This is the generalization of (1), $\frac{1}{r^{p-2}}$ being the hyperpotential function which satisfies the equation

$$V^2 \frac{1}{r^{p-2}} = 0.$$

Also

$$\begin{aligned} & f_n \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \dots \frac{\partial}{\partial x_p} \right) \phi (x_1^2 + x_2^2 + \dots + x_p^2) \\ &= \left\{ 2^n \phi^{(n)} (x_1^2 + \dots) + \frac{2^{n-1}}{2} \phi^{(n-1)} (x_1^2 + \dots) V^2 \right. \\ & \quad \left. + \frac{2^{n-2}}{2 \cdot 4} \phi^{(n-2)} (x_1^2 + \dots) V^4 + \dots \right\} f_n (x_1, x_2 \dots x_p) \dots (5). \end{aligned}$$

This is the extension of (2) to any number of variables.

The theorem (4) may be used, as in the case $p=3$, to find a general expression for the hyper-spherical harmonic which has given poles, but, as the method contains nothing new, I shall not proceed further with this extension. The theorem (5) is the most general differentiation theorem of those which I have given, as it contains all the others as particular cases; it may, I think, be regarded as fundamental in the theory of hyper-spherical or spherical harmonics of all orders.

Notes on Determinants. By J. E. CAMPBELL. Received November 29th, 1892. Read December 8th, 1892.

[In accordance with the late Professor Smith's notation, a determinant of the p^{th} class may be written

$$| a_{ik} \dots |.$$

The fact that a determinant of the second class (an ordinary determinat) is not altered if the vertical columns be written horizontally, is expressed by the identity

$$| a_{ij} | = | a_{ji} |.$$

For determinants of higher class it is known that any of the suffixes can be interchanged except the first; and, if the class be even,

the first suffix can also be interchanged with any other, but for determinants of odd class this is not true.

By considering a cubic determinant as an ordinary determinant in alternate numbers, I have tried to explain this essential distinction between determinants of odd and even classes.

If the element $a_{pqr\dots} = -a_{qpr\dots}$ and $a_{ppr} = 0$,

the determinant is called skew symmetrical.

It is easily seen that skew symmetrical determinants of even class and odd degree vanish identically. This is analogous to the well-known theorem in ordinary determinants; but there is no corresponding analogue to the theorem that skew symmetrical determinants of the second class and even degree are perfect squares.

The reasoning which establishes these propositions does not apply to skew symmetrical determinants of odd class.

By a different method it is shown that they vanish identically whether the class be even or odd.

It is next shown that, if we form any determinant of even class $2p$, from $2p$ ordinary determinants, in a manner analogous to that in the rule for the multiplication of two ordinary determinants, the determinant so formed is the product of the $2p$ determinants; and if any determinant of odd class $2p+1$ is formed from $2p+1$ ordinary determinants, the determinant so formed is the product of the last $2p$ of these ordinary determinants into the first, taken with all its signs positive.

A somewhat similar result is shown to hold for determinants of alternate numbers.

As an application, let

$$Z = \frac{a_1}{(x-a_1)(y-\beta_1)} + \dots + \frac{a_n}{(x-a_n)(y-\beta_n)},$$

and let (p, q) denote $\frac{a^{p+q}z}{p! q! dx^p dy^q}$.

By multiplying the arrays

$$\left\| \begin{array}{cccc} \frac{a_1}{x-a_1} & \dots & \dots & \dots \\ \frac{a_1}{(x-a_1)^2} & \dots & \dots & \dots \end{array} \right\|, \quad \left\| \begin{array}{cccc} \frac{1}{y-\beta_1} & \dots & \dots & \dots \\ \frac{1}{(y-\beta_1)^2} & \dots & \dots & \dots \end{array} \right\|,$$

we get
$$\begin{vmatrix} (0, 0), & (0, 1) \\ (1, 0), & (1, 1) \end{vmatrix} = \Sigma \frac{a_p a_q (a_p - a_q)(\beta_p - \beta_q)}{(x - a_p)^2 (x - \beta_p)^2 (x - a_q)^2 (x - \beta_q)^2}.$$

