2} \left\{ \frac{u_1^2 \sin^2 \theta_1}{(v-u_1 \cos \theta_1)^2} + \frac{u_2^2 \sin^2 \theta_2}{(v-u_2 \cos \theta_2)^2} - 2 \frac{u_1 u_2 \cos \alpha - u_1 u_2 \cos \theta_1 \cos \theta_2}{(v-u_1 \cos \theta_1)(v-u_2 \cos \theta_2)} \right\}.$$

The numerator of the last term in this expression can be written in the form

$$2\{v(v-u_1 \cos \theta_1) + v(v-u_2 \cos \theta_2) - (v-u_1 \cos \theta_1)(v-u_2 \cos \theta_2) - v^2 + u_1 u_2 \cos \alpha\},$$

and thus we obtain, after an easy reduction,

$$E^2 = \frac{Q^2}{4K^2r^2a^2} \left\{ \frac{2(v^2 - u_1 u_2 \cos \alpha)}{(v-u_1 \cos \theta_1)(v-u_2 \cos \theta_2)} - \frac{v^2 - u_1^2}{(v-u_1 \cos \theta_1)^2} - \frac{v^2 - u_2^2}{(v-u_2 \cos \theta_2)^2} \right\}. \quad (3)$$

In the pulse, the magnetic energy per unit volume is equal to the electric energy per unit volume, and hence, remembering that the thickness of the pulse is $2a$, we find that W , the radiated energy, is given by

$$W = \frac{Kar^2}{2\pi} \int E^2 d\omega, \quad (4)$$

where $d\omega$ is an element of solid angle.

The last two terms in the expression for E^2 can be integrated at once. Thus we have

$$\begin{aligned} \int \frac{d\omega}{(v-u_1 \cos \theta_1)^2} &= 2\pi \int_0^\pi \frac{\sin \theta_1 d\theta_1}{(v-u_1 \cos \theta_1)^2} \\ &= \frac{4\pi}{v^2 - u_1^2}. \end{aligned}$$

The expression for W now becomes

$$W = \frac{Q^2}{8\pi a K} \left\{ \int \frac{2(v^2 - u_1 u_2 \cos \alpha) d\omega}{(v-u_1 \cos \theta_1)(v-u_2 \cos \theta_2)} - 8\pi \right\}. \quad (5)$$

Writing n_1 for u_1/v and n_2 for u_2/v , and noting that

$Q^2/2aK = U_0$, the electric energy of the sphere at rest, we have

$$W = U_0 \left\{ \frac{1 - n_1 n_2 \cos \alpha}{2\pi} \int \frac{d\omega}{(1 - n_1 \cos \theta_1)(1 - n_2 \cos \theta_2)} - 2 \right\}. \quad (6)$$

The only difficulty lies in the evaluation of the integral in this expression. Since the integral is a function of n_1 , n_2 , and α , we shall write

$$F(n_1, n_2, \alpha) = \int \frac{d\omega}{(1 - n_1 \cos \theta_1)(1 - n_2 \cos \theta_2)}, \quad (7)$$

where the integration extends over the surface of the unit sphere of which $d\omega$ is an element. Thus

$$W = U_0 \{ (1 - n_1 n_2 \cos \alpha) (2\pi)^{-1} F(n_1, n_2, \alpha) - 2 \}. \quad (8)$$

§ 4. When the directions of $\mathbf{u}_1$ and $\mathbf{u}_2$ are parallel and in the same sense, so that $\cos \alpha = 1$, the value of F is easily found, for we now have $\theta_2 = \theta_1$ and can take $2\pi \sin \theta_1 d\theta_1$ as the element of surface of the unit sphere. Thus

$$\begin{aligned} F(n_1, n_2, 0) &= 2\pi \int_0^\pi \frac{\sin \theta_1 d\theta_1}{(1 - n_1 \cos \theta_1)(1 - n_2 \cos \theta_1)} \\ &= \frac{2\pi}{n_1 - n_2} \left\{ n_1 \int_0^\pi \frac{\sin \theta_1 d\theta_1}{1 - n_1 \cos \theta_1} - n_2 \int_0^\pi \frac{\sin \theta_1 d\theta_1}{1 - n_2 \cos \theta_1} \right\} \\ &= \frac{2\pi}{n_1 - n_2} \log \frac{(1 + n_1)(1 - n_2)}{(1 - n_1)(1 + n_2)}. \quad (9) \end{aligned}$$

When $\mathbf{u}_1$ and $\mathbf{u}_2$ are parallel but directed in *opposite* directions, so that $\cos \alpha = -1$, we have $\cos \theta_2 = -\cos \theta_1$, and hence the value of F can be found from (9) by changing the sign of n_2 . Then

$$F(n_1, n_2, \pi) = \frac{2\pi}{n_1 + n_2} \log \frac{(1 + n_1)(1 + n_2)}{(1 - n_1)(1 - n_2)}. \quad (10)$$

Substituting these values of F in (8) we obtain the following expressions for the radiated energy:—

When $\mathbf{u}_1$ and $\mathbf{u}_2$ are parallel and in the *same* direction, so that $\cos \alpha = 1$,

$$W = U_0 \left\{ \frac{1 - n_1 n_2}{n_1 - n_2} \log \frac{(1 + n_1)(1 - n_2)}{(1 - n_1)(1 + n_2)} - 2 \right\}. \quad (11)$$

When $\mathbf{u}_1$ and $\mathbf{u}_2$ are parallel but in opposite directions, so that $\cos \alpha = -1$,

$$W = U_0 \left\{ \frac{1+n_1n_2}{n_1+n_2} \log \frac{(1+n_1)(1+n_2)}{(1-n_1)(1-n_2)} - 2 \right\}. \quad (12)$$

When the motion is exactly reversed, so that $\cos \alpha = -1$ and $n_2 = n_1$

$$W = U_0 \left\{ \frac{1+n_1^2}{n_1} \log \frac{1+n_1}{1-n_1} - 2 \right\}. \quad (13)$$

The first result (11) was given by Dr. Oliver Heaviside*. These results have also been deduced by Paul Hertz† on dynamical principles from the values of the energy and momentum of a charged sphere in steady rectilinear motion.

§ 5. We now pass on to the general case in which the initial and final velocities are inclined at any angle. The integration of $F(n_1, n_2, \alpha)$ is now somewhat complicated, but on account of the fundamental importance of the integral two independent methods of evaluation are given below.

§ 6. The first method of evaluating $F(n_1, n_2, \alpha)$ is due to Mr. G. T. Bennett. Since this method is partly geometrical, it will be convenient to modify the integral so as to exhibit the geometrical quantities involved. We see, at once, from (7) that

$$F\left(\frac{R}{h_1}, \frac{R}{h_2}, \alpha\right) = \frac{h_1 h_2}{R^2} \cdot J, \quad (14)$$

where

$$J = \int \frac{R^2 d\omega}{(h_1 - R \cos \theta_1)(h_2 - R \cos \theta_2)}. \quad (15)$$

Here θ_1 and θ_2 are the angles between the radius defined by $d\omega$ and two fixed radii drawn from the centre of a sphere of radius R , and thus we can write

$$J = \int \frac{R^2 d\omega}{z_1 z_2}, \quad (16)$$

where z_1, z_2 are the perpendiculars drawn from any point on the surface of the sphere upon two fixed planes which cut the two fixed radii at right angles at distances h_1 and h_2 from the centre of the sphere. Since $n_1 < 1$ and $n_2 < 1$ we suppose

* 'Nature,' Nov. 6, 1902.

† *Untersuchungen über unstetige Bewegungen eines Electrons*. Göttingen, 1904. See also M. Abraham, *Theorie der Electricität*, vol. ii. p. 233.

that $h_1 > R$ and $h_2 > R$, and thus the planes do not cut the sphere.

If we denote the mean value of $1/z_1 z_2$ for the surface of the sphere by $[1/z_1 z_2]_S$, we have —

$$J = 4\pi R^2 \left[\frac{1}{z_1 z_2} \right]_S \dots \dots \dots (17)$$

The first step is to cut the sphere in a circle by a plane perpendicular to both the fixed planes and to find the mean value of $1/z_1 z_2$ for this circle. The case in which the two fixed planes are parallel has been discussed in § 4. We may therefore now suppose that their line of intersection is at a finite distance from the centre of the sphere.

§ 7. Let the plane of the paper be normal to the line of intersection of the two fixed planes and let it cut the sphere in a circle of radius r , the centre of the circle being at a distance L from the line of intersection. Let OH_1 , OH_2 (fig. 1) be sections of the two fixed planes by the plane of

Fig. 1.

the paper, and let $CH_1 = h_1$ and $CH_2 = h_2$ be the perpendiculars from C , the centre of the circle, upon OH_1 and OH_2 . Then the angles H_1CH_2 and H_1OH_2 are each equal to α . If P be any point on the circle and PN_1 , PN_2 be perpendiculars from P upon the lines OH_1 , OH_2 , then $PN_1 = z_1$ and $PN_2 = z_2$.

