XV.—On a Test for Continuity. By W. H. Young, Sc.D., F.R.S.
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(MS. received February 20, 1908. Read March 16, 1908.)

§ 1. THE usual method of proving that a function defined as the limit of a sequence of continuous* functions is continuous is by proving that the convergence is uniform. This method may fail owing to the presence of points at which the convergence is non-uniform although the limiting function is continuous.† In such a case it would be necessary to apply a further test, *e.g.* that of Arzelà (“uniform convergence by segments”).

In some cases the continuity may be proved directly by means of a totally different principle, without reference to modes of convergence at all. It is, in fact, a necessary and sufficient condition for the continuity of a function that it should be possible to express it at the same time as the limit of a monotone *ascending* and of a monotone *descending* sequence‡ of continuous functions.

This test may or may not, of course, be an easy one to apply in any particular problem, but in certain cases, both in theory and practice, the application is almost immediate.

The object of the present note is to call attention to the test, and show how it may be applied in a few simple cases.

§ 2. For instance, take the continued fraction

$$\frac{u_1}{v_1 + \frac{u_2}{v_2 + \dots}}$$

where the u 's and the v 's are essentially positive continuous functions.

* Throughout this paper the word “continuous” will be used to mean “bounded and continuous” unless the contrary is stated.

† A point of non-uniform convergence where the limiting function is continuous is what I call an “invisible point of non-uniform convergence.”

‡ $f_1, f_2, \dots$ are said to form a monotone *ascending* sequence if

$$f_1 \leq f_2 \leq f_3 \leq \dots,$$

$f_1, f_2, \dots$ being functions of any number of variables. The theorem on which the test depends is that “the limit of an *ascending* sequence of continuous functions is a *lower* semi-continuous function, and that the limit of a *descending* sequence of continuous functions is an *upper* semi-continuous function. A function which is the limit of both an ascending and a descending sequence is therefore both lower and upper semi-continuous, *i.e.* it is continuous.” It may be added that there are no invisible points of non-uniform convergence, or divergence, in the case of monotone sequences of continuous functions.—W. H. Young, “On Monotone Sequences of Continuous Functions,” *Camb. Phil. Soc. Proc.*, Lent Term, 1908.

The odd convergents form a monotone descending sequence having the continued fraction as limit, so that the latter is an upper semi-continuous function. The even convergents form a monotone ascending sequence having the continued fraction as limit, so that it is lower semi-continuous, and therefore, being also upper semi-continuous, is a continuous function, and this whatever the number of variables on which the u 's and the v 's depend.

§ 3. Again, the u 's having the same meaning as before, the infinite series

$$u_1 - u_2 + u_3 - u_4 + \dots,$$

if the sum at each point is definite, necessarily represents a continuous function when the u 's form a monotone descending sequence. In fact, it is the limit of the ascending sequence of functions constituted by the partial sums of the series

$$(u_1 - u_2) + (u_3 - u_4) + \dots \quad (1)$$

and also of the descending sequence constituted by the partial sums of the series

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots \quad (2)$$

It should be noted that in the case of such a series "term-by-term integration" is allowable. In fact, each of the series (1) and (2) is so integrable, since it is monotone and represents a continuous function. This is therefore the case with the original series.

§ 4. If we omit the condition that the series

$$u_1 - u_2 + \dots$$

should have a definite sum at each point, the monotone ascending sequence of continuous functions $f_1, f_2, \dots$, where

$$f_n = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n}),$$

defines a limiting function f which is $\geq u_1 - u_2$ and is *lower* semi-continuous.

On the other hand, the monotone *descending* sequence of continuous functions $g_1, g_2, \dots$, where

$$g_n = u_1 - (u_2 - u_3) - \dots - (u_{2n-2} - u_{2n-1}),$$

defines a limiting function g which is $\leq u_1$ and is *upper* semi-continuous. Moreover, since

$$f_n \leq g_n, \\ u_1 - u_2 \leq f \leq g \leq u_1 \quad (1)$$

Thus at each point P there are two possible values for the sum of the series, according as we proceed to the limit by taking an even or an odd

The argument of § 3 shows, in fact, that if the series (3) converges, the series (1) represents a continuous bounded function in the closed interval $(0, r)$.