Suppose now $n = 1$; we get that the primitive of

$$\begin{vmatrix} (0, 0), & (0, 1) \\ (1, 0), & (1, 1) \end{vmatrix}$$

is
$$Z = \frac{a_1}{(x - a_1)(y - \beta_1)}.$$

Similarly, by multiplying

$$\left\| \begin{array}{c} \frac{a_1}{(x - a_1)} \dots\dots \\ \frac{a_1}{(x - a_1)^2} \dots\dots \\ \frac{a_1}{(x - a_1)^3} \dots\dots \end{array} \right\|, \quad \left\| \begin{array}{c} \frac{1}{y - \beta_1} \dots\dots \\ \frac{1}{(y - \beta_1)^2} \dots\dots \\ \frac{1}{(y - \beta_1)^3} \dots\dots \end{array} \right\|,$$

we get that the primitive of

$$\begin{vmatrix} (0, 0), & (0, 1), & (0, 2) \\ (1, 0), & (1, 1), & (1, 2) \\ (2, 0), & (2, 1), & (2, 2) \end{vmatrix} = 0$$

is
$$Z = \frac{a_1}{(x - a_1)(y - \beta_1)} + \frac{a_2}{(x - a_2)(y - \beta_2)}.$$

Similar primitives are obtained for differential equations which are in the form of determinants of higher class.

A further application is obtained by taking powers of different invariantive symbols, of which (123) is the simplest for the ternary quantie.

The resulting invariants are seen to be determinants of some even class.]

In accordance with Prof. Sylvester's umbral notation, a cubic determinant is written

$$\begin{vmatrix} 1, & 1, & 1 \\ 2, & 2, & 2 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ n, & n, & n \end{vmatrix},$$

and denotes the sum of all possible products

$$a_{111} \cdot a_{222} \dots a_{nnn},$$

obtained by giving the terms in the second and third columns every possible permutation, and changing the sign with each permutation.

Similarly determinants of the fourth class may be written

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ 2, & 2, & 2, & 2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ n, & n, & n, & n \end{vmatrix};$$

and so on for determinants of higher class.

Shorter ways of writing these would be

$$|a_{ijk}|, \quad |a_{ijkl}|, \quad \dots\dots$$

The fact that a determinant of the second class (an ordinary determinant) is not altered if the vertical rows be written horizontally, is expressed by the identity

$$|a_{ij}| = |a_{ji}|.$$

It is shown by Prof. Lloyd Tanner (*Proc. Lond. Math. Soc.*, Vol. x.), that

$$|a_{ijkl}| = |a_{ikjl}|, \quad \text{and} \quad |a_{ijk}| = |a_{ikj}|;$$

and generally that any of the suffixes can be interchanged except the first.

For determinants of even class, the first suffix can also be interchanged with any other, but this is not true for determinants of odd class; thus

$$|a_{ijkl}| = |a_{jilk}|;$$

but $|a_{ijk}|$ is not equal to $|a_{jik}|$.

To consider more closely the case of determinants of odd class, take a cubic determinant, or its equivalent, an ordinary determinant of alternate numbers (Scott, *Proc. Lond. Math. Soc.*, Vol. xi.), $|a_{ij}|$.

I shall show that $|a_{ij}| = \{a_{ji}\}$,

where $\{ \}$ denotes a determinant with all the signs taken positive.

Consider any term in $|a_{ij}|$, $a_{13} \cdot a_{21} \cdot a_{32}$; to this there will be a complementary term of the same sign

$$a_{12} \cdot a_{23} \cdot a_{31}.$$

Corresponding to these, in the signless determinant $\{a_{ij}\}$, we have

$$a_{31} \cdot a_{12} \cdot a_{23},$$

and its complementary $a_{21} \cdot a_{32} \cdot a_{13}.$

To bring these to the forms $a_{12} \cdot a_{23} \cdot a_{31}$, and $a_{13} \cdot a_{21} \cdot a_{32}$, respectively, the alternate numbers must be inverted precisely the same number of times that the symbols were to deduce 13.21.32 from 11.22.33. The signs therefore of $a_{12} \cdot a_{23} \cdot a_{31}$ and $a_{13} \cdot a_{21} \cdot a_{32}$, in $\{a_{ij}\}$, will be the same as the signs of the corresponding terms in $|a_{ij}|$; so that we have

$$|a_{ij}| = \{a_{ij}\}.$$

Just as it was shown that cubic determinants were equivalent to ordinary determinants of alternate numbers, it may be shown that determinants of odd class generally are equivalent to ordinary determinants of odd products of sets of alternate numbers, and even-classed determinants to ordinary determinants of even products of alternate numbers. The reasoning applied to cubic determinants will then explain the essential distinction between the odd and even classes.