Now draw PM_1 and PM_2 perpendicular to CH_1 and CH_2 . Then $CM_1 = h_1 - z_1$ and $CM_2 = h_2 - z_2$. Since M_1 and M_2 lie on a circle of which CP is the diameter, and since M_1M_2 subtends the angle α at C , it follows that M_1M_2 subtends an angle 2α at the middle point of CP . Hence

$$M_1M_2 = CP \sin \alpha = r \sin \alpha.$$

Since $CO=L$, we find in a similar manner

$$H_1H_2=L \sin \alpha.$$

From the triangles M_1CM_2 and H_1CH_2 we have

$$r^2 \sin^2 \alpha = (h_1 - z_1)^2 + (h_2 - z_2)^2 - 2(h_1 - z_1)(h_2 - z_2) \cos \alpha,$$

$$L^2 \sin^2 \alpha = h_1^2 + h_2^2 - 2h_1h_2 \cos \alpha;$$

and hence

$$(L^2 - r^2) \sin^2 \alpha = 2z_1z_2 \cos \alpha + 2z_1(h_1 - h_2 \cos \alpha) + 2z_2(h_2 - h_1 \cos \alpha) - z_1^2 - z_2^2.$$

Dividing by z_1z_2 , we obtain

$$\frac{(L^2 - r^2) \sin^2 \alpha}{z_1z_2} = 2 \cos \alpha + \frac{2(h_1 - h_2 \cos \alpha)}{z_2} + \frac{2(h_2 - h_1 \cos \alpha)}{z_1} - \frac{z_1}{z_2} - \frac{z_2}{z_1}. \quad (18)$$

The problem of finding the mean value of $1/z_1z_2$ for the circle is thus reduced to finding the mean values of $1/z_1$, $1/z_2$, z_1/z_2 and z_2/z_1 for the same circle.

If $[1/z_1]_C$ denote the mean value of $1/z_1$ for the circle of radius, r , and if ϕ_1 denote the angle PCH_1 , we have

$$\left[\frac{1}{z_1} \right]_C = \frac{1}{\pi r} \int_0^\pi \frac{rd\phi_1}{h_1 - r \cos \phi_1},$$

since the integral from π to 2π is equal to that from 0 to π . Putting $\tan \frac{1}{2}\phi_1 = x$ and therefore $\cos \phi_1 = (1 - x^2)/(1 + x^2)$, we find

$$\begin{aligned} \left[\frac{1}{z_1} \right]_C &= \int_0^\infty \frac{2dx/\pi}{h_1 - r + (h_1 + r)x^2} = \frac{2/\pi}{(h_1^2 - r^2)^{\frac{1}{2}}} \left[\tan^{-1} x \sqrt{\frac{h_1 + r}{h_1 - r}} \right]_0^\infty \\ &= \frac{1}{(h_1^2 - r^2)^{\frac{1}{2}}} = \frac{1}{t_1}, \quad \dots \dots \dots (19) \end{aligned}$$

where t_1 is the length of the tangent from H_1 to the circle,

$$\text{Similarly} \quad [1/z_2]_C = 1/t_2. \quad \dots \dots \dots (20)$$

$$\text{Now} \quad \frac{z_2}{z_1} = \frac{PN_2}{PN_1} = \frac{PN_1 \cos \alpha + ON_1 \sin \alpha}{PN_1}.$$

But there are two positions of P for which PN_1 or z_1 has the same value, and the average value of ON_1 for these two positions is OH_1 . Thus we find for the whole circle

$$\begin{aligned} \left[\frac{z_2}{z_1} \right]_C &= \cos \alpha + OH_1 \sin \alpha \left[\frac{1}{z_1} \right]_C \\ &= \cos \alpha + \sin \alpha \cdot OH_1/t_1. \end{aligned}$$

Similarly, since

$$\frac{z_1}{z_2} = \frac{PN_1}{PN_2} = \frac{PN_2 \cos \alpha - ON_2 \sin \alpha}{PN_2},$$

we find

$$\left[\frac{z_1}{z_2} \right]_C = \cos \alpha - \sin \alpha \cdot OH_2/t_2.$$

$$\text{But } OH_1 \sin \alpha = CH_2 - CH_1 \cos \alpha = h_2 - h_1 \cos \alpha$$

$$OH_2 \sin \alpha = CH_2 \cos \alpha - CH_1 = h_2 \cos \alpha - h_1,$$

and thus

$$[z_2/z_1]_C = \cos \alpha + (h_2 - h_1 \cos \alpha)/t_1 \quad . \quad . \quad (21)$$

$$[z_1/z_2]_C = \cos \alpha + (h_1 - h_2 \cos \alpha)/t_2 \quad . \quad . \quad (22)$$

When we substitute in (18) the mean values shown in (19), (20), (21), and (22), we obtain

$$(L^2 - r^2) \sin^2 \alpha \left[\frac{1}{z_1 z_2} \right]_C = \frac{h_1 - h_2 \cos \alpha}{t_2} + \frac{h_2 - h_1 \cos \alpha}{t_1}.$$

$$\text{But } h_2 - h_1 \cos \alpha = OH_1 \sin \alpha = \sin \alpha \cdot (L^2 - h_1^2)^{\frac{1}{2}} \quad . \quad (23)$$

$$h_2 \cos \alpha - h_1 = OH_2 \sin \alpha = \sin \alpha (L^2 - h_2^2)^{\frac{1}{2}}, \quad . \quad . \quad (24)$$

and hence

$$\left[\frac{1}{z_1 z_2} \right]_C = \frac{1}{(L^2 - r^2) \sin \alpha} \left\{ \frac{1}{t_1} (L^2 - h_1^2)^{\frac{1}{2}} - \frac{1}{t_2} (L^2 - h_2^2)^{\frac{1}{2}} \right\}. \quad (25)$$

§ 8. We can now find the mean value of $1/z_1 z_2$ for the surface of the sphere. If we measure x from the centre of the sphere parallel to the line of intersection of the two fixed planes, the area of the zone on the sphere defined by x and $x + dx$ is $2\pi R dx$, and thus

$$\left[\frac{1}{z_1 z_2} \right]_S = \frac{1}{2\pi R^2} \int_0^R \left[\frac{1}{z_1 z_2} \right]_C 2\pi R dx.$$

Now

$$r^2 = R^2 - x^2,$$

and hence

$$t_1^2 = h_1^2 - r^2 = h_1^2 - R^2 + x^2,$$

$$t_2^2 = h_2^2 - r^2 = h_2^2 - R^2 + x^2.$$

Thus we have

$$\left[\frac{1}{z_1 z_2} \right]_S = \frac{1}{R \sin \alpha} \int_0^R \frac{dx}{L^2 - R^2 + x^2} \left\{ \frac{(L^2 - h_1^2)^{\frac{1}{2}}}{(h_1^2 - R^2 + x^2)^{\frac{1}{2}}} - \frac{(L^2 - h_2^2)^{\frac{1}{2}}}{(h_2^2 - R^2 + x^2)^{\frac{1}{2}}} \right\}.$$

The integration can be effected by aid of the formula

$$\int \frac{(p^2 - q^2)^{\frac{1}{2}} dx}{(p^2 + x^2)(q^2 + x^2)^{\frac{1}{2}}} = \frac{1}{2p} \log \frac{p(q^2 + x^2)^{\frac{1}{2}} + x(p^2 - q^2)^{\frac{1}{2}}}{p(q^2 + x^2)^{\frac{1}{2}} - x(p^2 - q^2)^{\frac{1}{2}}}$$

if we put $p^2 = L^2 - R^2$, $q^2 = h^2 - R^2$,

and $p^2 - q^2 = L^2 - h^2$,

where h stands for either h_1 or h_2 . We thus obtain

$$\left[\frac{1}{z_1 z_2} \right]_s = \frac{1}{2R \sin \alpha (L^2 - R^2)^{\frac{1}{2}}} \left[\log A - \log B \right]_0^R,$$

where

$$A = \frac{(L^2 - R^2)^{\frac{1}{2}}(h_1^2 - R^2 + x^2)^{\frac{1}{2}} + x(L^2 - h_1^2)^{\frac{1}{2}}}{(L^2 - R^2)^{\frac{1}{2}}(h_1^2 - R^2 + x^2)^{\frac{1}{2}} - x(L^2 - h_1^2)^{\frac{1}{2}}}.$$

Thus

$$\left[\log A \right]_0^R = \log \frac{h_1(L^2 - R^2)^{\frac{1}{2}} + R(L^2 - h_1^2)^{\frac{1}{2}}}{h_1(L^2 - R^2)^{\frac{1}{2}} - R(L^2 - h_1^2)^{\frac{1}{2}}}.$$

For B we have only to substitute h_2 for h_1 . Hence

$$\left[\log \frac{A}{B} \right]_0^R = \log \frac{\{h_1(L^2 - R^2)^{\frac{1}{2}} + R(L^2 - h_1^2)^{\frac{1}{2}}\} \{h_2(L^2 - R^2)^{\frac{1}{2}} - R(L^2 - h_2^2)^{\frac{1}{2}}\}}{\{h_1(L^2 - R^2)^{\frac{1}{2}} - R(L^2 - h_1^2)^{\frac{1}{2}}\} \{h_2(L^2 - R^2)^{\frac{1}{2}} + R(L^2 - h_2^2)^{\frac{1}{2}}\}}.$$

