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“Hurst's Triangular Prismatic Formula for Earthwork compared with the Prismoidal Formula.”

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IN this communication references to Hurst's “Handbook of Formulæ, Tables, and Memoranda,” apply to the 12th edition, 1879; and references to Molesworth's “Pocket-Book of useful Formulæ, and Memoranda,” apply to the 19th edition, 1879.

In the first of these Handbooks Hurst's Triangular Prismatic Formula is given at p. 257 as follows:—

“To measure the solidity of earthwork over large areas of irregular depth:—Divide the surface into triangles, and multiply the horizontal area of each by one-third of the sum of the vertical depths taken at the angles, and the result will equal the solidity.”

This is quoted by Molesworth, p. 52, but without the following important note by Hurst:—“The surfaces of the triangles must be true planes, or they must be taken so small as to approximate to planes.”

The well-known Prismoidal Formula is given by Molesworth at p. 46 as follows:—

$$\frac{[\text{Sum of areas of both ends} + (\text{area of middle} \times 4)] \times \text{length}}{6}$$

The convenience and simplicity of Hurst's triangular prismatic formula is incontestable; but it has been looked upon with disfavour, owing to the different results obtained according to the manner in which a quadrangular figure may be divided by a diagonal into two triangles, and to the difference between either of those results and the result obtained by the prismoidal formula. The object of this Paper is to point out the reason of these divergences and to provide a remedy.

Hurst's formula is undoubtedly mathematically correct, provided, as he takes care to explain, the surface of each triangle is a true plane, and any three points of a surface can be connected by a true plane. But in practice, levels being taken for convenience at stated intervals, on parallel lines at stated distances from one

another, the surface to be dealt with is primarily divided into quadrangular figures; and the subsequent division into triangles is purely arbitrary. Of course, if the engineer knew that a line between any two angles would touch the surface all along, that line would represent the proper diagonal to be drawn on the plan, and there would be no reason to further investigate the application of Hurst's method; but, as a rule, he cannot know this.

As, however, except under the rare condition of the whole surface of the quadrangle being a true plane, the result obtained by Hurst's method, if the quadrangle be divided by a diagonal between any two angles, will differ from that obtained by dividing the quadrangle by a diagonal between the other two angles, a mean result, easily arrived at, is a desideratum. Though such a mean may not be based upon a strictly mathematical principle, yet if only a mean, it should be welcome; and since it can be shown that the mean corresponds with the effect of the prismoidal formula, which has received universal acceptance, it should be deemed entitled to general adoption.

The mean is most easily obtained if the area be divided into parallelograms, and it is not often in practice an area would be otherwise divided. In these cases the earthwork (whether filling or excavation) would consist of prisms whose ends are parallelograms. The formula for the solidity, giving the mean of the two ways in which Hurst's formula may be applied, is simply:—

$$(Z) \quad \frac{\text{Sum of the four depths}}{4} \times \text{area.}$$

In the examples which follow, the depths are enclosed within circles, and are taken more divergent from one another than they would generally be in practice, in order to more severely test the applicability of the above formula.

Example 1.—Plan of rectangle, area = 72.

Required the solidity of Fig. 1, depth at each angle being given.

(a) By Hurst's formula, if divided as in Fig. 2:—

$$\text{Solidity} = \left\{ \begin{array}{l} \frac{6 + 18 + 11}{3} \times \frac{72}{2} = 420 \\ + \frac{6 + 11 + 9}{3} \times \frac{72}{2} = 312 \\ \hline \text{Total solidity} = 732 \end{array} \right.$$

(b) By Hurst's formula if divided as in Fig. 3:—

$$\text{Solidity} = \left\{ \begin{array}{l} \frac{9 + 6 + 18}{3} \times \frac{72}{2} = 396 \\ \frac{18 + 11 + 9}{3} \times \frac{72}{2} = 456 \end{array} \right.$$

Total solidity = 852

(c) Mean of a and $b = 792$.

(d) By new formula (quadrangular) Z :—

$$\text{Solidity} = \frac{18 + 11 + 9 + 6}{4} \times 72 = 792, \text{ i.e., the mean of } a \text{ and } b.$$

(e) By prismoidal formula, observing the following directions in Hurst's Handbook, p. 257:—

"The sections are to be taken parallel to each other, and the area of the middle section to be calculated from the mean dimensions, and not by averaging the areas of the ends."

The figure then becomes as follows, the depths (12) and (10) being calculated respectively as the means of $18 + 6$ and $11 + 9$:—

$$\begin{array}{l} \frac{18 + 11}{2} \times 6 = 87 = \text{sectional area at one end,} \\ \frac{6 + 9}{2} \times 6 = 45 = \text{,, ,, other end,} \\ \frac{12 + 10}{2} \times 6 = 66 = \text{,, ,, in middle.} \end{array}$$

$$\begin{array}{r}
 87 \\
 + 45 \\
 + (66 \times 4 =) 264 \\
 \hline
 396 \\
 \times 12 \text{ the length.} \\
 \hline
 6)4752
 \end{array}$$

Solidity = $\frac{792}{1}$ *i.e.*, the same as by the quadrangular formula (Z).

The prismoidal formula might be applied by taking the sections in the other direction (*i.e.* longitudinally), but the result would be precisely the same.

Example 2.—Plan of rhombus, area = 60.

Required the solidity of Fig. 5, depth at each angle being given.

FIG. 5.

(a) By Hurst's formula, if divided as in Fig. 6:—

$$\text{Solidity} = 210 + 190 = 400.$$

FIG. 6.