If the element $a_{ijk\dots} = -a_{jik\dots}$ and $a_{iik} = 0$,

the determinant is called skew symmetrical.

It is then easily shown that a skew symmetrical determinant of even class and odd degree vanishes.

For

$$a_{ijkl} = -a_{jilk},$$

and the degree is odd; therefore

$$|a_{ijkl}| = -|a_{jilk}|;$$

and therefore

$$|a_{ijkl}| = 0,$$

since $|a_{ijkl}|$ also equals $|a_{jilk}|$.

This theorem is analogous to the corresponding one in ordinary determinants; but we have no theorem for higher determinants analogous to the one that skew symmetrical determinants of even degree are perfect squares.

This reasoning would not apply to determinants of odd class, since we have not the fundamental formula

$$|a_{ik}| = |a_{jik}|.$$

I shall show, however, by another method, that skew symmetrical determinants of odd class vanish, whether their degree be even or odd.

Consider a cubic determinant, or its equivalent, an ordinary determinant of alternate numbers.

Let $a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44} \cdot a_{55}$ be the leading term. Every other term is derived from this by cyclical permutations of groups of the elements.

Take any permutation 245.31. This generates the term $a_{13} \cdot a_{25} \cdot a_{31} \cdot a_{42} \cdot a_{54}$, and from this, by the permutation 452, we get another term $a_{13} \cdot a_{24} \cdot a_{31} \cdot a_{45} \cdot a_{52}$. Since the number of terms in the cycle is odd, we can write this in the form $a_{13} \cdot a_{52} \cdot a_{31} \cdot a_{24} \cdot a_{45}$ without altering the sign. Add now these terms $a_{13} \cdot a_{52} \cdot a_{31} \cdot a_{24} \cdot a_{45}$ and $a_{13} \cdot a_{25} \cdot a_{31} \cdot a_{42} \cdot a_{54}$; they cancel one another, since

$$a_{25} + a_{52} = 0, \quad a_{24} + a_{42} = 0, \quad a_{45} + a_{54} = 0.$$

Similarly, we see that every other term which appears will have a term cancelling it; unless all the cycles of the group are mere inversions; and these also disappear for

$$a_{pq} \cdot a_{qp} = -(a_{pq})^2 = 0,$$

since a_{pq} is an alternate number. The only term now left in $a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44} \cdot a_{55}$, and each factor of this is zero.

Similarly, the theorem may be proved for any determinant of odd class; the essential part of the proof consisting in noticing that the term obtained by a right-handed cyclical permutation of a group (exceeding two) is of the same sign as the term obtained by the corresponding left-handed permutation.

If the determinant had been only half skew, that is, if

$$a_{ijk} = -a_{jik},$$

but a_{iik} not zero, the determinant would have reduced to $a_{11} \cdot a_{22} \dots a_{nn}$ in alternate numbers; that is, to the ordinary determinant

$$\begin{vmatrix} a_{111} & a_{112} & \dots & a_{11n} \\ a_{221} & a_{222} & \dots & a_{22n} \\ \dots & \dots & \dots & \dots \\ a_{nn1} & a_{nn2} & \dots & a_{nnn} \end{vmatrix}.$$

If a cubic determinant be formed with the elements

$$(ijk) = \sum_{p=1}^{p=n} a_{ip} \cdot b_{jp} \cdot c_{kp},$$

it is equal to the product of the ordinary determinants $|b_{ik}|$, $|c_{ik}|$, and the signless determinant $\{a_{ik}\}$.

