Inductance and Resistance in Telephone and other Circuits

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LVII. *Inductance and Resistance in Telephone and other Circuits.* By J. W. NICHOLSON, M.A., D.Sc.; Trinity College, Cambridge*.

I. EFFECTIVE INDUCTANCE.

A GENERAL formula for the effective inductance of a circuit consisting of two long parallel wires has been given by the author †, and is suitable for cases in which the current distribution in either wire is greatly affected by the frequency of alternation. In its general form, although its limitations are clearly defined, the result is not well adapted, in the absence of tables, to rapid calculation. The main object of the present paper is to examine certain important cases in detail, and to obtain formulæ capable of immediate use. A calculation of the effective resistance is also made in each case. A problem to which attention has been mainly directed, which includes several practical cases of great interest, is that of the simple telephone circuit, in which the leads are not twisted round each other in order to annul the inductive effects of the earth and of neighbouring circuits.

In the proof of the general formula, the influence of electrostatic capacity was ignored. This imposes a great limitation upon the types of circuit for which the expression may be used. An estimation of the maximum capacity causing no alteration to a given order of accuracy is given in this paper.

Throughout the investigation, only iron and copper wires, as the two extreme cases, are considered. The large permeability of iron completely changes the character of the effect of frequency on its self-induction, as compared with other metals. To all metals except iron greatly used in practice, the formulæ developed for copper wires may be applied with a nearly identical order of accuracy.

Let a be the radius of either wire, c their distance apart, and (μ, σ) their permeability and resistivity. They are equal in all respects.

* Read June 11, 1909.

† Phil. Mag. Feb. 1909; Proc. Phys. Soc. vol. xxi.

The value of the inductance per unit length is then

$$L = 4 \log \frac{c}{a} + \frac{4\mu}{x} \cdot \frac{\text{ber } x \text{ ber}' x + \text{bei } x \text{ bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} + M, \quad (1)$$

where, if $n/2\pi$ be the frequency,

$$x = 2a(\pi\mu n/\sigma)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, M and N being real,

$$c^2(M + \iota N) = \frac{4a^2(\log a/c - \mu J_0/kaJ_0')(1 - \mu J_1/kaJ_1')}{(\log ha - \mu J_0/kaJ_0')(1 + \mu J_1/kaJ_1')}. \quad (3)$$

The Bessel functions have an argument $ka = x\iota^{\frac{1}{2}}$, and if V be the velocity of propagation of electromagnetic disturbances in the outer medium,

$$h = n^2 a^2 c / V^2. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

M may be described as the correction for closeness of the wires.

The effective resistance per unit length is given by

$$R = \frac{4\pi u}{x} \cdot \frac{\text{ber } x \text{ bei}' x - \text{bei } x \text{ ber}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} - Nn. \quad (5)$$

Now the functions $\text{ber } x$, $\text{bei } x$ are usually defined by

$$J_0(x\iota^{\frac{1}{2}}) = \text{ber } x + \iota \text{ bei } x,$$

so that

$$J_1(x\iota^{\frac{1}{2}}) = -\iota^{-\frac{1}{2}}(\text{ber}' x + \iota \text{ bei}' x),$$

$$J_1'(x\iota^{\frac{1}{2}}) = -\iota(\text{ber}'' x + \iota \text{ bei}'' x),$$

and

$$\frac{\mu}{ka} \cdot \frac{J_1}{J_1'} = \frac{\mu}{x} \cdot \frac{\text{ber}' x + \iota \text{ bei}' x}{\text{ber}'' x + \iota \text{ bei}'' x},$$

whence on reduction

$$(1 - \mu J_1/kaJ_1')/(1 + \mu J_1/kaJ_1') = (E + \iota F)/D, \quad (6)$$

where

$$\left. \begin{aligned} Ex^2 &= (x \text{ber}'' x)^2 + (x \text{bei}'' x)^2 - (\mu \text{ber}' x)^2 - (\mu \text{bei}' x)^2, \\ Fx &= 2\mu(\text{ber}' x \text{bei}'' x - \text{bei}' x \text{ber}'' x), \\ Dx^2 &= (x \text{ber}'' x)^2 + (x \text{bei}'' x)^2 + (\mu \text{ber}' x)^2 + (\mu \text{bei}' x)^2 \\ &\quad + 2\mu x(\text{ber}' x \text{ber}'' x + \text{bei}' x \text{bei}'' x). \end{aligned} \right\} \quad (7)$$

But if accents denote differentiations with respect to x ,

$$y = \text{ber } x + \iota \text{ ber } x$$

is a solution of

$$xy'' + y' = \omega x'',$$

so that

$$x \text{ber}'' x + \omega \text{bei}'' x + \text{ber}' x + \iota \text{bei}' x = \omega (\text{ber } x + \iota \text{bei } x),$$

and therefore

$$\left. \begin{aligned} x \text{ber}'' x &= -\text{ber}' x - x \text{bei } x, \\ x \text{bei}'' x &= x \text{ber } x - \text{bei}' x, \end{aligned} \right\} \dots \dots (8)$$

from which may be deduced

$$\left. \begin{aligned} \text{ber}' x \text{bei}'' x - \text{bei}' x \text{ber}'' x &= \text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x, \\ \text{ber}' x \text{ber}'' x + \text{bei}' x \text{bei}'' x &= \text{ber } x \text{bei}' x - \text{bei } x \text{ber}' x \\ &\quad - x^{-1}(\text{ber}' x)^2 - x^{-1}(\text{bei}' x)^2, \\ (x \text{ber}'' x)^2 + (x \text{bei}'' x)^2 &= (\text{ber}' x)^2 + (\text{bei}' x)^2 + (x \text{ber } x)^2 \\ &\quad + (x \text{bei } x)^2 - 2x(\text{ber } x \text{bei}' x - \text{bei } x \text{ber}' x). \end{aligned} \right\} \dots (9)$$