We now substitute for $(L^2 - h_1^2)^{\frac{1}{2}}$ and $(L^2 - h_2^2)^{\frac{1}{2}}$ the values given in (23) and (24), and then carry out the multiplications in the numerator and denominator of the quantity under the logarithm. Replacing $h_1^2 + h_2^2 - 2h_1 h_2 \cos \alpha$ by $L^2 \sin^2 \alpha$, we then easily find

$$\begin{aligned} \left[\frac{1}{z_1 z_2} \right]_s &= \frac{1}{2R \sin \alpha (L^2 - R^2)^{\frac{1}{2}}} \left[\log \frac{A}{B} \right]_0^R \\ &= \frac{1}{2R \sin \alpha (L^2 - R^2)^{\frac{1}{2}}} \log \frac{h_1 h_2 - R^2 \cos \alpha + R(L^2 - R^2)^{\frac{1}{2}} \sin \alpha}{h_1 h_2 - R^2 \cos \alpha - R(L^2 - R^2)^{\frac{1}{2}} \sin \alpha} \end{aligned} \quad (26)$$

where $L^2 \sin^2 \alpha = h_1^2 + h_2^2 - 2h_1 h_2 \cos \alpha$.

We have thus found the mean value of $1/z_1 z_2$ for points on the surface of the sphere.

Mr. Bennett has supplied the following remarks:—"The quantity under the logarithm in (26) is the cross-ratio of the pencil formed by the two planes together with the two tangent planes to the sphere through their line of intersection; and the result takes a simpler form if reciprocated with respect to the sphere itself. The perpendiculars from a variable point of the sphere to fixed planes are then replaced by (constant multiples of) perpendiculars drawn from fixed points within the sphere to a variable tangent plane. The result of the

mean-value problem may then be expressed concisely as follows :—

A sphere being given, AB being a fixed chord, and CD being fixed points on the chord, the mean value of the reciprocal of the product of the distances of C and D from a variable tangent plane is $\log (ABCD) \div AB \cdot CD$."

§ 9. To apply (26) to the electrical problem, we put

$$R=1, \quad h_1=1/n_1=v/u_1, \quad h_2=1/n_2=v/u_2,$$

It will be convenient to write

$$n_1^2 + n_2^2 - 2n_1n_2 \cos \alpha = m^2 = w^2/v^2, \quad \dots \quad (27)$$

where w is the change of the velocity of the sphere, so that u_2 is the resultant of u_1 and w . Then

$$L^2 \sin^2 \alpha = \frac{1}{n_1^2} + \frac{1}{n_2^2} - \frac{2 \cos \alpha}{n_1 n_2} = \frac{m^2}{n_1^2 n_2^2}$$

and

$$(L^2 - R^2) \sin^2 \alpha = \frac{m^2 - n_1^2 n_2^2 \sin^2 \alpha}{n_1^2 n_2^2}.$$

By § 6 we have

$$F(n_1, n_2, \alpha) = \frac{4\pi}{n_1 n_2} \left[\frac{1}{z_1 z_2} \right]_s$$

provided h_1, h_2 and R have the values just written. Inserting those values in (26) we have

$$F(n_1, n_2, \alpha) = \frac{2\pi}{(m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \log \frac{1 - n_1 n_2 \cos \alpha + (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{1 - n_1 n_2 \cos \alpha - (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \dots \quad (28)$$

§ 10. We now proceed to give a second investigation of the value of

$$F(n_1, n_2, \alpha) = \int \frac{d\omega}{(1 - n_1 \cos \theta_1)(1 - n_2 \cos \theta_2)}$$

where the integration is taken over a sphere of unit radius.

Let the plane containing the radius OP and the radius Ou_1 make an angle ϕ with the plane containing Ou_1 and Ou_2 , as shown in fig. 2. Then we have

$$\cos \theta_2 = \cos \theta_1 \cos \alpha + \sin \theta_1 \sin \alpha \cos \phi,$$

while

$$d\omega = \sin \theta_1 d\theta_1 d\phi.$$

Fig. 2.

Hence

$$F = \iint \frac{\sin \theta_1 d\theta_1 d\phi}{(1-n_1 \cos \theta_1)(1-n_2 \cos \alpha \cos \theta_1 - n_2 \sin \alpha \sin \theta_1 \cos \phi)}$$

where ϕ goes from 0 to 2π and θ_1 goes from 0 to π .

By § 7 we see that

$$\begin{aligned} \int_0^{2\pi} \frac{d\phi}{1-n_2 \cos \alpha \cos \theta_1 - n_2 \sin \alpha \sin \theta_1 \cos \phi} \\ = \frac{2\pi}{\{(1-n_2 \cos \alpha \cos \theta_1)^2 - n_2^2 \sin^2 \alpha \sin^2 \theta_1\}^{\frac{1}{2}}} \end{aligned}$$

provided that the result is *real*. But

$$\begin{aligned} (1-n_2 \cos \alpha \cos \theta_1)^2 - n_2^2 \sin^2 \alpha \sin^2 \theta_1 \\ = (\cos \alpha - n_2 \cos \theta_1)^2 + \sin^2 \alpha (1-n_2^2), \end{aligned}$$

and hence this quantity is always positive, since $n_2^2 < 1$. The integration is therefore valid.

Replacing $\sin^2 \theta_1$ by $1 - \cos^2 \theta_1$ in the ϕ -integral, we find

$$F = \int_0^\pi \frac{2\pi \sin \theta_1 d\theta_1}{(1-n_1 \cos \theta_1)(1-n_2^2 \sin^2 \alpha - 2n_2 \cos \alpha \cos \theta_1 + n_2^2 \cos^2 \theta_1)^{\frac{1}{2}}}.$$

To reduce the integral to a simpler form, put

$$1-n_1 \cos \theta_1 = 1/x, \quad \cos \theta_1 = (x-1)/n_1 x.$$

$$\text{Then} \quad \sin \theta_1 d\theta_1 = -dx/n_1 x^2,$$

and $x = (1-n_1)^{-1}$ when $\theta = 0$ and $x = (1+n_1)^{-1}$ when $\theta = \pi$. Making these substitutions, we easily find

$$F = 2\pi \int_A^B X^{-\frac{1}{2}} dx,$$

where

$$\begin{aligned} X &= p^2 x^2 - 2x n_2 (n_2 - n_1 \lambda) + n_2^2 \\ p^2 &= n_1^2 + n_2^2 - 2n_1 n_2 \lambda - n_1^2 n_2^2 + n_1^2 n_2^2 \lambda^2 \quad \dots \quad (29) \end{aligned}$$

$$\lambda = \cos \alpha \quad \dots \quad (30)$$

and

$$A = (1+n_1)^{-1}, \quad B = (1-n_1)^{-1}.$$

Hence

$$F = \frac{2\pi}{p} \log Y,$$

where

$$\log Y = \left[\log \{p^2 x - n_2(n_2 - n_1 \lambda) + pX^{\frac{1}{2}}\} \right]_A^B.$$

Using (29), we easily find that when $x=B=(1-n_1)^{-1}$

$$X^{\frac{1}{2}}=n_1(1-n_2\lambda)(1-n_1)^{-1},$$

and that when $x=A=(1+n_1)^{-1}$

$$X^{\frac{1}{2}}=n_1(1+n_2\lambda)(1+n_1)^{-1}.$$

Thus

$$Y = \frac{(1+n_1)\{p^2-n_2^2+n_1n_2\lambda+n_1n_2^2-n_1^2n_2\lambda+pn_1(1-n_2\lambda)\}}{(1-n_1)\{p^2-n_2^2+n_1n_2\lambda-n_1n_2^2+n_1^2n_2\lambda+pn_1(1+n_2\lambda)\}}.$$

When we substitute for p^2 from (29), we find that both numerator and denominator contain n_1 as a factor. When this factor is removed, we have

$$Y = \frac{(1+n_1)\{n_1+n_2^2-n_1n_2^2-n_2\lambda-n_1n_2\lambda+n_1n_2^2\lambda+p(1-n_2\lambda)\}}{(1-n_1)\{n_1-n_2^2-n_1n_2^2-n_2\lambda+n_1n_2\lambda+n_1n_2^2\lambda+p(1-n_2\lambda)\}}.$$

On carrying out the multiplications, we find that the five terms on the right side of (29) occur in both numerator and denominator. Replacing these terms by p^2 , we obtain

$$\begin{aligned} Y &= \frac{(1-n_1n_2\lambda)(n_1-n_2\lambda)+p(1-n_1n_2\lambda+n_1-n_2\lambda)+p^2}{(1-n_1n_2\lambda)(n_1-n_2\lambda)+p(1-n_1n_2\lambda-n_1+n_2\lambda)-p^2} \\ &= \frac{(1-n_1n_2\lambda+p)(n_1-n_2\lambda+p)}{(1-n_1n_2\lambda-p)(n_1-n_2\lambda+p)} \\ &= \frac{1-n_1n_2\lambda+p}{1-n_1n_2\lambda-p}. \end{aligned}$$

Since, by (29) and (27), $p^2 = m^2 - n_1^2 n_2^2 \sin^2 \alpha$ and since $\lambda = \cos \alpha$, we obtain, finally,

$$F(n_1, n_2, \alpha) = \frac{2\pi}{(m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \log \frac{1 - n_1 n_2 \cos \alpha + (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{1 - n_1 n_2 \cos \alpha - (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}}.$$

This result agrees with that found in § 9.