The cubic determinant is

$$\Pi_{i=1}^{i=n} \sum_{j,k=1}^{j,k=n} (ijk) \epsilon_j \epsilon_k,$$

where $\epsilon_1, \epsilon_2 \dots \epsilon_1, \epsilon_2 \dots$ are two independent sets of alternate units (Scott, *Theory of Determinants*, VII., §3).

$$\sum_{j=1}^{j=n} (ijk) \epsilon_j \epsilon_k = \sum_{p=1}^{p=n} a_{ip} c_{kp} \beta_p \epsilon_k,$$

where β_p is the alternate number $b_{1p}\epsilon_1 + b_{2p}\epsilon_2 + \dots + b_{np}\epsilon_n$.

$$\sum_{j,k=1}^{j,k=n} (ijk) \epsilon_j \epsilon_k = \sum_{p=1}^{p=n} a_{ip} \beta_p \gamma_p,$$

where γ_p is the alternate number $c_{1p}\epsilon_1 + c_{2p}\epsilon_2 + \dots + c_{np}\epsilon_n$.

Therefore the cubic determinant

$$= \Pi_{i=1}^{i=n} (a_{i1}\beta_1\gamma_1 + a_{i2}\beta_2\gamma_2 + \dots + a_{in}\beta_n\gamma_n).$$

Now $\beta\gamma$, being the product of two alternate numbers, is a concurrent number, so that the determinant reduces to

$$\{a_{ik}\} \beta_1\gamma_1 \beta_2\gamma_2 \dots \beta_n\gamma_n,$$

and

$$\beta_1\beta_2 \dots \beta_n = |b_{ik}|,$$

and

$$\gamma_1\gamma_2 \dots \gamma_n = |c_{ik}|;$$

so that, finally, the cubic determinant

$$= \{a_{ik}\} \times |b_{ik}| \times |c_{ik}|.$$

If a determinant of the fourth class be formed with the elements

$$(ijkl) = \sum_{p=1}^{p=n} a_{ip} b_{jp} c_{kp} d_{lp},$$

it may be shown by a similar method that it is the product of four ordinary determinants

$$|a_{ik}| \times |b_{ik}| \times |c_{ik}| \times |d_{ik}|;$$

for it is reduced to the product

$$\Pi_{i=1}^{i=n} (a_{i1}\beta_1\gamma_1\delta_1 + a_{i2}\beta_2\gamma_2\delta_2 + \dots + a_{in}\beta_n\gamma_n\delta_n),$$

and $\beta \cdot \gamma \cdot \delta$, being the product of three alternate numbers, is itself an alternate number.

And so generally, if the class p of the determinant formed by this method is odd, it is the product of $p-1$ ordinary determinants and a signless determinant $\{a_{ik}\}$, where a is the letter of the first determinant of the auxiliary system; and, if the class p is even, the determinant is the product of the p auxiliary determinants.

If the number of rows and columns in the auxiliary matrices be not equal, similar reasoning leads to analogous results to those in the ordinary multiplication of matrices.

A rule is given by the late Mr. Spottiswoode (*Proc. Lond. Math. Soc.*, February, 1876) for the multiplication of two determinants of alternate numbers. A more convenient rule (with a view to this extension) would be:—

The product of two such determinants = $\pm$ the corresponding compound determinant taken with all the signs positive.

The general rule for multiplication is then the same as for determinants of ordinary numbers, if the resulting determinant of the p^{th} class is taken with all its signs positive.

This may easily be deduced from the former method by employing sets of concurrent units instead of alternate ones.

$$\text{Let } Z = \frac{a_1}{(x-a_1)(y-\beta_1)} + \frac{a_2}{(x-a_2)(y-\beta_2)} + \dots + \frac{a_n}{(x-a_n)(y-\beta_n)},$$

and let (pq) denote $\frac{\partial^{p+q} Z}{p! q! \partial x^p \partial y^q}$. Then