Writing

$$\left. \begin{aligned} P &= \text{ber}^2 x + \text{bei}^2 x, \\ \alpha P &= (\text{ber}' x)^2 + (\text{bei}' x)^2, \\ \beta P &= \text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x, \\ \gamma P &= \text{ber } x \text{bei}' x - \text{bei } x \text{ber}' x. \end{aligned} \right\} \dots \dots (10)$$

Then

$$Ex^2/P = \alpha(1 - \mu^2) + x^2 - 2\gamma x,$$

$$Fx/P = 2\mu\beta,$$

$$Dx^2/P = \alpha(1 - \mu)^2 + x^2 - 2x\gamma(1 - \mu),$$

where

$$D\lambda_1 = E + \iota F. \dots \dots (11)$$

For the special case $\mu = 1$; $D = P$, so that

$$\lambda_1 = 1 - 2(\gamma - \iota\beta)/x, \dots \dots (12)$$

a very simple result, which is true in practice for all but iron wires. Again, on reduction,

$$\log a/c - \mu J_0/kaJ_0' = \log a/c - \mu(\beta - \iota\gamma)/\alpha x,$$

$$\log ha - \mu J_0/haJ_0' = \log ha - \mu(\beta - \iota\gamma)/\alpha x,$$

and their quotient becomes

$$(A + \iota B)/C,$$

where

$$\left. \begin{aligned} x^2 A &= \alpha x^2 \log ha \cdot \log a/c - \mu \beta x \log ha^2/c + \mu^2, \\ x B &= \mu \gamma \log hc, \\ x^2 C &= \alpha x^2 \log^2 ha - 2\mu \beta x \log ha + \mu^2, \end{aligned} \right\}. \quad (13)$$

and it has been noticed that $\beta^2 + \gamma^2 = \alpha$ identically.

With these values, it is found that, M and N being given by (3),

$$c^2 CD(M + \iota N) = 4a^2(E + \iota F)(A + \iota B).$$

Thus

$$\left. \begin{aligned} M &= 4a^2(AE - BF)/c^2 CD, \\ N &= 4a^2(AF + BE)/c^2 CD. \end{aligned} \right\}. \quad (14)$$

Moreover,

$$\frac{CD}{P} = \left(\alpha \log^2 ha - \frac{2\mu\beta}{x} \log ha + \frac{\mu^2}{x^2} \right) \left(\frac{\alpha}{x^2} \overline{1 - \mu^2} + 1 - \frac{2\gamma}{x} \overline{1 - \mu} \right). \quad (15)$$

$$\begin{aligned} \frac{AE - BF}{P} &= \left(\alpha \log \frac{a}{c} \log ha - \frac{\mu\beta}{x} \log \frac{ha^2}{c} + \frac{\mu^2}{x^2} \right) \left(\frac{\alpha - \alpha\mu^2}{x^2} \right. \\ &\quad \left. + 1 - \frac{2\gamma}{x} \right) - \frac{2\mu^2\beta\gamma}{x^2} \log hc. \quad (16) \end{aligned}$$

The succeeding reduction depends upon the particular circumstances of the wires. We now write $\mu=1$, and suppose that the wires are of copper. The only case needing separate treatment is that of iron, where μ is very large.

Inductance when $\mu=1$. High frequency.

Quoting the value of M in (14), the inductance becomes, when $\mu=1$,

$$L = 4 \log_e \frac{c}{a} + \frac{4\beta}{\alpha x} + \frac{4a^2(AE - BF)}{c^2 CD}, \quad (17)$$

where

$$\left. \begin{aligned} \frac{AE - BF}{P} &= \left(\alpha \log \frac{a}{c} \log ha - \frac{\beta}{x} \log \frac{ha^2}{c} + \frac{1}{x^2} \right) \\ &\quad \left(1 - \frac{2\gamma}{x} \right) - \frac{2\beta\gamma}{x^2} \log hc, \\ \frac{CD}{P} &= \alpha \log^2 ha - \frac{2\beta}{x} \log ha + \frac{1}{x^2}. \end{aligned} \right\}. \quad (18)$$

Asymptotic formulæ for the functions (α, β, γ) have been developed by Dr. A. Russell*.

* Proc. Phys. Soc. vol. xxii.