§ 11. We can now return to the electrical problem and obtain an expression for the energy radiated when the velocity of the sphere is impulsively changed.

Since, by (8)

$$W = U_0 \{ (2\pi)^{-1} (1 - n_1 n_2 \cos \alpha) F(n_1, n_2, \alpha) - 2 \}$$

we have, by (28)

$$\begin{aligned} W/U_0 &= \\ &= \frac{1 - n_1 n_2 \cos \alpha}{(m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \log \frac{1 - n_1 n_2 \cos \alpha + (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{1 - n_1 n_2 \cos \alpha - (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} - 2. \end{aligned}$$

In terms of the velocities the expression becomes

$$\frac{W}{U_0} = \frac{v^2 - u_1 u_2 \cos \alpha}{(v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \log \frac{v^2 - u_1 u_2 \cos \alpha + (v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{v^2 - u_1 u_2 \cos \alpha - (v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}}} - 2.$$

It is easily seen that the values of W found in § 4 for $\alpha=0$ and for $\alpha=\pi$ may be deduced from the general expression for W .

It is convenient to write W/U_0 in the form

$$\frac{W}{U_0} = \frac{1}{Z} \log \frac{1+Z}{1-Z} - 2,$$

where

$$Z = \frac{(v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{v^2 - u_1 u_2 \cos \alpha}.$$

The expression for W/U_0 has no meaning when Z , which is *positive*, is greater than unity. When Z is less than unity, we have

$$\frac{W}{U_0} = 2 \left\{ \frac{Z^2}{3} + \frac{Z^4}{5} + \frac{Z^6}{7} + \dots \right\}.$$

Since all the terms of this series are positive, W/U_0 increases as Z increases.

If we write $u_1^2 + u_2^2 - 2u_1 u_2 \cos \alpha$ for w^2 , we find

$$Z^2 = 1 - \frac{(v^2 - u_1^2)(v^2 - u_2^2)}{(v^2 - u_1 u_2 \cos \alpha)^2}.$$

Hence Z increases as α increases from 0 to π . Thus for given values of u_1 and u_2 , the radiated energy, W , increases as the angle between the directions of u_1 and u_2 increases, since both u_1 and u_2 are less than v .

For given values of u_1 and u_2 , the maximum value of Z occurs when $\alpha=\pi$. Then we have

$$Z_\pi = \frac{v(u_1 + u_2)}{v^2 + u_1 u_2} = 1 - \frac{(v - u_1)(v - u_2)}{v^2 + u_1 u_2},$$

and thus Z_π is positive and less than unity. Collecting these results we see that for all values of α and for all permissible values of u_1 and u_2 , Z is positive and less than unity.

§ 12. Since the loss of energy arises from the change in the velocity of the sphere, it will be convenient to express W as directly as possible in terms of w , the change of velocity. As we require two other quantities to fix the circumstances of the problem, we take u and ψ where $2u$ is the resultant of u_1 and u_2 and ψ is the angle between the positive

directions of $\mathbf{u}$ and $\mathbf{w}$. For convenience, we write

$$u = nv. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The relations which u_1 , u_2 , and α bear to u , w , and ψ are shown in fig. 3.

From the three equations

$$w^2 = u_1^2 + u_2^2 - 2u_1u_2 \cos \alpha,$$

$$u_1^2 = u^2 + \frac{1}{4}w^2 - uw \cos \psi,$$

$$u_2^2 = u^2 + \frac{1}{4}w^2 + uw \cos \psi,$$

we obtain

$$u_1u_2 \cos \alpha = u^2 - \frac{1}{4}w^2,$$

$$\text{or} \quad n_1n_2 \cos \alpha = n^2 - \frac{1}{4}m^2.$$

Writing down two equivalent expressions for the area of the parallelogram, we have

$$u_1u_2 \sin \alpha = uw \sin \psi,$$

or

$$n_1n_2 \sin \alpha = nm \sin \psi.$$

Hence

$$v^2w^2 - u_1^2u_2^2 \sin^2 \alpha = w^2(v^2 - u^2 \sin^2 \psi),$$

or

$$m^2 - n_1^2n_2^2 \sin^2 \alpha = m^2(1 - n^2 \sin^2 \psi).$$

When these values are inserted in the results of § 11, we obtain

$$\frac{W}{U_0} = \frac{1 - n^2 + \frac{1}{4}m^2}{m(1 - n^2 \sin^2 \psi)^{\frac{1}{2}}} \log \frac{1 - n^2 + \frac{1}{4}m^2 + m(1 - n^2 \sin^2 \psi)^{\frac{1}{2}}}{1 - n^2 + \frac{1}{4}m^2 - m(1 - n^2 \sin^2 \psi)^{\frac{1}{2}}} - 2,$$

or

$$\frac{W}{U_0} = \frac{v^2 - u^2 + \frac{1}{4}w^2}{w(v^2 - u^2 \sin^2 \psi)^{\frac{1}{2}}} \log \frac{v^2 - u^2 + \frac{1}{4}w^2 + w(v^2 - u^2 \sin^2 \psi)^{\frac{1}{2}}}{v^2 - u^2 + \frac{1}{4}w^2 - w(v^2 - u^2 \sin^2 \psi)^{\frac{1}{2}}} - 2.$$

§ 13. We now pass on to calculate the momentum radiated in the pulse generated by an impulsive change of velocity of the sphere. It will not be necessary to employ the integral calculus, since the momentum can be deduced on dynamical principles from quantities which are already known.

To change the velocity of the sphere impulsively from $\mathbf{u}_1$ to $\mathbf{u}_2$ it is necessary to apply a force $\mathbf{F}_{12}$ to the sphere as long as the pulse is passing over the sphere. When the pulse is clear of the sphere the force is no longer required. While the pulse is passing over the sphere, $\mathbf{F}_{12}$ probably changes both in direction and in magnitude, but we are not concerned

Fig. 3.

with the value of F_{12} at any instant, as we shall eliminate F_{12} from the equations.

Let U_1 , T_1 , and U_2 , T_2 be the electric and magnetic energies of the sphere when moving with the velocities u_1 and u_2 , and let W_1 be the energy radiated when the velocity u_1 is suddenly destroyed, and W_2 the energy radiated when u_2 is destroyed. Since the force which stops the sphere does no work, the total energy in the electromagnetic field is unchanged. Before the sphere is stopped the energy is $U_1 + T_1$, and after it is stopped the energy is $U_0 + W_1$, where U_0 is the electrostatic energy. Hence

$$\begin{aligned} U_1 + T_1 &= U_0 + W_1 \\ \text{and} \quad U_2 + T_2 &= U_0 + W_2. \end{aligned}$$

The values of W_1 and W_2 , which may be deduced from the results of § 4, are given by

$$\begin{aligned} W_1 &= U_0 \left(\frac{v}{u_1} \log \frac{v+u_1}{v-u_1} - 2 \right) \\ &= U_0 \left(\frac{2n_1^2}{3} + \frac{2n_1^4}{5} + \frac{2n_1^6}{7} + \dots \right). \quad (32) \end{aligned}$$

$$W_2 = U_0 \left(\frac{v}{u_2} \log \frac{v+u_2}{v-u_2} - 2 \right). \quad (33)$$

Since $u_1 = n_1 v$, we can write

$$W_1 = U_0 \left\{ \frac{1}{n_1} \log \frac{1+n_1}{1-n_1} - 2 \right\}. \quad (34)$$

If we prefer to do so, we may express W_1 in terms of $\frac{1}{2}m_0v^2$, by the formula $U_0 = \frac{3}{2} \cdot \frac{1}{2}m_0v^2$, where m_0 is the electromagnetic mass for infinitesimal speeds.

TABLE I.—Values of $\frac{W_1}{U_0}$.

n_1 .	W_1/U_0 .	n_1 .	W_1/U_0 .
0.1	0.00671	0.6	0.31049
0.2	0.02733	0.7	0.47800
0.3	0.06346	0.8	0.74653
0.4	0.11824	0.85	0.95565
0.5	0.19722	0.9	1.27160

§ 14. Now consider the impulsive change of velocity from $\mathbf{u}_1$ to $\mathbf{u}_2$. Before the force $\mathbf{F}_{12}$ acted the energy in the electromagnetic field was $U_1 + T_1$, and after the force has acted the energy is $U_2 + T_2 + W$. Hence, by § 13, the gain of energy is $W_2 - W_1 + W$, and this must be equal to the work done by the force. Thus, in vector notation,

$$\begin{aligned} W_2 - W_1 + W &= \int \mathbf{F}_{12} \mathbf{u}_2 dt \\ &= \mathbf{u}_2 \int \mathbf{F}_{12} dt, \quad (35) \end{aligned}$$

since $\mathbf{u}_2$ is constant, the last expression being the scalar product of $\mathbf{u}_2$ and $\int \mathbf{F}_{12} dt$.