$$(0, 0) = \frac{a_1}{(x-a_1)(y-\beta_1)} + \dots,$$

$$-(0, 1) = \frac{a_1}{(x-a_1)(y-\beta_1)^2} + \dots,$$

$$(-1)^{p+q} (p, q) = \frac{a_1}{(x-a_1)^{p+1} (y-\beta_1)^{q+1}} + \dots$$

By multiplying the rectangular arrays

$$\left\| \begin{array}{ccc} \frac{a_1}{x-a_1}, & \frac{a_2}{x-a_2}, & \dots \\ \frac{a_1}{(x-a_1)^2}, & \frac{a_2}{(x-a_2)^2}, & \dots \\ \dots & \dots & \dots \end{array} \right\| \times \left\| \begin{array}{ccc} \frac{1}{y-\beta_1}, & \frac{1}{y-\beta_2}, & \dots \\ \frac{1}{(y-\beta_1)^2}, & \frac{1}{(y-\beta_2)^2}, & \dots \\ \dots & \dots & \dots \end{array} \right\|,$$

we have the series

$$\begin{vmatrix} (0, 0), & (0, 1) \\ (1, 0), & (1, 1) \end{vmatrix} \equiv \Sigma \frac{a_p a_q (a_p - a_q)(\beta_p - \beta_q)}{(x - a_p)^2 (y - \beta_p)^2 (x - a_q)^2 (y - \beta_q)^2},$$

$$\begin{vmatrix} (0, 0), & (0, 1), & (0, 2) \\ (1, 0), & (1, 1), & (1, 2) \\ (2, 0), & (2, 1), & (2, 2) \end{vmatrix}$$

$$\equiv \Sigma \frac{a_p a_q a_r \rho^4 (a_p a_q a_r) \rho^4 (\beta_p \beta_q \beta_r)}{(x - a_p)^3 (y - \beta_p)^3 (x - a_q)^3 (y - \beta_q)^3 (x - a_r)^3 (y - \beta_r)^3},$$

and similar functions, where the summation extends to all values of p, q, r selected from the numbers 1, 2, 3 ... n .

Let $n = 2$; we have then

$$\begin{vmatrix} (0, 0), & (0, 1), & (0, 2) \\ (1, 0), & (1, 1), & (1, 2) \\ (2, 0), & (2, 1), & (2, 2) \end{vmatrix} = 0,$$

so that the primitive of this equation is

$$Z = \frac{a_1}{(x - a_1)(y - \beta_1)} + \frac{a_2}{(x - a_2)(y - \beta_2)},$$

where $a_1, a_2, \alpha_1, \beta_1, \alpha_2, \beta_2$, are arbitrary constants.

This was suggested by a paper of Mr. Forsyth's in the *Messenger of Mathematics* (February, 1888), in which, by a very simple method, similar equations were obtained for the case of one independent variable.

So, by multiplying together four matrices, we should obtain

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \end{vmatrix}$$

$$\equiv \Sigma \frac{a_p a_q (\alpha_p - \alpha_q)(\beta_p - \beta_q)(\gamma_p - \gamma_q)(\delta_p - \delta_q)}{(x_1 - a_p)^2 (x_2 - \beta_p)^2 (x_3 - \gamma_p)^2 (x_4 - \delta_p)^2 (x_1 - a_q)^2 (x_2 - \beta_q)^2 (x_3 - \gamma_q)^2 (x_4 - \delta_q)^2},$$

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \\ 2, & 2, & 2, & 2 \end{vmatrix}$$

$$\equiv \sum \frac{\alpha_p \alpha_q \alpha_r \rho^4 (a_p, a_q, a_r) \rho^4 (\beta_p, \beta_q, \beta_r) \rho^4 (\gamma_p, \gamma_q, \gamma_r) \rho^4 (\delta_p, \delta_q, \delta_r)}{(x_1 - a_p)^3 (x_2 - \beta_p)^3 (x_3 - \gamma_p)^3 (x_4 - \delta_p)^3 (x_1 - a_q)^3 (x_2 - \beta_q)^3 (x_3 - \gamma_q)^3 (x_4 - \delta_q)^3} \\ \times (x_4 - \delta_r)^3 (x_1 - a_r)^3 (x_2 - \beta_r)^3 (x_3 - \gamma_r)^3 (x_4 - \delta_r)^3$$

where $\rho^4(a_p, a_q, a_r)$ denotes the product

$$(a_p - a_q)(a_p - a_r)(a_q - a_r),$$

and where the expressions on the left denote determinants of the fourth class, and 2nd, 3rd, ... degrees, whose elements

$$(p, q, r, s) = \frac{\partial^{p+q+r+s} Z}{p! q! r! s! \partial x_1^p \cdot \partial x_2^q \cdot \partial x_3^r \cdot \partial x_4^s}.$$