(b) By Hurst's formula if divided as in Fig. 7:—

$$\text{Solidity} = 220 + 250 = 470.$$

FIG. 7.

(c) Mean of *a* and *b* = 435.

(d) By new quadrangular formula (Z):—

$$\text{Solidity} = \frac{10 + 7 + 8 + 4}{4} \times 60 = 435, \text{ i.e., the mean of } a \text{ and } b.$$

(e) By prismoidal formula, the depths ($8\frac{1}{2}$) and (6) being calculated, the former as the mean of $10 + 7$ and the latter as the mean of $4 + 8$:—

Sectional area at one end = 70 ;

„ „ at other end = 75 ;

„ „ in middle = $72\frac{1}{2}$.

Solidity = 435, i.e., the same as by the quadrangular formula (Z).

The same result would have been obtained if the sectional lines had been taken as 6 long instead of 10, and the length of the figure as 10 instead of 6.

When the area of the surface is a trapezoid, not often occurring in practice, a somewhat different method has to be adopted.

Taking the two parallel sides of the trapezoid, as the respective bases of the two triangles into which the trapezoid may have been divided, the formula, to obtain a mean result, will be as follows :—

$$(Y) \text{ Solidity} = \frac{\text{Mean of the two depths opposite base} + \text{sum of the two depths at the base}}{3} \times \left\{ \begin{array}{l} \text{area of} \\ \text{triangle} \end{array} \right.$$

The solidity in each triangle must be calculated separately.

Example 3.—Plan of trapezoid, Fig. 9, area of smaller triangle = 24, area of larger triangle = 60.

Required the solidity, the depth at each angle being given.

(a) By Hurst's formula, if divided as shown by the dotted diagonal:—

$$\frac{9 + 11 + 3}{3} \times 24 = 184$$

$$\frac{3 + 7 + 9}{3} \times 60 = 380$$

$$\text{Total solidity} = \underline{\underline{564}}$$

(b) By Hurst's formula, if divided thus:—

FIG. 10.

$$\frac{11 + 9 + 7}{3} \times 24 = 216$$

$$\frac{11 + 3 + 7}{3} \times 60 = 420$$

$$\text{Total solidity} = \underline{\underline{636}}$$

(c) Mean of a and $b = 600$.

(d) By new formula (Y):—

$$\frac{\frac{7 + 3}{2} + 11 + 9}{3} \times 24 = 200$$

$$\frac{\frac{11 + 9}{2} + 3 + 7}{3} \times 60 = 400$$

$$\text{Solidity} = \underline{\underline{600}}, \text{ i.e., the mean of } a \text{ and } b.$$

(e) By prismoidal formula, the depths 7 and 8 being calculated,

FIG. 11.

the former as the mean of $11 + 3$, and the latter as the mean of $9 + 7$:—

Sectional area at one end = 40;

„ „ at other end = 50;

„ „ in middle = $52\frac{1}{2}$.

Solidity = 600, *i.e.*, the same as by the formula (Y).

In the case of a trapezium, no simpler method has been found than to take the mean of the two results, arrived at from the application of Hurst's formula to the two triangles obtained by drawing the diagonal in one direction, and to the two other triangles obtained by drawing the diagonal in the other direction. The trapezium is, however, a figure to be avoided in areas of earthwork; the surface should be divided into parallelograms and triangles, the triangles being formed only when necessitated by the irregular direction of the boundaries. Formula (Z) should be applied to the parallelograms, and Hurst's formula to the triangles.

A trapezium may be divided into a trapezoid and a triangle, an additional level being taken on the ground, at the point where the line drawn parallel to one side of the trapezium from one of the angles cuts the side opposite to the angle. Formula (Y) may be applied to the trapezoid, and Hurst's formula to the triangle.

When a surface is divided into parallelograms of equal area, the application of formula (Z) is much simplified by placing the depths in columns; the depths which belong to only one parallelogram in one column, those which are common to two parallelograms in another, and those which are common to four parallelograms in a third column. The sums of the second and third columns are multiplied by 2 and 4 respectively, and the products added to the sum of the first column, and then multiplied by one-fourth of the area of one parallelogram, give the total solidity.

In calculations for earthwork, it will be found preferable to use the actual reduced levels, or those levels minus an easily deducted whole number, rather than depths, each one of which is a probable source of error, as it involves a more or less complicated subtraction; besides, when the finished surface is to be sloped, a separate calculation is necessary to ascertain the reduced level of the finished surface at each point.

For instance, suppose that the lowest reduced levels of a surface to be excavated are between 80 and 90 on the datum, then if 80 be deducted from each reduced level, and the remainder used as a depth, the result will be the solidity if the surface be excavated down to 80 on datum. The difference between 80 and the reduced level to which the surface is to be finished, if used as a depth in

connection with the whole area, will, in one single calculation, give the contents to be either added to or deducted from the previously ascertained solidity, as the case may require, in order to arrive at the total or net solidity. Even if the finished surface is to be sloped, the mean reduced level will generally be seen almost at a glance, and one calculation for the addition or deduction will be sufficient.

It is obvious that quantities of filling may be arrived at in a similar manner.

Besides the benefit of avoiding a multiplicity of minor calculations afforded by this method, there is also the advantage of all the actual reduced levels used in the computations being preserved and recorded in the measurement book. In the use of depths only, there is no record of the reduced levels.

The Paper is illustrated by sketches in the manuscript from which the eleven Figs. have been prepared.