When the argument x is not less than 8, then to four significant figures, if $\lambda = x\sqrt{2}$ *, so that, in terms of the characteristics of the two wires,

$$\lambda = 2a \left(\frac{2\pi n}{\sigma} \right)^{\frac{1}{2}}, \dots \dots \dots (19)$$

we may write

$$\left. \begin{aligned} \alpha &= 1 - \frac{1}{\lambda} + \frac{1}{2\lambda^2} + \frac{3}{4\lambda^3}, \\ \beta \sqrt{2} &= 1 - \frac{1}{\lambda} - \frac{1}{4\lambda^2}, \\ \gamma \sqrt{2} &= 1 + \frac{1}{4\lambda^2} + \frac{1}{2\lambda^3}, \end{aligned} \right\} \dots \dots (20)$$

and after reduction

$$\left. \begin{aligned} \frac{AE-BF}{P} &= \left(1 - \frac{3}{\lambda} + \frac{5}{2\lambda^2} - \frac{3}{4\lambda^3} \right) \log \frac{a}{c} \log ha \\ &\quad - \frac{2}{\lambda} \left(1 - \frac{3}{\lambda} - \frac{1}{4\lambda^2} \right) \log ha + \frac{1}{\lambda} \left(1 - \frac{5}{\lambda} \right. \\ &\quad \left. + \frac{15}{4\lambda^2} \right) \log hc + \frac{2}{\lambda^2} \left(1 - \frac{2}{\lambda} \right), \\ \frac{CD}{P} &= \left(1 - \frac{1}{\lambda} + \frac{1}{2\lambda^2} + \frac{3}{4\lambda^3} \right) \log^2 ha - \frac{2}{\lambda} \left(1 - \right. \\ &\quad \left. \frac{1}{\lambda} - \frac{1}{4\lambda^2} \right) \log ha + \frac{2}{\lambda^2}. \end{aligned} \right\} \dots (21)$$

We proceed to examine the limiting frequency for which the four-figure accuracy of these results is preserved. For a copper wire of low resistance as ordinarily used, the resistivity may be taken as 1696 C.G.S. If f be the frequency of alternation, $n = 2\pi f$, leading to $x = \frac{1}{5} af^{\frac{1}{2}}$ approximately.

Thus $x = 8$ leads to the determining relation

$$af^{\frac{1}{2}} = 40. \dots \dots \dots (22)$$

If $af^{\frac{1}{2}}$ is less than 40, the four-figure accuracy, in so far as it depends on x , is lost. When f is only 400 per second, the limiting diameter of a wire becomes as large as 4 centimetres, so that the formulæ are unsuited for practice until

* This modification is more convenient for purposes of printing.

the frequency is really great. The usual range of diameter for wires applied to such purposes as telephonic communication is from 1 to 3 millimetres in the case of cables, and to 5.7 millimetres in that of overhead wires. The frequency in cases of transmission of speech will not be greater than 2000. Taking $f=1600$ as perhaps the real maximum, the limiting radius of a wire becomes 1 centimetre. The results therefore cannot have a great accuracy in such a case, and the formula (43) below must be used. But it will appear that the corresponding formula for iron leads can be so employed.

The present results have a two-figure accuracy even when $x=3$, leading to a radius of 3.75 millimetres, and therefore will apply to many cases even of telephony when thicker wires are used. In such cases, their use is preferable to that of (43), in that the calculation is more rapid.

It is necessary not to overlook the other sources of error. Firstly, in the proof of (1)*, h^2a^2 was neglected in comparison with unity. In the most unfavourable case furnished by the telephone, $f=1600$, $a=3$ millimetres, leading to $h^2a^2=10^{-14}$ approximately.

Thus this source of error needs no further consideration in such a case. But in fact, the neglect of this quantity is always legitimate. For if f be the frequency, $h^2a^2=4f^2 10^{-20}$ for a wire of a centimetre radius. It can only therefore affect the fourth figure if $f=20$ million, and the second if $f=200$ million, and such a frequency is never used in combination with so thick a wire.

Secondly, a^4/c^4 was neglected in comparison with unity. Now even in ordinary telephone construction there is usually about $2\frac{1}{2}$ millimetres of paper and air between the wires, so that in the case of the cable, taking an extreme radius of 1.3 millimetres, $c=4a$, and only the third figure is affected. For overhead wires, the distance apart is about 30 centimetres.

The limiting practical closeness of the wires, when a knowledge of the self-induction is required, is, I believe, attained in Mr. A. Campbell's experiments on variable mutual inductances†. The error may then be so great as one or two per cent.

* Phil. Mag. Feb. 1909.

† Phil. Mag. Jan. 1908.

Effect of a Small Capacity.

Heaviside * has divided circuits into five classes, with the following determining properties:—

(1) Submarine cables proper, in which the capacity is the main factor, and whose treatment must follow the electrostatic theory.

(2) Short lines of low frequency, in which self-induction and resistance determine the effects. This class includes the majority of short telephone circuits.

(3) A class in which capacity and self-induction are equally important. These circuits are very difficult in theory.

(4) Circuits of high frequency, but small capacity and resistance, and with the inductance so chosen that signals may travel without distortion. For this class the electrostatic theory cannot be applied however long the circuit.

(5) Circuits in which distortionless propagation is obtained by allowing an electric leakage.

The effect of leakage is small in all but the fifth case, and will be ignored.