If the series (3) diverge to the value $+\infty$, the series (1) still represents a continuous function $f(x)$ from 0 to r , the latter not included, by the preceding article. At each point $x < r$ we have

$$f(x) = (a_0 - a_1x) + (a_2x^2 - a_3x^3) + \dots \dots \dots \quad (4)$$

while the right-hand side of this equation, being the limit of a monotone increasing sequence of continuous functions, represents a lower semi-continuous function in the closed interval $(0, r)$. The value of this lower semi-continuous function at r being, however, $+\infty$, the function is continuous there, so that the series represents a continuous but unbounded function $f(x)$ throughout the closed interval $(0, r)$.

Similarly, if the series (3) diverge to the value $-\infty$, using the grouping (2) of the preceding article, and the fact that an upper semi-continuous function is continuous where it has the value $-\infty$, we prove the continuity of $f(x)$ in the closed interval $(0, r)$.

Finally, let the series (3) oscillate between the limits of indeterminacy L and U . Then, as in the case when the series (3) had the value $+\infty$, we get the equation (4), whose right-hand side represents a *lower* semi-continuous function in the closed interval $(0, r)$. The value of this function at r , being one of the values the series (3) is capable of assuming, is $\geq L$. But this value is $\leq$ the limits of the values of the lower semi-continuous function in the neighbourhood, that is, is $\leq$ the limits of $f(x)$ as x approaches r . This shows that all such limits are $\geq L$. Similarly, using the grouping employed when the series (3) had the value $-\infty$, we show that all such limits are $\leq U$. Thus all such limits lie between L and U , both included.

§ 7. We now proceed to give two examples, the first constituting another example of the application of § 3; and the second, of the original test of § 1.

Example 1.—Consider the integral

$$U(y) = \int_0^\infty \frac{\sin x}{x+y} dx,$$

where y is any positive quantity. This may be written

$$\int_0^\pi + \int_\pi^{2\pi} + \dots = u_1(y) - u_2(y) + \dots$$

Here the u 's are essentially positive and continuous functions of y and form a monotone descending sequence. Thus, by § 3, $U(y)$ is a continuous

function of y , and its integral may be got by "term-by-term integration," and therefore by integration under the integral sign. Thus, for example,

$$\int_0^\infty \sin x \cdot \log \frac{x+1}{x} dx = \int_0^1 U(y) dy.$$

Example 2.—To prove that the infinite product

$$\cos x \cdot \cos \frac{1}{2}x \cdot \dots \cdot \cos \frac{1}{n}x \cdot \cos \frac{1}{n+1}x \cdot \dots$$

is convergent and defines a bounded continuous function of x for all finite values of x .

Let

$$F_n(x) = \cos x \cdot \cos \frac{1}{2}x \cdot \dots \cdot \cos \frac{1}{n}x,$$

then $F_n(x)$, being the product of n continuous functions, is a continuous function of x for all finite values of x . Also, since the cosine of a quantity lying in the closed interval $(0, \frac{\pi}{2}) \geq 0$ and ≤ 1 , $F_1, F_2, \dots$ form a monotone decreasing sequence, whose limit is therefore an upper semi-continuous function of x in the closed interval $(0, \frac{\pi}{2})$, lying between 0 and 1, both inclusive: this limit is, however, the infinite product in question.