Similarly if $\mathbf{F}_{21}$ be the force required to change the velocity from $\mathbf{u}_2$ to $\mathbf{u}_1$, we have

$$W_1 - W_2 + W = \mathbf{u}_1 \int \mathbf{F}_{21} dt, \quad (36)$$

since the energy radiated is the same for the change from $\mathbf{u}_2$ to $\mathbf{u}_1$ as for the change from $\mathbf{u}_1$ to $\mathbf{u}_2$.

From these equations of energy we pass to the equations of momentum. When the sphere is in steady motion, the resultant momentum of the electromagnetic field is in the same direction as the velocity of the sphere. If $\mathbf{M}_1$ be the momentum when the velocity is $\mathbf{u}_1$ and $\mathbf{M}_2$ when the velocity is $\mathbf{u}_2$, it is known that *

$$. . . \mathbf{M}_1 = \frac{U_0}{2u_1} \left\{ \frac{v^2 + u_1^2}{vu_1} \log \frac{v + u_1}{v - u_1} - 2 \right\} . . . (37)$$

$$\mathbf{M}_2 = \frac{U_0}{2u_2} \left\{ \frac{v^2 + u_2^2}{vu_2} \log \frac{v + u_2}{v - u_2} - 2 \right\} . . . (38)$$

Since $u_1 = n_1 v$, we can write

$$\begin{aligned} \mathbf{M}_1 &= \frac{U_0}{v} \left[\frac{1}{2} \left(1 + \frac{1}{n_1^2} \right) \log \frac{1 + n_1}{1 - n_1} - \frac{1}{n_1} \right] \\ &= \frac{U_0}{v} \left\{ \frac{4n_1}{1.3} + \frac{8n_1^3}{3.5} + \frac{12n_1^5}{5.7} + \dots \right\}. \quad (39) \end{aligned}$$

If we prefer to do so, we may express $\mathbf{M}_1$ in terms of $m_0 v$ by means of the formula $U_0/v = 3m_0 v/4$, where m_0 is the electromagnetic mass for infinitely slow motion.

* These results may be deduced from § 19 of my previous paper, Phil. Mag. Jan. 1907.

TABLE II.—Values of $\frac{\mathbf{M}_1 v}{U_0}$.

u_1 .	$\mathbf{M}_1 v/U_0$.	u_1 .	$\mathbf{M}_1 v/U_0$.
0.1	0.1339	0.6	0.9519
0.2	0.2710	0.7	1.2087
0.3	0.4153	0.8	1.5652
0.4	0.5715	0.85	1.8183
0.5	0.7465	0.9	2.1787

Let $\mathbf{P}$ be the momentum in the pulse, so that

$$\mathbf{P} = \frac{\mu K}{4\pi} \iiint \mathbf{V} \mathbf{E} \mathbf{H} \, dx \, dy \, dz, \quad . \quad . \quad . \quad (40)$$

where $\mathbf{V} \mathbf{E} \mathbf{H}$ denotes the vector product of the electric and magnetic forces and the integration extends throughout the volume of the pulse. Then, since the momentum of the electromagnetic field is changed from $\mathbf{M}_1$ to $\mathbf{M}_2 + \mathbf{P}$ by the action of the force $\mathbf{F}_{12}$,

$$\mathbf{M}_2 - \mathbf{M}_1 + \mathbf{P} = \int \mathbf{F}_{12} dt. \quad . \quad . \quad . \quad (41)$$

When the change of velocity of the sphere is effected in the reverse direction, both $\mathbf{E}$ and $\mathbf{H}$ in the pulse are simply reversed in direction without change of magnitude, as appears from (1) and (2), and hence $\mathbf{P}$ remains unchanged. Hence we have

$$\mathbf{M}_1 - \mathbf{M}_2 + \mathbf{P} = \int \mathbf{F}_{21} dt. \quad . \quad . \quad . \quad (42)$$

Combining these equations with (35) and (36), and writing

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2,$$

where $\mathbf{P}_1$ is in the same direction as $\mathbf{u}_1$ and $\mathbf{P}_2$ in the same direction as $\mathbf{u}_2$, we have

$$u_2(\mathbf{M}_2 - \mathbf{M}_1 + \mathbf{P}_1 + \mathbf{P}_2) = W_2 - W_1 + W$$

$$u_1(\mathbf{M}_1 - \mathbf{M}_2 + \mathbf{P}_1 + \mathbf{P}_2) = W_1 - W_2 + W.$$

Working out the scalar products, we obtain

$$P_1 \cos \alpha + P_2 = M_1 \cos \alpha - M_2 + (W_2 - W_1 + W)u_2^{-1} = X_1$$

$$P_1 + P_2 \cos \alpha = M_2 \cos \alpha - M_1 + (W_1 - W_2 + W)u_1^{-1} = X_2.$$

Solving these equations, we have

$$P_1 = \frac{X_2 - X_1 \cos \alpha}{\sin^2 \alpha}, \quad P_2 = \frac{X_1 - X_2 \cos \alpha}{\sin^2 \alpha}.$$

The magnitude of P is given by

$$\begin{aligned} P^2 &= P_1^2 + P_2^2 + 2P_1P_2 \cos \alpha \\ &= P_2X_1 + P_1X_2 \\ &= \frac{X_1^2 + X_2^2 - 2X_1X_2 \cos \alpha}{\sin^2 \alpha}. \end{aligned}$$

If we substitute for M_1 , M_2 , W_1 , and W_2 the values given in (37), (38), (32), and (33), and write

$$\log \frac{v+u_1}{v-u_1} = L_1, \quad \log \frac{v+u_2}{v-u_2} = L_2,$$

and make a slight re-arrangement of the factor of L_1 , we find that

$$\begin{aligned} P_2 + P_1 \cos \alpha &= \frac{W}{u_2} + \frac{U_0}{v} \left[\left\{ \frac{(v^2 - u_1^2) \cos \alpha}{2u_1^2} - \frac{v^2 - u_1u_2 \cos \alpha}{u_1u_2} \right\} L_1 \right. \\ &\quad \left. + \frac{v^2 - u_2^2}{2u_2^2} L_2 + \frac{v(u_1 - u_2 \cos \alpha)}{u_1u_2} \right]. \quad (43) \end{aligned}$$

Similarly, we find

$$\begin{aligned} P_1 + P_2 \cos \alpha &= \frac{W}{u_1} + \frac{U_0}{v} \left[\left\{ \frac{(v^2 - u_2^2) \cos \alpha}{2u_2^2} - \frac{v^2 - u_1u_2 \cos \alpha}{u_1u_2} \right\} L_2 \right. \\ &\quad \left. + \frac{v^2 - u_1^2}{2u_1^2} L_1 + \frac{v(u_2 - u_1 \cos \alpha)}{u_1u_2} \right]. \quad (43a) \end{aligned}$$

We can, without difficulty, write out the value of P^2 , but the expression becomes very complicated in the general case. We shall therefore consider its value in special cases only.

The component parallel to u_1 of the momentum in the pulse is $P_1 + P_2 \cos \alpha$. If we make $u_2 = 0$, we obtain the momentum in the pulse which is formed when u_1 , the initial velocity of the sphere, is suddenly destroyed. The expression for $P_1 + P_2 \cos \alpha$ (not $P_2 + P_1 \cos \alpha$) becomes indeterminate when we put $u_2 = 0$, but this difficulty can be overcome by first expanding L_2 in the form $2n_2 + 2n_2^3/3 + \dots$, and then putting $n_2 = 0$ in the terms which do not cancel. Using the value of W_1 given by (34) and using $u_1 = n_1v$, we find for

the resultant momentum in the pulse formed on stopping the sphere

$$P_1 = \frac{U_0}{v} \left\{ \left(\frac{3}{2n_1^2} - \frac{1}{2} \right) \log \frac{1+n_1}{1-n_1} - \frac{3}{n_1} \right\} \quad \dots \quad (44)$$

$$= \frac{U_0}{v} \left\{ \frac{4n_1^3}{3 \cdot 5} + \frac{8n_1^5}{5 \cdot 7} + \frac{12n_1^7}{7 \cdot 9} + \dots \right\}.$$

This result agrees with that given in § 19 of my earlier paper. In Table III. the numerical values of $P_1 v / U_0$ are given as well as the ratio of P_1 to M_1 , *i. e.* the ratio of the momentum carried off in the pulse to the momentum of the field before the velocity is destroyed. It will be easily seen from (39) and (44) that, when n_1 is very small, $P_1 / M_1 = n_1^2 \cdot 5$. On the other hand, when n_1 approaches unity, only the logarithmic parts of the expressions for P_1 and M_1 need be considered, and we find that although both P_1 and M_1 tend to infinity, their ratio P_1 / M_1 tends to unity.