Let an impressed force Ee^{nt} act at one end of a circuit consisting of two parallel wires, with terminal apparatus whose inductance, capacity, and resistance are neglected. Let z denote distance of a point P from the end at which the force acts. If R , L , C be the resistance, inductance,

and capacity per unit length, and (ι_1, V) the current and potential at a point P; then

$$\left. \begin{aligned} C \partial V / \partial t &= -\partial \iota_1 / \partial z \\ L \partial \iota_1 / \partial t + R \iota_1 &= -\partial V / \partial z \end{aligned} \right\}, \quad \cdot \cdot \quad (23)$$

where

$$\partial / \partial t = n.$$

The solution of these equations is found to be

$$\iota_1 = (E/I)e^{\iota(n t - \alpha)}, \quad \cdot \cdot \cdot \quad (24)$$

* 'Electrical Papers,' vol. ii.

where

$$\left. \begin{aligned} I &= L^{\frac{1}{2}} C^{-\frac{1}{2}} (1 + R^2/n^2 L^2)^{\frac{1}{2}} (\cosh 2lQ_1 - \cos 2lQ_2)^{\frac{1}{2}} \\ Q_1, Q_2 &= n(\frac{1}{2}LC)^{\frac{1}{2}} \{ (1 + R^2/n^2 L^2)^{\frac{1}{2}} \mp 1 \}^{\frac{1}{2}}. \end{aligned} \right\} \quad (25)$$

With the value of α we are not concerned. For the effect of leakage, reference may be made to Heaviside's paper*.

$$\text{If } C = 0, \quad I = l(R^2 + L^2 n^2)^{\frac{1}{2}}, \quad . \quad . \quad . \quad (26)$$

which is the ordinary impedance assumed to be valid in the previous paper. In the electrostatic theory, we write $L=0$, which is obviously unjustifiable if R is small or n very great, so that circuits of the first kind lose their character in these cases.

We proceed to estimate the correction to be made in the value of L , to take account of a small capacity, regarding only the first order term. Writing

$$\lambda_1^2, \lambda_2^2 = (1 + R^2/n^2 L^2)^{\frac{1}{2}} \mp 1,$$

then on reduction,

$$\begin{aligned} \cosh 2lQ_1 - \cos 2lQ_2 &= LCn^2 l^2 (\lambda_1^2 + \lambda_2^2) \{ 1 + \frac{1}{6} LCn^2 l^2 (\lambda_1^2 - \lambda_2^2) \} \\ &= 2Cn l^2 (R^2 + L^2 n^2)^{\frac{1}{2}} (1 - \frac{1}{3} l^2 n^2 LC), \end{aligned}$$

so that

$$I = l\sqrt{2} \cdot (R^2 + L^2 n^2)^{\frac{1}{2}} (1 - \frac{1}{6} l^2 n^2 LC).$$

If L' be the equivalent induction when this small capacity is taken into account,

$$R^2 + L'^2 n^2 = (R^2 + L^2 n^2) (1 - \frac{1}{3} l^2 n^2 LC)$$

$$\text{or } (L' - L)/L = -l^2 C(R^2 + L^2 n^2)/6L \text{ approximately.} \quad (27)$$

This equation serves to limit the types of circuit to which the uncorrected inductance formulæ may be applied, in so far as error due to capacity is concerned. For example, if a three-figure accuracy is required, it is necessary that

$$l^2 C(R^2 + L^2 n^2)/6L \nless 10^{-4}, \quad . \quad . \quad . \quad (28)$$

where l is the length of either wire of the circuit.

* *Loc. cit.*

We have also neglected

$$\frac{1}{90} l^4 n^4 L^2 C^2 (\lambda_1^4 + \lambda_2^4 - \lambda_1^2 \lambda_2^2) = \frac{1}{90} l^4 n^4 L^2 C^2 (4 + R^2 / l^2 n^2)$$

in comparison with unity. This restricts the capacity for which the correction (27) may be used. Thus, for low resistance,

$$4l^4 n^2 C^2 \gtrsim 9 \cdot 10^{-5}.$$

This equation will ordinarily supply the lower limit for C. Thus for frequencies such that

$$nC \gtrsim (5l^2)^{-1} 10^{-2} \quad . \quad . \quad . \quad (29)$$

the uncorrected (for capacity) formulæ may be used provided that C is also so small as to make

$$Cn^2 L \gtrsim 6l^{-2} 10^{-4}, \quad . \quad . \quad . \quad (30)$$

again neglecting R.

For a capacity of one microfarad per kilometre, $C = 10^{-20}$. In a case like that of the Atlantic cable the capacity is about a quarter of this amount, and the limiting frequency, with l in centimetres, is about $l^{-2} 10^{18}$. This excludes such a cable altogether from the investigation, although a shorter cable of the same radius of wire and capacity gradient can be within its scope. For example, a frequency of so much as 10 million can be treated if l is not greater than 1.2 kilometres, for a suitable range of inductance given by (30). The range of inductance is of course dependent mainly on the distance between the wires.

Obviously all short telephone circuits satisfy both (29) and (30), and their capacity needs no consideration.

Copper Wires with High Frequency.

With the above restrictions as regards capacity and frequency (the latter being of little practical import), writing in (17),

$$\begin{aligned} \frac{\beta}{\alpha x} &= \frac{1}{\lambda} \left(1 - \frac{1}{\lambda} - \frac{1}{4\lambda^2} \right) \left(1 - \frac{1}{\lambda} + \frac{1}{2\lambda^2} \right) \\ &= \frac{1}{\lambda} \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) \quad . \quad . \quad . \quad (31) \end{aligned}$$