Again, by the product form of $\sin x$, the infinite product

$$f(x) = \left(1 - \frac{x^2}{4}\right) \left(1 - \frac{x^2}{9}\right) \left(1 - \frac{x^2}{16}\right) \cdot \dots = \frac{\sqrt{2}}{\pi x} \sin \frac{\pi x}{\sqrt{2}}$$

is a continuous function of x throughout the closed interval $(0, 1)$, therefore the same is true of each of the functions

$$f_n(x) = \cos x \cdot \cos \frac{1}{2}x \cdot \dots \cdot \cos \frac{1}{n-1}x \cdot \left(1 - \frac{x^2}{n^2}\right) \left(1 - \frac{x^2}{(n+1)^2}\right) \cdot \dots$$

which form a monotone increasing sequence, since

$$\cos y \geq 1 - \frac{1}{2}y^2.$$

The limit of this latter sequence is therefore a lower semi-continuous function of x in the closed interval $(0, \frac{\pi}{2})$; but this limit is the same infinite product as before, and is thus a continuous function of x in the closed interval $(0, \frac{\pi}{2})$, lying between 0 and 1, *a fortiori* between -1 and $+1$. In the interval $(\frac{\pi}{2}, \pi)$ the same argument applies when we omit the

first factor $\cos x$, and it applies in each subsequent interval of the same length, omitting in succession 2, 3, . . . factors of the infinite product. This shows that the product is a continuous function of x for all positive finite values and lies between -1 and $+1$. But the expression is unaltered by changing x into $-x$, so that the same is true for all negative finite values, which proves the required result.

§ 8. The use of the test in theoretical investigations is exemplified by the following proof of the continuity of an improper integral defined in the mode of De la Vallée-Poussin.

For brevity, it will be assumed that the function $f(x)$ to be integrated has a finite lower bound. The argument would require a repeated application if the lower as well as the upper bound were infinite.

Let $M_1, M_2, \dots$ be any set of constantly increasing positive constants having infinity as limit, and let the function $f_n(x)$ differ only from $f(x)$ at the points where $f(x) > M_n$, and at these points $f_n(x) = M_n$.

Assuming the functions $f_n(x)$ to be integrable, and denoting $\int_a^x f_n(x)dx$ by $F_n(x)$, De la Vallée-Poussin defines the improper integral $F(x)$ of $f(x)$ to be the limit of $F_n(x)$, provided this latter limit exist and is finite and independent of the particular sequence $M_1, M_2, \dots$.

Now we know that the proper integral $F_n(x)$ is a continuous function of x , and hence $F_1(x), F_2(x), \dots$ form a monotone increasing sequence of continuous functions, having $F(x)$ as limit, since the functions $f_1(x), f_2(x), \dots$ form a monotone *increasing* sequence having $f(x)$ as limit.

Denote by $F(b)$ the integral from a to b of $f(x)$, and let

$$G_n(x) = F(b) - \int_x^b f_n(x)dx.$$

Then it is plain that the functions $G_1(x), G_2(x), \dots$ form a monotone *decreasing* sequence, having $F(x)$ for limit. Hence $F(x)$ is continuous.

It will be noticed that the argument depends on $F(b)$ being finite. In fact, if we admit integrals with infinite values, the theorem of continuity no longer holds. We still have the first part of the argument holding good, viz. that $F_1(x), F_2(x), \dots$ form a monotone increasing sequence, whence it follows that $F(x)$ is, at a point of discontinuity, lower semi-continuous.

Suppose, for definiteness, that $f(x)$ is a positive function, so that $F(x)$ is a monotone increasing function of x . Then if $F(x)$ is first of all finite, and then becomes infinite, the only way in which this can happen is that up to *and including* some value $x=c$, $F(x)$ is finite, and then suddenly becomes infinite. Moreover, this is the only kind of discontinuity possible, since $F(x)$ cannot jump up to a finite value.

Example.—

$$\begin{aligned} \text{Let } f(x) &= (1-x)^{-1} \quad \text{in the closed interval } (0, 1), \\ \text{and } &= (x-1)^{-2} \quad \text{when } x > 1 \text{ but } \leq 2. \end{aligned}$$

This function, being continuous but not bounded, and always positive, can be generated as the limit of such a sequence of functions as that used by De la Vallée-Poussin. We have, however,

$$\begin{aligned} F(x) &= 2 - 2(1-x)^{\frac{1}{2}} \quad \text{in the closed interval } (0, 1), \\ \text{and } &= +\infty, \quad \text{when } x > 1 \text{ but } \leq 2. \end{aligned}$$

It should be noticed, further, that the argument used in this article depends in no way on the form of the functions devised by De la Vallée-Poussin, but only on the fact that they form a monotone sequence.