TABLE III.—Values of $\frac{P_1 v}{U_0}$ and of $\frac{P_1}{M_1}$.

n_1 .	$P_1 v / U_0$.	P_1 / M_1 .	n_1 .	$P_1 v / U_0$.	P_1 / M_1 .
0	0	0	0.6	0.08308	0.08728
0.1	0.00027	0.00201	0.7	0.15699	0.12988
0.2	0.00221	0.00816	0.8	0.30113	0.19239
0.3	0.00780	0.01878	0.85	0.43029	0.23664
0.4	0.01977	0.03459	0.9	0.64711	0.29702
0.5	0.04237	0.05676	1.0	∞	1.00000

If the "impulse" of the force which stops the motion of the sphere be I_1 , then

$$M_1 - I_1 = P_1,$$

and hence, by (39) and (44),

$$I_1 = M_1 - P_1$$

$$= \frac{U_0}{v} \left\{ \left(1 - \frac{1}{n_1^2} \right) \log \frac{1+n_1}{1-n_1} + \frac{2}{n_1} \right\}.$$

Since $(1-n_1) \log(1-n_1)$ tends to zero as n_1 tends to unity, it follows that the limiting value of I is $2U_0/v$.

Thus it appears that if a charged sphere moving at the

speed of light be arrested by an obstacle, it only communicates a finite part of its infinite momentum to the obstacle. From Table II. we see that the momentum communicated to the obstacle is equal to that possessed by the sphere when moving at a speed of about 0.87 v .

The "impulse" of the force required to suddenly give the sphere a velocity equal to that of light is $P_1 + M_1$, and thus is infinite.

When the sphere is stopped, the pulse ceases to cut the sphere after a time $2a/v$, and then no force is required to hold the sphere at rest.

But when the sphere is started into motion at the speed of light, the sphere is always just enclosed in its own pulse, and consequently the force required to maintain its motion does not vanish until the motion has continued for an infinite time.

The component in the direction opposite to the sphere's motion of the electric force in the pulse is *

$$E = \frac{Qu \sin \theta}{2Kra(v - u \cos \theta)} \sin \theta$$

provided r be very great compared with a . When $u=v$ and θ is very small,

$$E = \frac{Q}{Kra}.$$

If x be measured along the direction of motion from the centre of the sphere, the retarding mechanical force experienced by the sphere in consequence of the action of the pulse tends to the value

$$\int_{-a}^{+a} \frac{Q}{Kra} \cdot \frac{Qdx}{2a} = \frac{Q^2}{Kar}$$

or to the value

$$\frac{Q^2}{Kart},$$

when t , the time since the sphere was started, tends to infinity. I stated this result in a review of Dr. Heaviside's 'Electromagnetic Theory' in the 'Physical Review,' July 1900, where, however, by some error the result is printed $Q^2/2art$ †.

When a charged sphere is suddenly set into motion with velocity u at the time $t=0$, the force required to maintain that velocity does not cease until $t=2a/(v-u)$, *i. e.* until the

* Phil. Mag. January 1907, p. 124.

† The printers did not furnish me with a "proof" of the review.

pulse generated by the sudden change of velocity no longer cuts the sphere. Dr. Paul Hertz* has obtained a complete solution for the force required in starting the sphere at any time from $t=0$ to $t=2a/(v-u)$, and, by extending his method, I have found the force required to stop the sphere†.

§ 15. The momentum in the pulse can also be obtained by direct integration. Since in the pulse $\mathbf{H} = v\mathbf{K}\mathbf{V}\mathbf{R}\mathbf{E}$, it follows that $\mathbf{V}\mathbf{E}\mathbf{H} = v\mathbf{K}\mathbf{V}\mathbf{E}\mathbf{V}\mathbf{R}\mathbf{E}$, where $\mathbf{R}$ is a unit-vector along the radius. But $\mathbf{E}$ is perpendicular to $\mathbf{R}$, and thus $\mathbf{V}\mathbf{E}\mathbf{H} = v\mathbf{K}\mathbf{E}^2\mathbf{R}$. Hence $\mathbf{V}\mathbf{E}\mathbf{H}$ is in the same direction as $\mathbf{R}$. Thus, by (40) the momentum in the pulse is

$$\mathbf{P} = \frac{\mathbf{K}}{4\pi v} \iiint \mathbf{E}^2 \mathbf{R} \, dx \, dy \, dz,$$

the integration extending throughout the volume of the pulse.

Since the thickness of the pulse is $2a$, it follows that, if r be the radius of the pulse and $d\omega$ an element of solid angle,

$$\mathbf{P} = \frac{\mathbf{K}r^2a}{2\pi v} \int \mathbf{E}^2 \mathbf{R} \, d\omega.$$

The value of $\mathbf{E}^2$ is given in (3). For brevity we write

$$\begin{aligned} v - u_1 \cos \theta_1 &= h_1, & v - u_2 \cos \theta_2 &= h_2, \\ v^2 - u_1 u_2 \cos \alpha &= S. \end{aligned}$$

Noting that $Q^2/2Ka = U_0$, we obtain

$$\mathbf{P} = \frac{U_0}{4\pi v} \int \left(\frac{2S}{h_1 h_2} - \frac{v^2 - u_1^2}{h_1^2} - \frac{v^2 - u_2^2}{h_2^2} \right) \mathbf{R} \, d\omega.$$

The component of $\mathbf{P}$ parallel to $\mathbf{u}_2$ is $P_2 + P_1 \cos \alpha$, with the notation of § 14, and thus, since the angle between $\mathbf{R}$ and $\mathbf{u}_2$ is θ_2 , we have

$$P_2 + P_1 \cos \alpha = \frac{U_0}{4\pi v} \int \left(\frac{2S}{h_1 h_2} - \frac{v^2 - u_1^2}{h_1^2} - \frac{v^2 - u_2^2}{h_2^2} \right) \cos \theta_2 \, d\omega.$$

* *Untersuchungen über unstetige Bewegungen eines Electrons.*

† "On the Force required to stop a Moving Electrified Sphere," Proc. Royal Society, A. vol. lxxix. p. 550. In this paper I have given a sketch of Dr. Hertz's method and have stated his results.

Now $\cos \theta_2 = (v - h_2)/u_2$, and hence

$$\int \frac{\cos \theta_2 d\omega}{h_1 h_2} = \frac{1}{u_2} \left\{ v \int \frac{d\omega}{h_1 h_2} - \int \frac{d\omega}{h_1} \right\}.$$

But, by (7), $v \int \frac{d\omega}{h_1 h_2} = \frac{1}{v} F(n_1, n_2, \alpha)$

and $\int \frac{d\omega}{h_1} = 2\pi \int_0^\pi \frac{\sin \theta_1 d\theta_1}{v - u_1 \cos \theta_1} = \frac{2\pi}{u_1} L_1$,

where $L_1 = \log \frac{v + u_1}{v - u_1}$ and $L_2 = \log \frac{v + u_2}{v - u_2}$.

With the notation of § 10,

$$\cos \theta_2 = \cos \theta_1 \cos \alpha + \sin \theta_1 \sin \alpha \cos \phi,$$

and thus

$$\int \frac{\cos \theta_2 d\omega}{h_1^2} = \iint \frac{(\cos \theta_1 \cos \alpha + \sin \theta_1 \sin \alpha \cos \phi) \sin \theta_1 d\theta_1 d\phi}{h_1^2},$$

where ϕ goes from 0 to 2π and θ_1 from 0 to π . The term involving $\cos \phi$ vanishes on integration with respect to ϕ . Writing $(v - h_1)/u_1$ for $\cos \theta_1$, we obtain

$$\begin{aligned} \int \frac{\cos \theta_2 d\omega}{h_1^2} &= \frac{2\pi \cos \alpha}{u_1} \left\{ v \int_0^\pi \frac{\sin \theta_1 d\theta_1}{h_1^2} - \int_0^\pi \frac{\sin \theta_1 d\theta_1}{h_1} \right\} \\ &= \frac{2\pi \cos \alpha}{u_1} \left\{ \frac{2v}{v^2 - u_1^2} - \frac{L_1}{u_1} \right\}. \end{aligned}$$

Writing $(v - h_2)/u_2$ for $\cos \theta_2$, we obtain

$$\begin{aligned} \int \frac{\cos \theta_2 d\omega}{h_2^2} &= \frac{2\pi}{u_2} \left\{ v \int_0^\pi \frac{\sin \theta_2 d\theta_2}{h_2^2} - \int_0^\pi \frac{\sin \theta_2 d\theta_2}{h_2} \right\} \\ &= \frac{2\pi}{u_2} \left\{ \frac{2v}{v^2 - u_2^2} - \frac{L_2}{u_2} \right\}. \end{aligned}$$

Collecting these results, and substituting for $F(n_1, n_2, \alpha)$ the value derived from (8), we have

$$\begin{aligned} P_2 + P_1 \cos \alpha &= \frac{W}{u_2} + \frac{U_0}{v} \left[\left\{ \frac{(v^2 - u_1^2) \cos \alpha}{2u_1^2} - \frac{v^2 - u_1 u_2 \cos \alpha}{u_1 u_2} \right\} L_1 \right. \\ &\quad \left. + \frac{v^2 - u_2^2}{2u_2^2} L_2 + \frac{v(u_1 - u_2 \cos \alpha)}{u_1 u_2} \right]. \end{aligned}$$

This result is identical with (43), which was obtained in § 14 by dynamical principles.