Then with the values of (21),

$$L = 4 \log_e \frac{c}{a} + \frac{4}{\lambda} \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) + \frac{4a^2}{c^2 CD} (AE - BF),$$

where $\lambda = 2a(2\pi\mu n/\sigma)^{\frac{1}{2}} = 4\pi a(8/\sigma)^{\frac{1}{2}},$

with a four-figure accuracy so far as λ is concerned, if $\lambda > 8\sqrt{2}$. But a further simplification may be introduced. Since ha is a very small magnitude in actual cases, its logarithm is large and negative. Thus if $\rho = \log_e ha$, the functions may be expanded in a descending series of powers of ρ , and

$$\frac{CD}{P\rho^2} = 1 - \frac{1}{\lambda} \left(1 + \frac{2}{\rho} \right) + \frac{1}{\lambda^2} \left(\frac{1}{2} + \frac{2}{\rho} + \frac{2}{\rho^2} \right) + \frac{1}{\lambda^3} \left(\frac{3}{4} + \frac{1}{2\rho} \right)$$

or

$$\frac{P\rho^2}{CD} = 1 + \frac{1}{\lambda} \left(1 + \frac{2}{\rho} \right) + \frac{1}{\lambda^2} \left(\frac{1}{2} + \frac{2}{\rho} + \frac{2}{\rho^2} \right) - \frac{1}{\lambda^3} \left(\frac{3}{4} + \frac{1}{2\rho} \right), \quad (32)$$

whence, after considerable reduction

$$\begin{aligned} \frac{\rho^2}{CD} (AE - BF) &= \rho \log_e \frac{a}{c} - \frac{1}{\lambda} \left(\rho + 4\rho \log_e \frac{a}{c} + 3 \log_e \frac{a}{c} \right) \\ &\quad + \frac{1}{4\rho^2\lambda^3} \left\{ 8\rho - 8\rho^2 + 19\rho^3 - (8 - 8\rho + 3\rho^2) \log_e \frac{a}{c} \right\}. \end{aligned}$$

The vanishing of the coefficient of λ^{-2} is curious.

Finally, for a pair of copper wires, the main error being of relative magnitude a^4/c^4 when they are close together, if $\lambda < 8\sqrt{2}$,

$$\lambda = 4\pi a (f/\sigma)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (33)$$

if f is the frequency, then

$$\begin{aligned} L &= 4 \log_e \frac{c}{a} + \frac{4}{\lambda} \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) + \frac{4a^2}{c^2} \frac{\log_e \frac{a}{c}}{\log_e ha} \\ &\quad - \frac{4a^2}{c^2\lambda\rho^2} \left(\rho + 4\rho \log_e \frac{a}{c} + 3 \log_e \frac{a}{c} \right) \\ &\quad + \frac{a^2}{c^2\lambda^3\rho^4} \left\{ 8\rho - 8\rho^2 + 19\rho^3 - (8 - 8\rho + 3\rho^2) \log_e \frac{a}{c} \right\}, \quad (34) \end{aligned}$$

where $\rho = \log_e ha$, and the limiting capacity is given by (29). For most useful purposes it may be greatly shortened, according to the value of λ .

Iron Wires with High Frequency.

When iron is used, there is no object to be served in obtaining so accurate a formula, on account of the vagaries in the magnetic behaviour of the metal. But the approximate value of μ is known, for Lord Rayleigh* has shown that for feeble magnetizing forces of periodic character, μ usually lies between 90 and 100, and becomes much greater as the forces are increased. It is therefore lawful to take μ as of order 10^2 . In the formulæ (20), (α, β, γ) never become large, and therefore whatever the value of x (or λ), the approximation may be conducted by expanding the functions in inverse powers of μ .

Thus if $\rho = \log ha$ as before,

$$\begin{aligned} \frac{CD}{P} &= \frac{\alpha\mu^4}{x^4} \left(1 - 2\beta\rho\frac{x}{\mu} + \alpha\rho^2\frac{x^2}{\mu^2}\right) \left(1 - \frac{2}{\mu}\left(1 - \frac{\gamma x}{\alpha}\right) + \frac{1}{\mu^2}\left(1 - \frac{2\gamma x}{\alpha} + \frac{x^2}{\alpha^2}\right)\right) \\ &= \frac{\mu^4\alpha}{x^4} \left\{1 - \frac{2}{\mu}\left(\beta\rho x + 1 - \frac{\gamma x}{\alpha}\right) + \frac{1}{\mu^2}\left(1 + \alpha\rho^2 x^2 - \frac{2\gamma x}{\alpha}\right.\right. \\ &\quad \left.\left.+ \frac{x^2}{\alpha} + 4\beta\rho x - 4\beta\gamma\rho\frac{x^2}{\alpha}\right)\right\} \\ \text{or} \\ \frac{P\mu^4\alpha}{x^4 CD} &= 1 + \frac{2}{\mu}\left(1 + \beta\rho x - \frac{\gamma x}{\alpha}\right) + \frac{1}{\mu^2}\left(3 + 4\beta^2\rho^2 x^2 + 4\beta\rho x + \frac{4\gamma^2 x^2}{\alpha^2}\right. \\ &\quad \left.- 4\beta\gamma\rho\frac{x^2}{\alpha} - \frac{x^2}{\alpha} - \frac{6\gamma x}{\alpha} - \alpha\rho^2 x^2\right). \quad (35) \end{aligned}$$

This neglects the ratio x^3/μ^3 . Taking a wire of gauge $\frac{1}{2}$ cm., an approximate practical limit for overhead wires, and writing, as an ordinary value for iron, $\sigma = 10,000$ c.g.s.,

$$x = \frac{3}{20}f^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

if $\mu = 100$. Thus even when $f = 10,000$ per second, neglect of x^3/μ^3 only affects the third figure of CD , and therefore at most the fourth of L , when the ratio a^2/c^2 is noticed. This process is justifiable whether the ascending or descending series are used for (α, β, γ) .