§ 9. We now deduce two subsidiary tests which are sometimes useful. The first applies to series and the second to integrals. Both follow immediately from the principle enunciated in § 1. They are analogous to certain known tests for uniform convergence, but more general in form.

Theorem.—*If*

$$u_1 + u_2 + \dots$$

is a series of continuous functions of any number of independent variables which is convergent at all points of a certain region, then it represents a continuous function provided we can find a second series

$$U_1 + U_2 + \dots$$

where the U's are positive continuous functions, whose sum is continuous, and are such that

$$|u_i| \leq U_i$$

for all values of i and each point of the region considered.

For the function in question is the limit of the monotone ascending sequence $f_1, f_2, \dots$, where

$$f_n = u_1 + u_2 + \dots + u_{n-1} - U_n - U_{n+1} - \dots$$

and is evidently continuous.

It is also the limit of the monotone descending sequence $g_1, g_2, \dots$ where

$$g_n = u_1 + u_2 + \dots + u_{n-1} + U_n + U_{n+1} + \dots$$

and is evidently continuous.

Hence the result follows.

Theorem.—*If $u(x, y)$ is a continuous function of the ensemble (x, y) , and $\int_a^x u(x, y) dx$ is convergent for every value of y considered, and if we*

can find a positive continuous function $U(x, y)$ such that $\int_a^\infty U(x, y)dx$ is a continuous function of y , and also

$$|u(x, y)| \leq U(x, y)$$

for all values of $x \geq a$ and each value of y , then $\int_a^\infty u(x, y)dx$ is a continuous function of y .

We have stated and shall prove this theorem for a single variable y ; the argument is, however, quite general, and applies equally when there are n variables y . We may add, it also applies when there are m variables x , the integral being then the m -ple integral, and proper limitations are made with respect to the nature of the infinite region of integration.

The integral in question is, in fact, the limit of the monotone ascending sequence of continuous functions $f_1, f_2, \dots$ where

$$f_n(y) = \int_a^{x_n} u(x, y)dx - \int_{x_n}^\infty U(x, y)dx,$$

$a, x_1, x_2, \dots$ being a previously selected monotone ascending sequence of quantities having $+\infty$ as limit.

The integral is also the limit of the monotone descending sequence of continuous functions $g_1, g_2, \dots$ where

$$g_n(y) = \int_a^{x_n} u(x, y)dx + \int_{x_n}^\infty U(x, y)dx.$$

Hence the result follows.

§ 10. The theorem of § 5 gives us the following obvious modification of the tests of the preceding article for continuity at a point.

Theorem.—If

$$u_1 + u_2 + \dots$$

is a series of functions of any number of variables which are continuous at a point P where the series is convergent, then it represents a (multi-valued) function which is continuous at P , provided we can find a second series

$$U_1 + U_2 + \dots$$

where the U 's are positive functions, which, as well as their sum, are continuous at P , and are such that, at all points of a certain neighbourhood of P ,

$$|u_1| \leq U_1$$

for all values of i .

Theorem.—If $u(x, y)$ is a continuous function of the ensemble (x, y) for the value $y = y_0$, and $\int_a^\infty u(x, y)dx$ is convergent for $y = y_0$, then, if we can find a positive function $U(x, y)$, continuous for $y = y_0$, and such that $\int_a^\infty U(x, y)dx$ is continuous at $y = y_0$, while

$$|u(x, y)| \leq U(x, y)$$

for all values of $x \leq a$ and each value of y in a certain neighbourhood of y , $\int_a^\infty u(x, y) dx$ is continuous at the point $y = y_0$.

As an example of the application of the last test, we see that, $f(x, y)$ being a positive function continuous with respect to the ensemble (x, y) , if $\int_0^\infty f(x, y) dx$ and $\int_0^\infty f(x, y) \sin x \cdot dx$ are both convergent at $y = y_0$, and the former is a continuous function of y at y_0 , so is the latter.