Thus we get: the primitive of

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \\ 2, & 2, & 2, & 2 \end{vmatrix} = 0$$

is
$$Z = \frac{a_1}{(x_1 - a_1)(x_2 - \beta_1)(x_3 - \gamma_1)(x_4 - \delta_1)},$$

the primitive of

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \\ 2, & 2, & 2, & 2 \end{vmatrix}$$

is

$$Z = \frac{a_1}{(x_1 - a_1)(x_2 - \beta_1)(x_3 - \gamma_1)(x_4 - \delta_1)} + \frac{a_2}{(x_1 - a_2)(x_2 - \beta_2)(x_3 - \gamma_2)(x_4 - \delta_2)},$$

and so on.

Similar reasoning would apply to determinants of any even class. The results for determinants of odd class are not quite so symmetrical.

In Mr. Forsyth's paper he shows that his functions are homographic invariants; but this is easily seen (from the form of the primitive) not to be true when the differential equations are partial, as here.

$$(12)", \quad \{(23)(31)(12)\}", \quad \{(23)(31)(12)(14)(24)(34)\}", \quad \dots$$

are invariants of the even binary quantio.

It is shown, amongst other matter, in the papers of Messrs. Cayley and Sylvester in the *Cambridge and Dublin Mathematical Journal*, 1852, that these may be expressed as commutants. The following is, perhaps, a slightly different way of looking at the matter.

The binary quantio being

$$(a_1x_1 + a_2x_2)^n + (\beta_1x_1 + \beta_2x_2)^n + \dots,$$

we can form, from

$$\begin{vmatrix} a_1 & \beta_1 & \dots & \dots \\ a_2 & \beta_2 & \dots & \dots \end{vmatrix}^n,$$

a determinant of the n^{th} class and second degree, which will be equal to the sum

$$\sum \begin{pmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \end{pmatrix}^n.$$

Now each of the expressions $\begin{vmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \end{vmatrix}$ is unaltered by linear transformation, and the determinant of the n^{th} class is at once seen to be a rational function of the coefficients of the binary quantio; it is therefore an invariant.

So, from

$$\begin{vmatrix} a_1^2 & \beta_1^2 & \dots \\ a_1a_2 & \beta_1\beta_2 & \dots \\ a_2^2 & \beta_2^2 & \dots \end{vmatrix}^n$$

we can generate a determinant of the n^{th} class and third degree,

$$\begin{aligned} &\equiv \sum \begin{vmatrix} a_1^2 & \beta_1^2 & \gamma_1^2 \\ a_1a_2 & \beta_1\beta_2 & \gamma_1\gamma_2 \\ a_2^2 & \beta_2^2 & \gamma_2^2 \end{vmatrix}^n \\ &\equiv \sum \begin{vmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \end{vmatrix}^n \times \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix}^n \times \begin{vmatrix} \gamma_1 & a_1 \\ \gamma_2 & a_2 \end{vmatrix}^n; \end{aligned}$$

and, as each of these expressions is unaltered by linear transformation, the determinant will be an invariant of

$$(a_1x_1 + a_2x_2)^{2n} + (\beta_1x_1 + \beta_2x_2)^{2n} + \dots;$$

and so generally for binary quantics.

For ternary quantics, $(123)^n$ is an invariative symbol if n be even.