§ 16. When w , the change of velocity, is small compared with $(v^2 - u^2)^{\frac{1}{2}}$, where $2\mathbf{u}$ is the resultant of $\mathbf{u}_1$ and $\mathbf{u}_2$, we can obtain approximate expressions for W and for $\mathbf{P}$ which may be useful for many purposes.

The energy radiated when w is small is easily deduced from the expression given in § 12. Since, when y/x is less than 1,

$$\frac{x}{y} \log \frac{x+y}{x-y} - 2 = 2 \left(\frac{y^2}{3x^2} + \frac{y^4}{5x^4} + \dots \right),$$

it follows that, when m or w/v is small,

$$W = U_0 \frac{2m^2(1 - n^2 \sin^2 \psi)}{3(1 - n^2 + \frac{1}{4}m^2)^2}.$$

But $U_0 = Q^2/2Ka = \mu v^2 Q^2/2a$, and thus as far as terms in w^2 ,

$$W = \frac{2U_0 w^2(1 - n^2 \sin^2 \psi)}{3v^2(1 - n^2)^2} = \frac{\mu Q^2 w^2(1 - n^2 \sin^2 \psi)}{3a(1 - n^2)^2},$$

a value identical with that obtained when w is treated as small throughout*. Thus the effect of a given change of velocity depends upon the initial velocity. If W_0 be the energy radiated when the velocity is changed from 0 to w , we have†, as far as w^2 ,

$$W_0 = \frac{\mu Q^2 w^2}{3a};$$

and hence

$$\frac{W}{W_0} = \frac{1 - n^2 \sin^2 \psi}{(1 - n^2)^2}.$$

The following table shows the value of W/W_0 for a few values of n and of ψ .

It will be seen from the table that the angle between the initial velocity and the change of velocity has only a small effect when the initial velocity is small. For initial velocities less than $\frac{1}{2}v$ the energy radiated does not differ by as much as 10 per cent. from the energy radiated when the initial

* Phil. Mag. Jan. 1907, p. 146.

† Phil. Mag. Jan. 1907, p. 131 (or § 13 above).

velocity is zero, the small change of velocity being the same in either case.

TABLE IV.—Values of $\frac{W}{W_0}$.

n .	$\psi=0^\circ$. $\psi=180^\circ$.	$\psi=30^\circ$. $\psi=150^\circ$.	$\psi=60^\circ$. $\psi=120^\circ$.	$\psi=90^\circ$.
0.1	1.020	1.018	1.013	1.010
0.2	1.085	1.074	1.053	1.042
0.3	1.208	1.180	1.126	1.099
0.4	1.417	1.361	1.247	1.190
0.5	1.778	1.667	1.444	1.333
0.6	2.441	2.222	1.782	1.562
0.7	3.845	3.374	2.432	1.961
0.8	7.716	6.481	4.012	2.778
0.85	12.986	10.640	5.949	3.604
0.9	27.701	22.091	10.873	5.263

§ 17. In § 32 of my previous paper I obtained approximate expressions for the components of the radiated momentum, in directions parallel and perpendicular to $\mathbf{u}$, by treating $\mathbf{w}$ as small from the beginning. It will be convenient to express these results in terms of $m_0 = 2\mu Q^2/3a$, the electromagnetic mass of the sphere for infinitesimal speeds.

If P_u^* be the component in the direction of $\mathbf{u}$, then

$$\begin{aligned}
 P_u^* &= \frac{m_0 w^2}{v} \cdot \frac{3}{8n^4} \left[\log \frac{1+n}{1-n} - \frac{2n(3-5n^2)}{3(1-n^2)^2} \right. \\
 &\quad \left. - \sin^2 \psi \left\{ \frac{3+n^2}{2} \log \frac{1+n}{1-n} - \frac{n(9-12n^2-n^4)}{3(1-n^2)^2} \right\} \right] \\
 &= \frac{m_0 w^2}{v} (A - B \sin^2 \psi),
 \end{aligned}$$

where A and B are functions of n .

For numerical calculations it is convenient to express P_u^* in the form of a series, which we may use when n is small. Thus

$$\begin{aligned}
 P_u^* &= \frac{m_0 w^2}{v} \left[\left\{ \frac{1.2}{5} n + \frac{2.3}{7} n^3 + \frac{3.4}{9} n^5 + \dots \right\} \right. \\
 &\quad \left. - \frac{1}{2} \sin^2 \psi \left\{ \frac{3}{3.5} n + \frac{34}{5.7} n^3 + \frac{117}{7.9} n^5 + \frac{276}{9.11} n^7 + \dots \right\} \right].
 \end{aligned}$$

* In the previous paper (Phil. Mag. Jan. 1907) the momentum in the direction of $\mathbf{u}$ was denoted by P_1 and the momentum perpendicular to $\mathbf{u}$ by P_2 .

If P_t be the transverse component of the momentum, *i. e.* the component perpendicular to $\mathbf{u}$ in the direction making an acute angle with $\mathbf{w}$,

$$P_t = \frac{m_0 w^2}{16v} \cdot \frac{3 \sin 2\psi}{16n^4} \left[\frac{2n(3-2n^2)}{3(1-n^2)} - \log \frac{1+n}{1-n} \right] \\ = \frac{m_0 w^2}{v} C \sin 2\psi,$$

where C is a function of n .

Expressing the result in a series, we have

$$P_t = \frac{m_0 w^2}{v} \cdot \frac{\sin 2\psi}{4} \left[\frac{n}{5} + \frac{2n^3}{7} + \frac{3n^5}{9} + \frac{4n^7}{11} + \dots \right].$$

I have verified that the values of P_1 and P_2 given in § 14 lead to the above formula for P_u by expanding all the quantities in (43) in powers of w as far as w^2 , the process involving a good deal of labour. The verification of the formula for P_t requires the expansions to be carried as far as w^3 , and this led to expressions which were so complicated that I abandoned the work.

Tables V. and VI., which have been prepared for me by my friend and former colleague R. S. Cole, show the values of $vP_u/m_0 w^2$ and of $vP_t/m_0 w^2$ for a few values of n and of ψ .

TABLE V.—Values of $\frac{vP_u}{m_0 w^2} = A - B \sin^2 \psi$.

n .	A.	B.	$vP_u/m_0 w^2$.				
			$\psi=0^\circ$. $\psi=180^\circ$.	$\psi=30^\circ$. $\psi=150^\circ$.	$\psi=45^\circ$. $\psi=135^\circ$.	$\psi=60^\circ$. $\psi=120^\circ$.	$\psi=90^\circ$.
0.1	0.0409	0.0105	0.0409	0.0382	0.0356	0.0330	0.0304
0.2	0.0873	0.0242	0.0873	0.0813	0.0752	0.0692	0.0631
0.3	0.1468	0.0457	0.1468	0.1354	0.1240	0.1125	0.1011
0.4	0.2322	0.0835	0.2322	0.2114	0.1905	0.1696	0.1487
0.5	0.3695	0.1559	0.3695	0.3305	0.2915	0.2525	0.2135
0.6	0.6204	0.3099	0.6204	0.5430	0.4655	0.3880	0.3105
0.7	1.1680	0.6924	1.1680	0.9948	0.8217	0.6486	0.4755
0.8	2.7651	1.9461	2.7651	2.2786	1.7921	1.3056	0.8190
0.85	5.0427	3.8667	5.0427	4.0760	3.1094	2.1427	1.1760
0.9	11.6575	9.7421	11.6575	9.2220	6.7865	4.3510	1.9154

TABLE VI.—Values of $\frac{vP_t}{m_0w^2} = C \sin 2\psi$.

n.	C.	vPt/m_0w^2 .			
		$\psi = 0^\circ$, $\psi = 90^\circ$.	$\psi = 15^\circ$, $\psi = 75^\circ$.	$\psi = 30^\circ$, $\psi = 60^\circ$.	$\psi = 45^\circ$.
0.1	0.0051	0	0.0025	0.0044	0.0051
0.2	0.0106	0	0.0053	0.0092	0.0106
0.3	0.0172	0	0.0086	0.0149	0.0172
0.4	0.0256	0	0.0128	0.0222	0.0256
0.5	0.0375	0	0.0187	0.0325	0.0375
0.6	0.0560	0	0.0280	0.0485	0.0560
0.7	0.0888	0	0.0444	0.0769	0.0888
0.8	0.1606	0	0.0803	0.1391	0.1606
0.85	0.2382	0	0.1191	0.2063	0.2382
0.9	0.4039	0	0.2020	0.3498	0.4039

§ 18. I shall now consider the energy and the momentum which are radiated when the impulsive change of velocity affects the direction but not the magnitude of the velocity. If the direction of the velocity is changed by the angle α , we have

$$u_2 = u_1 = n_1 v,$$

$$w^2 = 2u_1^2(1 - \cos \alpha) = 4n_1^2 v^2 \sin^2 \frac{1}{2}\alpha,$$

$$(v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}} = 2n_1 v^2 \sin \frac{1}{2}\alpha (1 - n_1^2 \cos^2 \frac{1}{2}\alpha)^{\frac{1}{2}}.$$

Hence, by § 11, we have

$$\frac{W}{U_0} = \frac{1}{Z} \log \frac{1+Z}{1-Z} - 2,$$

where

$$Z = \frac{2n_1 \sin \frac{1}{2}\alpha (1 - n_1^2 \cos^2 \frac{1}{2}\alpha)^{\frac{1}{2}}}{1 - n_1^2 \cos \alpha}.$$

When n_1 is small, the numerical calculation of W/U_0 is best made by the series

$$\frac{W}{U_0} = 2 \left\{ \frac{Z^2}{3} + \frac{Z^4}{5} + \frac{Z^6}{7} + \dots \right\},$$

which, by § 11, is always valid when n_1 is less than unity.