Addendum (received 9th March 1908).

§ 11. The theorem quoted in § 5 (footnote), on which the test given in that article depends, is a particular case of the following more general theorem :—*

“A monotone decreasing (increasing) sequence of functions $f_1, f_2, \dots$ which are upper (lower) semi-continuous at P has for limit a function f which is also upper (lower) semi-continuous at P .”

This latter theorem also may, of course, be used as a test for continuity at a definite point. The necessary and sufficient test for continuity at a point P will then be the possibility of expressing f both as the limit of a monotone decreasing sequence of functions which are upper semi-continuous (or in particular continuous) at P , and of a monotone increasing sequence of functions which are lower semi-continuous (or in particular continuous) at P .

§ 12. It should be noticed that the methods given above may be still further generalised. We have seen that to prove a function continuous throughout an interval, it is sufficient to show that it is the limit both of a monotone increasing and of a monotone decreasing sequence of continuous functions. The theorem quoted in the preceding article shows that *to prove that a function is pointwise discontinuous, it is sufficient to show that it is the limit both of a monotone increasing and of a monotone decreasing sequence of pointwise discontinuous functions.*

The theorem quoted, however, may be taken to apply not only to a fundamental interval, but to any fundamental perfect set; hence, if it is desired to show that a function is pointwise discontinuous with respect not only to an interval, but with respect to every perfect set in that interval, it is sufficient to show that it is the limit both of a monotone increasing and of a monotone decreasing sequence of functions with the same property.

It is a simplification in this case, however, that, in applying the test, we may use semi-continuous functions. To prove that a function is upper (lower) semi-continuous, we only need to show that it is the limit of a single

* *Loc. cit.*; for proof see *Mess. of Math.*, 1908.

monotone sequence of semi-continuous functions, viz. of a monotone decreasing (increasing) sequence of upper (lower) semi-continuous functions. These are particular cases of functions which are pointwise discontinuous with respect to every perfect set, and have, moreover, the advantage that they may be generated as the limits of monotone sequences of *continuous* functions. *To prove that a function is pointwise discontinuous with respect to every perfect set, it follows that it is sufficient to show that it is the limit of (1) a monotone decreasing sequence of lower semi-continuous functions, and (2) of a monotone increasing sequence of upper semi-continuous functions.*

It is interesting to remark that what I call *the upper function* of a sequence of continuous functions (that is, the function which at each point has the value of the highest possible limit approached by the values of the continuous functions there) satisfies the condition (1), while *the lower function* satisfies (2). This may be proved in the manner indicated in Hobson's *Functions of a Real Variable*, p. 552, line 22 seq., where it is shown that the upper derivate, which is a particular case of an upper function, is the limit of a monotone decreasing sequence of functions $w_1, w_2, \dots$ each of which is the limit of a monotone increasing sequence of continuous functions $v_1, v_2, \dots$.

It follows that *an upper (lower) function of a sequence of continuous* functions, and in particular an upper lower derivate, is upper (lower) semi-continuous except at the points of a set of the first category* (viz. the points at which one at least of the semi-continuous functions of the monotone sequence is discontinuous). This is true, moreover, not only with respect to the continuum, but with respect to every perfect set.

This, again, gives us a new proof of Baire's theorem, that (in the case when the upper and lower functions coincide) the limit of a sequence of continuous functions is pointwise discontinuous with respect to every perfect set. The converse, proved also by Baire, viz. that any function which is pointwise discontinuous with respect to every perfect set is the limit of a continuous function, shows that the conditions above given are not only sufficient but necessary for the function in question to have the required property.

* It may be added that this result can be still further generalised. The general result is that the upper (lower) function of a sequence of lower (upper) semi-continuous functions is upper (lower) semi-continuous except at the points of a set of the first category. Further, this is true with respect to the continuum or any perfect set.

(Issued separately April 10, 1908.)