* Scientific Papers, iii.

In a similar manner,

$$-\frac{(\text{AE}-\text{BF})}{\text{P}} = + \frac{\alpha\mu^4}{x^4} \left(1 - \frac{\beta x}{\mu} \log \frac{ha^2}{c} + \frac{\alpha x^2}{\mu^2} \log ha \log \frac{a}{c} \right) \left(1 - \frac{\alpha + x^2 - 2\gamma x}{\alpha\mu^2} \right) \\ + 2\mu^2 \frac{\beta\gamma}{x^2} \log hc ;$$

and after some reduction,

$$-\frac{\text{AE}-\text{BF}}{\text{CD}} = 1 + \frac{1}{\mu} \left(2 + \beta\rho x - \frac{2\gamma x}{\alpha} - \beta x \log \frac{a}{c} \right) \\ + \frac{1}{\mu^2} \left(2 + 2\beta\rho x - \alpha\rho^2 x^2 + 2\beta^2\rho^2 x^2 + \frac{4\gamma^2 x^2}{\alpha^2} - \frac{4\gamma x}{\alpha} - \frac{2x^2}{\alpha^2} \right) \\ + \frac{1}{\mu^2} \log \frac{a}{c} \cdot (\alpha\rho^2\rho - 2\beta^2 x^2\rho - 2\beta x).$$

In the extreme case above, $\frac{x}{\mu^2} \gg \frac{1}{600}$, and the final error consequent upon its neglect is of order $(6000)^{-1}$ relative to unity even when the wires have limiting practical closeness.

Accordingly,

$$-\left(\frac{\text{AE}-\text{BF}}{\text{CD}}\right) = 1 + \frac{1}{\mu} \left(2 + \beta\rho x - \frac{2\gamma x}{\alpha} - \beta x \log \frac{a}{c} \right) \\ + \frac{1}{\mu^2} \left(2\beta^2\rho^2 x^2 - \alpha\rho^2 x^2 + \frac{4\gamma^2}{\alpha^2} x^2 - \frac{2x^2}{\alpha} + x^2\rho(\alpha - 2\beta^2) \log \frac{a}{c} \right).$$

Consider now the case in which $x > 8$. In the coefficient of μ^{-2} , the first order values of $(\alpha\beta\gamma)$ may be used, with the second order in that of μ^{-1} .

Under these conditions, the coefficient of μ^{-2} is found to vanish, leading to the very simple result, to order μ^{-3} ,

$$-\left(\frac{\text{AE}-\text{BF}}{\text{CD}}\right) = 1 + \frac{1}{\mu} \left\{ 2 + \frac{1}{2}\lambda \left(1 - \frac{1}{\lambda} \right) \log hc - \lambda \left(1 + \frac{1}{\lambda} \right) \right\} \\ = 1 + (\lambda - 1)(\log hc - 2)/2\mu,$$

and thus for iron wires with $\lambda > 8\sqrt{2}$,

$$\text{L} = 4 \log_e \frac{c}{a} + \frac{4\mu}{\lambda} \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) - \frac{4a^2}{c^2} - \frac{2a^2}{c^2\mu} (\lambda - 1)(\log hc - 2), \quad (37)$$

where

$$\lambda = 4\pi a \left(\frac{f\mu}{\sigma} \right)^{\frac{1}{2}}, \quad \text{and} \quad \frac{a^2}{c^2} \frac{\lambda}{\mu^2} \text{ is neglected.}$$

On account of μ , this formula may be applied, for any given frequency, to a wire of only about a quarter the corresponding limiting radius in the case of copper. For example, for a frequency 800, and radius 2 millimetres, the value of x becomes 5, with $\mu=100$. Thus a three-figure accuracy is obtained for the most unfavourable case of the telephone circuit. Accordingly, for a telephone circuit of iron wires not twisted together, this result is of universal application.

Iron Wires with Small Frequency.

This case is not very important, but the result may be given at once. Thus if x is not greater than 2, we may neglect the term involving μ^{-2} , and write, to three significant figures,

$$L = 4 \log \frac{c}{a} + \frac{4\mu\beta}{\alpha x} - \frac{4a^2}{c^2} - \frac{4a^2}{c^2\mu} \left(\beta x \log hc + 2 - \frac{2\gamma x}{\alpha} \right). \quad (38)$$

The values of (α, β, γ) suited to this case have been given by Russell*.

$$\text{If} \quad z = \frac{x}{2} = a\pi \left(\frac{2\mu f}{\sigma} \right)^{\frac{1}{2}},$$

$$\left. \begin{aligned} \alpha &= z^2 \left(1 - \frac{5}{12} z^4 + \frac{143}{720} z^8 - \frac{7661}{4^2 \cdot 7!} z^{12} \right) \\ \beta &= \frac{1}{2} z^3 \left(1 - \frac{11}{24} z^4 + \frac{473}{3 \cdot 6!} z^8 - \frac{304107}{4 \cdot 12^2 \cdot 7!} z^{12} \right) \\ \gamma &= z \left(1 - \frac{1}{3} z^4 + \frac{19}{5!} z^8 - \frac{687}{7 \cdot 6^4} z^{12} \right) \end{aligned} \right\}. \quad (39)$$