If the denominator of each term of the last series be expanded in powers of n_1^2 , we obtain a *double* series in n_1^2 . This can be rearranged in the form of a *single* power series in n_1^2 ,

provided that the double series remains convergent when the modulus of every term is substituted for the term itself. This will be the case if

$$\frac{4n_1^2 \sin^2 \frac{1}{2} \alpha (1 + n_1^2 \cos^2 \frac{1}{2} \alpha)}{(1 - n_1^2 |\cos \alpha|)^2} < 1,$$

where $|\cos \alpha|$ is the modulus of $\cos \alpha$.

The quantity on the left increases with n_1^2 , and hence the inequality will be satisfied if n_1 be less than the least real value obtained from the equation

$$4n_1^2 \sin^2 \frac{1}{2} \alpha (1 + n_1^2 \cos^2 \frac{1}{2} \alpha) = (1 - n_1^2 |\cos \alpha|)^2.$$

When α is not greater than $\frac{1}{2}\pi$ this equation leads to $(\sqrt{2}-1)^{\frac{2}{3}}$, or 0.643... as the value of n . When α is greater than $\frac{1}{2}\pi$, the value is $\sqrt{2}-1$, or 0.414.... Hence, if $n < \sqrt{2}-1$ the double series can certainly be rearranged as a single series for all values of α .

On rearranging this series, we obtain

$$\frac{W}{U_0} = 8n_1 S^2 \left\{ \frac{1}{3} + (5-3S^2) \frac{n_1^2}{3 \cdot 5} + (35-42S^2+16S^4) \frac{n_1^4}{3 \cdot 5 \cdot 7} \right. \\ \left. + (105-189S^2+144S^4-40S^6) \frac{n_1^6}{5 \cdot 7 \cdot 9} + \dots \right\}, \quad (45)$$

where $S = \sin \frac{1}{2} \alpha$.

Mr. R. S. Cole has prepared Table VII., giving the values of W/U_0 for some values of n_1 and of α .

TABLE VII.—Values of $\frac{W}{U_0}$.

n_1 .	$\alpha=0^\circ$.	$\alpha=30^\circ$.	$\alpha=60^\circ$.	$\alpha=90^\circ$.	$\alpha=120^\circ$.	$\alpha=150^\circ$.	$\alpha=180^\circ$.
0.1	0	0.0018	0.0067	0.0134	0.0201	0.0250	0.0268
0.2	0	0.0074	0.0276	0.0547	0.0818	0.1013	0.1084
0.3	0	0.0177	0.0650	0.1281	0.1895	0.2333	0.2492
0.4	0	0.0338	0.1235	0.2404	0.3415	0.4293	0.4572
0.5	0	0.0588	0.2118	0.4050	0.5325	0.7038	0.7465
0.6	0	0.0983	0.3464	0.6454	0.9082	1.0820	1.1423
0.7	0	0.1653	0.5623	1.0087	1.3779	1.6125	1.6922
0.8	0	0.2968	0.9475	1.6030	2.1021	2.4040	2.5043
0.85	0	0.4221	1.2773	2.0704	2.6433	2.9804	3.0911
0.9	0	0.6540	1.8161	2.7765	3.4285	3.8008	3.9216

The radiated momentum is, by symmetry, along the direction bisecting the angle between the initial and final velocities. In this case the two components P_1 and P_2 , which are parallel to u_1 and u_2 , have equal values, and thus

$$P = 2P_1 \cos \frac{1}{2}\alpha.$$

Hence, since $M_2 = M_1$ and $W_2 = W_1$, we find by § 14 that

$$\begin{aligned} P &= 2 \cos \frac{1}{2}\alpha \frac{M_1(\cos \alpha - 1) + W/u_1}{1 + \cos \alpha} \\ &= \left(\frac{W}{u_1} - 2M_1 \sin^2 \frac{1}{2}\alpha \right) \sec \frac{1}{2}\alpha. \end{aligned}$$

Hence

$$\frac{Pv}{U_0} = \left[\frac{W}{n_1 U_0} - \frac{2M_1 v}{U_0} \sin^2 \frac{1}{2}\alpha \right] \sec \frac{1}{2}\alpha. \quad . \quad . \quad (46)$$

If we wish, we can express P in terms of $m_0 v$ by means of the formula $U_0/v = 3m_0/4$, where m_0 is the electromagnetic mass for infinitesimal speeds.

If we make use of the series (39) for M_1 and combine it with the series (45) for W/U_0 , we obtain the series

$$\begin{aligned} \frac{Pv}{U_0} &= 8n_1^2 S^2 \sqrt{1-S^2} \left\{ \frac{1}{5} + (26-16S^2) \frac{n_1^2}{3 \cdot 5 \cdot 7} \right. \\ &\quad \left. + (85-104S^2+40S^4) \frac{n_1^4}{5 \cdot 7 \cdot 9} + \dots \right\}, \end{aligned}$$

where, as before, $S = \sin \frac{1}{2}\alpha$. This series is certainly valid when $n_1 < 0.414 \dots$.

In Table VIII., which was prepared by Mr. A. J. Bamford of Emmanuel College, the values of Pv/U_0 for $n_1 = 0.1, n = 0.2$, and $n_1 = 0.3$ were calculated by aid of this series. For the remaining values of n_1 , Pv/U_0 was calculated by (46) from the values of W/U_0 given in Table VII. and those of $M_1 v/U_0$ given in Table II.

When $\alpha = \pi$, so that the motion is just reversed, the value of P given by (46) becomes indeterminate since then $\cos \frac{1}{2}\alpha = 0$ and $W = 2u_1 M_1$, as appears from (13) and (39). But in this case symmetry demands that P should vanish.

TABLE VIII.—Values of $\frac{Pv}{U_0}$.

n_1 .	$\alpha=0^\circ$.	$\alpha=30^\circ$.	$\alpha=60^\circ$.	$\alpha=90^\circ$.	$\alpha=120^\circ$.	$\alpha=150^\circ$.	$\alpha=180^\circ$.
0.1	0	0.00010	0.00035	0.00057	0.00060	0.00039	0
0.2	0	0.00087	0.00289	0.00468	0.00493	0.00316	0
0.3	0	0.00312	0.01031	0.01654	0.01723	0.01095	0
0.4	0	0.0082	0.0266	0.0417	0.0430	0.0264	0
0.5	0	0.0182	0.0581	0.0898	0.0905	0.0564	0
0.6	0	0.0376	0.1171	0.1751	0.1716	0.1046	0
0.7	0	0.0768	0.2297	0.3285	0.3108	0.1859	0
0.8	0	0.1670	0.4640	0.6202	0.5596	0.3257	0
0.85	0	0.2619	0.6854	0.8733	0.7646	0.4381	0
0.9	0	0.4501	1.0722	1.2817	1.0828	0.6091	0

In conclusion I desire to thank those friends who have assisted me in the preparation of this paper. But for the help of Mr. G. T. Bennett I should have given up the problem as unmanageable. To Mr. R. S. Cole and to my colleague Mr. T. G. Bedford, and to Mr. A. J. Bamford I am indebted for much help in the heavy arithmetical work involved in the construction and verification of the numerical tables. Mr. Bamford has also verified the mathematical work in the paper.

IV. *A Modified Theory of Gravitation.*

By C. V. BURTON, *D.Sc.**

1. **S**O many questions arise in connexion with any theory which aims at assigning a dynamical basis for gravitation, that it may conduce to clearness if some elements of this paper are first presented without mathematical treatment.

The theory now put forward is pulsatory, and may in fact be regarded as a development of that which has been suggested by Prof. Hicks †. The æther, even where modified

* Communicated by the Physical Society : read May 8, 1908.

† W. M. Hicks, "On the Problem of two Pulsating Spheres in a Fluid," *Proc. Camb. Phil. Soc.* iii. p. 277 (Oct. 1879).