Another such symbol is derived from the condition that six points lie on a conic; it is

$$\begin{vmatrix} \frac{d^2}{dx_1^2} & \frac{d^2}{dy_1^2} & \dots\dots\dots \\ \frac{d^2}{dx_1 dx_2} & \frac{d^2}{dy_1 dy_2} & \dots\dots\dots \\ \frac{d^2}{dx_1 dx_3} & \frac{d^2}{dy_1 dy_3} & \dots\dots\dots \\ \frac{d^2}{dx_2^2} & \frac{d^2}{dy_2^2} & \dots\dots\dots \\ \frac{d^2}{dx_2 dx_3} & \frac{d^2}{dy_2 dy_3} & \dots\dots\dots \\ \frac{d^2}{dx_3^2} & \frac{d^2}{dy_3^2} & \dots\dots\dots \end{vmatrix}''.$$

It might more shortly be written

$$(1, 2, 3, 4, 5, 6)_2'' \text{ or } (6)_2''.$$

Similarly, from the condition that ten points should lie on a cubic, is derived the operator $(10)_3''$.

All such invariants can be expressed as determinants of the n^{th} class.

To see this we have only to write the ternary quantic

$$(a_1 x_1 + a_2 x_2 + a_3 x_3)^n + (\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)^n + \dots,$$

and for $\frac{d^2}{dx_1^2}$ write a_1^2 , for $\frac{d^2}{dx_1 dx_2}$ write $a_1 a_2$, and so on.

Then the same method as applied to the binary quantic applies to the ternary.

Thus $(1, 2, 3)_2''$, for the conic $\equiv (a_1, \beta_2, \gamma_3)_2''$, which vanishes if the conic can be thrown into the form of the sum of two squares.

$(1, 2, 3)_4''$, applied to the quartic, gives the invariant A (Salmon, *Higher Plane Curves*, Art. 293).

$(1, 2, 3, 4, 5, 6)_3''$ applied to the quartic, gives the invariant B .

It is an ordinary determinant, and it vanishes if the six lines $a=0$, $\beta=0$, $\gamma=0$, $\delta=0$, $\epsilon=0$, $\theta=0$, touch a conic, or (as a particular case) if the quartic can be expressed as the sum of five fourth powers.

$(10)_3^2$, applied to a curve of the sixth degree, gives an invariant which vanishes if the sides of what might be called the decagon of reference touch a curve of the third class.

So $(15)_4^2$, applied to a curve of the eighth degree, gives an invariant whose vanishing is the condition that the sides of the quindecagon of reference touch a curve of the fourth class.

It would not be so easy to interpret geometrically the invariants which are in the form of determinants of higher class.

A similar method will apply to equations in any number of variables.

Thursday, January 12th, 1893.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Mr. Pat Doyle, C.E., Editor of *Indian Engineering*, Calcutta, was elected a member, and Mr. Greenstreet was admitted into the Society.

The following communications were made:—

On the Application of Clifford's Graphs to Ordinary Binary Quantics—second part, Seminvariants: the President (Prof. Elliott in the chair).

On the Evaluation of a certain Surface-Integral, and its Application to the Expansion of the Potential of Ellipsoids in Series: Dr. Hobson.

Mr. Love made a brief statement on "The Vibrations of an Elastic Circular Ring."

A cabinet likeness of Lt.-Col. Campbell was received and placed in the Society's album.

The following presents were made to the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Band xvi., Stück 11.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. xi., No. 2, 1892; Coimbra.

"Principles of the Algebra of Physics," by A. Macfarlane; pamphlet, 8vo, Salem, Mass., 1891.

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On the Evaluation of a certain Surface-Integral, and its application to the Expansion, in Series, of the Potential of Ellipsoids.
By E. W. HOBSON. Received and read January 12th, 1893.

If V be any function of x, y, z , the coordinates of a point, the function being finite and continuous throughout a sphere of radius R whose centre is the origin, it is known that

$$\iint V dS = 4\pi R^2 \sum_{n=0}^{\infty} \frac{R^{2n}}{(2n+1)!} \nabla^{2n} V,$$

the integration being taken over the whole surface of the sphere, and $\nabla^{2n} V$ having its value at the origin; ∇^2 denotes Laplace's operator. This theorem has been applied by Mr. W. D. Niven* to the evaluation of a number of important definite integrals involving spherical harmonics, and to the development, in series, of the potentials of a uniform solid ellipsoid and of a homœoid.

I propose here to investigate a more general surface-integral theorem which includes the above, and which also furnishes a proof

* *Phil. Trans.*, 1879.