Moreover,

$$\left. \begin{aligned} \frac{\beta}{\alpha} &= \frac{1}{2} z \left(1 - \frac{1}{24} z^4 + \frac{13}{4320} z^8 - \frac{647}{12^2 \cdot 360 \cdot 56} z^{12} \right) \\ \frac{\gamma}{\alpha} &= z^{-1} \left(1 + \frac{1}{12} z^4 - \frac{1}{180} z^8 + \frac{11}{12 \cdot 28 \cdot 30} z^{12} \right) \end{aligned} \right\} \quad (40)$$

* *Loc. cit.*

Heaviside* has also given the value of β/α , and Lord Rayleigh† that of γ/α , by different modes of proof. Thus finally, when

$$z = a\pi(2\mu f/\sigma)^{\frac{1}{2}} \gg 1, \quad (41)$$

$$\begin{aligned} L = 4 \log \frac{c}{a} + \mu \left(1 - \frac{1}{24} z^4 + \frac{13}{4320} z^8 - \frac{647}{12^2 \cdot 360 \cdot 56} z^{12} \right) - \frac{4a^2}{c^2} \\ - \frac{4z^4 a^2}{c^2 \mu} \log_e hc \left\{ 1 - \frac{11}{24} z^4 + \frac{473}{3 \cdot 6!} z^8 - \frac{304107}{4 \cdot 12^2 \cdot 7!} z^{12} \right\} \\ + \frac{8a^2}{c^2 \mu} \left(1 + \frac{1}{6} z^4 + \frac{1}{90} z^8 - \frac{11}{28 \cdot 180} z^{12} \right). \quad (42) \end{aligned}$$

Small Frequency, Copper Wires.

In the notation of (18), with $\rho = \log_e ha$,

$$x^2 \frac{CD}{P} = 1 - 2\beta x \rho + \alpha x^2 \rho^2.$$

The frequencies for which this formula will ordinarily be required lie between 200 and 1600, and the radii between 1 and 3 millimetres. These values cause ρ to vary between about 13 and 16 only, and z between $\frac{1}{4}$ and unity, so that P/CD cannot be expanded in ascending or descending powers of ρ or z in general. The result may be most simply written in the form

$$\begin{aligned} L = 4 \log \frac{c}{a} + \frac{2}{z} \frac{\beta}{\alpha} + \frac{4a^2}{c^2} \left(1 - 2\beta \rho z - \frac{\gamma}{z} \right) / (1 - 4\beta \rho z + 4\alpha \rho^2 z^2) \\ + \frac{8a^2}{c^2} \log \frac{a}{c} \cdot (2\alpha \rho z^2 - 2\alpha \rho \gamma z - \beta z + 2\beta \gamma) / (1 - 4\beta \rho z + 4\alpha \rho^2 z^2), \quad (43) \end{aligned}$$

where $(\alpha \beta \gamma z)$ are given by (39), and $\rho = \log_e ha$.

The two final terms in the last numerator are only about $\frac{1}{50}$ of the other two, and may be ignored.

* Elec. Papers, ii. p. 64.

† Phil. Mag. xxii. p. 381 (1886); Scientific Papers, vol. ii.

II. EFFECTIVE RESISTANCE.

Copper Wires with High Frequency.

The limitations of all the formulæ below, except when otherwise stated, are the same as in the corresponding inductance formulæ. With the previous notation, the effective resistance per unit length of the two wires is

$$\begin{aligned} R &= \frac{4n\mu}{x} \frac{\text{ber } x \text{ bei}' x - \text{bei } x \text{ ber}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} - nN \\ &= \frac{4n\mu}{x} \cdot \frac{\gamma}{\alpha} - \frac{4a^2n}{c^2 \text{CD}} (\text{AF} + \text{BE}). \quad \dots \quad (44) \end{aligned}$$

where CD/P has the value in (15), and on reduction

$$\begin{aligned} \frac{\text{AF} + \text{BE}}{P} &= \frac{2\mu\beta}{x} \left(\alpha \log ha \log \frac{a}{c} - \frac{\mu\beta}{x} \log \frac{ha^2}{c} + \frac{\mu^2}{x^2} \right) \\ &\quad + \frac{\mu\gamma}{x} \log hc \left(1 - \frac{2\gamma}{x} + \frac{\alpha}{x^2} \overline{1 - \mu^2} \right). \quad \dots \quad (45) \end{aligned}$$

When $\mu=1$, and the frequency is high, so that, if $x\sqrt{2}=\lambda$, the formulæ (20) may be employed, then after further reduction,

$$\begin{aligned} \frac{\text{AF} + \text{BE}}{P} &= \frac{2}{\lambda} \log \frac{a}{c} \log ha \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) - \frac{4}{\lambda^2} \log ha \left(1 - \frac{2}{\lambda} \right) \\ &\quad + \frac{1}{\lambda} \log hc \left(1 - \frac{15}{4\lambda^2} \right) + \frac{4}{\lambda^3}, \end{aligned}$$

and with the help of (32), we ultimately obtain

$$\begin{aligned} \frac{\lambda\rho^2}{\text{CD}} (\text{AF} + \text{BE}) &= 2\rho \log \frac{a}{c} + \rho - \log \frac{a}{c} - \frac{1}{\lambda\rho} \left\{ \rho^2 + \log \frac{a}{c} (2 - 3\rho + 2\rho^2) \right\} \\ &\quad + \frac{1}{4\lambda^3\rho^3} \left\{ 8\rho - 8\rho^2 + 3\rho^3 - \log \frac{a}{c} (8 - 8\rho + 3\rho^2 + 2\rho^3) \right\} \end{aligned}$$

where $\rho = \log_e ha$.

Finally, for a pair of copper wires, with high frequency, and with the limitations of (34),

$$\begin{aligned} R &= \frac{4n}{\lambda} \left(1 + \frac{1}{\lambda} + \frac{3}{4\lambda^2} \right) - \frac{4a^2n}{\lambda c^2 \rho^2} \left\{ \rho + (2\rho - 1) \log \frac{a}{c} \right\} \\ &\quad + \frac{4a^2n}{\lambda^2 c^2 \rho^3} \left\{ \rho^2 + (2 - 3\rho + 2\rho^2) \log \frac{a}{c} \right\} \\ &\quad - \frac{a^2n}{\lambda^3 c^2 \rho^4} \left\{ 8\rho - 8\rho^2 + 3\rho^3 - (8 - 8\rho + 3\rho^2 + 2\rho^3) \log \frac{a}{c} \right\}. \quad (46) \end{aligned}$$

Iron Wires with High Frequency.

When μ is large, we may write from (45),

$$\frac{AF + BE}{P} = \frac{\mu^3}{x^3} \left(2\beta - \alpha\gamma \log hc - 2\beta^2 \frac{x}{\mu} \log \frac{ha^2}{c} \right), \quad (47)$$

and by (35),

$$\frac{P\mu^4\alpha}{x^4CD} = 1 + \frac{2}{\mu} \left(1 + \beta\rho x - \gamma \frac{x}{\alpha} \right),$$

so that

$$\begin{aligned} \frac{AF + BE}{CD} &= \left(\frac{2\beta}{\alpha} - \gamma \log hc \right) \\ &\quad - \frac{2x}{\alpha\mu^2} \left\{ \left(1 + \beta\rho x - \gamma \frac{x}{\alpha} \right) (2\beta - \alpha\gamma \log hc) + \beta^2 x \log \frac{ha^2}{c} \right\}. \end{aligned}$$

Neglecting x/μ^2 as before, and writing the first approximations to $(\alpha\beta\gamma)$ in the second term, and the second in the first, we finally obtain

$$\frac{AF + BE}{CD} = \frac{\lambda}{2\mu} \left(2 - \rho + \log \frac{a}{c} \right) - \frac{\lambda^2}{2\mu^2} \left(4\rho - 2 - \rho^2 + \rho \log \frac{a}{c} \right),$$

the term in $\lambda^0\mu^{-1}$ vanishing identically.

Thus the effective resistance becomes

$$\begin{aligned} R &= \frac{4n\mu}{\lambda} \left(1 + \frac{1}{\lambda} + \frac{3}{4\lambda^2} \right) - \frac{2a^2n\lambda}{c^2\mu} \left(2 - \rho + \log \frac{a}{c} \right) \\ &\quad + \frac{2a^2n\lambda^2}{c^2\mu^2} \left(4\rho - 2 - \rho^2 + \rho \log \frac{a}{c} \right). \quad (48) \end{aligned}$$

This result is much more accurate than the corresponding formula (37) for inductance, owing to the presence of μ in the numerator of the first term. Moreover, λ here denotes the previous λ multiplied by $\mu^{\frac{1}{2}}$. It is applicable to all iron circuits of importance in practice, for a frequency greater than about a hundred with a radius of 1 millimetre.

Iron Wires with Low Frequency.

This result has little interest, but may be obtained at once. For $(\alpha\beta\gamma)$ are all comparable with unity in this case, and x is not greater than 2. We therefore ignore x^2/μ^2 , and obtain

$$\frac{AF + BE}{CD} = \frac{x}{\mu} \left(\frac{2\beta}{\alpha} - \gamma \log hc \right). \quad (49)$$

This term also may be ignored unless a very high order of accuracy be desired, which can rarely happen owing to the uncertainty as to the value of μ . But if it be retained, the resistance becomes

$$R = \frac{2n\mu}{z^2} \left(1 - \frac{1}{12} z^4 - \frac{1}{180} z^6 \right) - \frac{8na^2 z^2}{\mu c^2} \left(1 - \frac{1}{24} z^4 + \frac{13}{4320} z^6 \right) + \frac{8na^2 z^2}{\mu c^2} \left(\rho - \log \frac{a}{c} \right) \left(1 - \frac{1}{3} z^4 + \frac{19}{120} z^6 \right). \quad (50)$$

where z has the value in (39), and the brackets may be shortened except when z is nearly unity.

When the frequency is smaller, this makes $R = 2\sigma/\pi a^2$, the value appropriate to steady currents.

Copper Wires with Low Frequency.

The formula suited to this case is, writing $\beta^2 + \gamma^2 = \alpha$, if $(z \alpha \beta \gamma)$ are defined as in (39), and under the same limitations,

$$R = \frac{2n}{z} \cdot \frac{\gamma}{\alpha} - \frac{4na^2}{zc^2} (\beta - 2\alpha\rho z + 2\gamma\rho z^2)/(1 - 4\beta\rho z + 4\alpha\rho^2 z^2) - \frac{8na^2}{c^2} \log \frac{a}{c} \left(\gamma^2 - \beta^2 - \gamma z + 2\alpha\beta\rho z \right)/(1 - 4\beta\rho z + 4\alpha\rho^2 z^2), \quad (51)$$

again reducing to $2\sigma/\pi a^2$ for steady currents.

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April 21, 1900.