IV. *On the Use of NEGATIVE QUANTITIES in the SOLUTION of PROBLEMS by ALGEBRAIC EQUATIONS.*
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BY the introduction of letters into algebra, to denote all the quantities, both known and unknown, involved in an equation, this very important advantage was gained, that the final equation exhibited both a general rule for the solution of all similar problems, and also the limitations within which such problems were possible.

THIS, however, could not be understood universally, if the signs + and — were not used in that extensive sense in which they are now taken. For there are innumerable problems, which require us to consider some of the quantities, as capable of existing in two opposite situations. Thus the distance of a star from the horizon, may include both its elevation at one time, and its depression at another. Hence, in the general investigation of this distance, two different cases arise, which may seem to require two different equations.

MATHEMATICIANS, however, came naturally to consider such opposite situations as analogous to addition and subtraction. And, upon this ground, they made the equation to which the problem was reduced in the one case, serve also for the other, in the following manner: If it was one of the unknown quantities which had changed its situation, they took a negative root; if it was one of the known quantities, they

changed its sign in the equation. It is evident how vast an accession was thus gained to the elegance and universality of algebraical solutions.

ALBERT GIRARD appears to have been the first who observed this use of negative roots. In his work, published in the year 1629, entitled, *Invention nouvelle en Algebre*, he mentions it with great distinctness, and as a matter unknown before *.

MONTUCLA, who acknowledges that he never saw the work of GIRARD, insists very strenuously that DES CARTES was the person, to whom we owe our knowledge of the nature and use of negative roots; that it was he, who first introduced them into geometry and algebra †. But without derogating from the very extraordinary merit of that philosopher, we must beg leave, in this matter, to differ somewhat from MONTUCLA. For, according to his own account, the work of GIRARD was published in 1629, whereas the *Geometry* of DES CARTES was not published till seven or eight years afterwards, in 1637 ‡.

WE may, perhaps, go farther, and add, that although DES CARTES indeed observed, that the shifting of a line or point from one side of a given line or point to the other, made no change in the equation, except in the signs + and —, yet he did not see, in its full extent, all the advantage to be derived from the use of negative quantities. This will appear not altogether improbable from the following circumstances: In the third book

* I HAVE not been able to procure the book; but the following passage, quoted by Dr HORSELEY, seems to warrant what has been asserted: “Jusques icy nous n’avons encore expliqué à quoy servent *les solutions par moins*, quand il y en a. La solution par moins s’explique en geometrie en retrogradant, et le moins recule là où le + avance.” And, after giving an instance, he adds; “Et ainsi faudra-t-il entendre de toutes solutions par moins; qui est une chose de consequence en geometrie incognue auparavant.” HORSELEY’S NEWTON, vol. I. p. 171. note (u).

† MONTUCLA Hist. de Math. vol. 2. p. 85.

‡ MONTUCLA Hist. de Math. vol. 2. p. 82. & 84.

book of his *Geometry*, where he treats expressly of the nature of equations, he takes no notice of those which have all their roots negative ; and, in the second book, where he mentions his new method of considering curves, he says nothing of negative abscissæ and ordinates, although the analysis of curves is, of all the parts of mathematics, that which most obviously suggests and requires the use of the negative roots.

IN fact, the use of negative quantities appears not to have been, either at this time, or for some time afterwards, familiar to mathematicians. This is evident from the *Elementa Curvarum* of JOHN DE WITT, a work, at that time, of some merit, but which will be considered, rather as affording a curious piece of information in the life of that great man, than as adding to a fame, which is so far superior to all literary eminence. In this work, which is published by SCHOOTEN at the end of his edition of DES CARTES's *Geometry*, no notice is taken of negative abscissæ and ordinates : So that the author is obliged to consider separately two or more equations, which every mathematician at present, from the view of negative quantities, which is now become familiar, would consider as one single equation, denoting for one and the same line referred to the same axis, the relation between the abscissæ and ordinates, through all the different cases of their situation with respect to each other, that is, through all the angles of the coordinates. Thus, in his second book, where he considers the lines of the first and second order, beyond which he does not go, he demonstrates, in the four first theorems, that the four following equations belong to straight lines : 1. $y = \frac{bx}{a}$;

2. $y = \frac{bx}{a} + c$; 3. $y = \frac{bx}{a} - c$; 4. $y = -\frac{bx}{a} + c$. But he does

not show, either in this or any other part of the work, that each of these equations, by changing the signs of y or x , gives the relation between the absciss and ordinate, when the one or the
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the other is taken in an opposite situation ; and that the second and fourth equation ought to be considered as one, and as belonging to the same straight line referred to the same axis.

EVEN long after this time, we find, that Dr WALLIS, although, in his *Algebra*, he considers at some length the meaning and use of negative quantities, yet, in his *Arithmetic of Infinites*, falls into a strange mistake, which leads us to suspect, that his notions on this subject were not perfectly clear. For, observing that his general expression for hyperbolic areas would, in certain cases, be a fraction with a negative denominator, he did not perceive, that this expressed the area on the other side of the ordinate, instead of the area sought ; but he contents himself with saying, that the denominator of the fraction being less than nothing, denoted the area to be more than infinite.

THAT even some time afterwards, the use of negative quantities had not become familiar, appears from the astonishment which Dr HALLEY expresses at his own discovery of a formula, which, by the mere change of the signs, gave the focus both of converging and diverging rays, whether reflected or refracted by convex or concave specula or lenses. And Mr MOLYNEUX speaks of the universality of HALLEY's formula as somewhat that resembled magic.

THUS it appears to have been long, before mathematicians ventured to employ negative quantities so freely, as we now perceive them to be employed. The reason of which probably was, that no satisfactory account had been given of the grounds upon which the conclusions drawn from them are founded. But the consistency of these conclusions, with all the most indisputable truths of the mathematical sciences, and the great beauty and advantage, derived from the very general solutions which are thus obtained, gradually established their use.

STILL, however, a complaint remains, which appears to be too well founded, that the Method of negative quantities, as has been the case with some other rules of the art, is supported,

ed, rather by induction and analogy, than by mathematical demonstration. But something more than this is to be expected from mathematicians; and their knowledge of algebra will still be considered as imperfect, if there be any of its operations which they cannot, in every letter and sign of it, explain as a series of just reasoning, capable of being expressed in common language. Now, it does not appear that our instructors in mathematics have enabled us to do this in the cases to which we allude, that is, where negative quantities are found, without any intimation of the quantities from which they are subtracted. In considering the abstract operations of addition, subtraction, multiplication, &c. we may indeed take negative quantities by themselves: For here the meaning is obvious; we are examining, in what manner such operations will affect those quantities, which are subtracted from other quantities. But when we come to apply these operations to the business of algebra, the resolution of equations, it is not so easy to perceive, nor does it appear to have been sufficiently explained, in what light we are to consider negative quantities, or how mathematicians are to be justified in the use which they make of them. The very vague and unsatisfactory, and often mysterious accounts of the matter, which are given even by writers of the greatest eminence, serve only to shew, that although they are satisfied of the certainty of the method, yet they perceive that something still remains which ought to be explained, and of which no good explanation has been given.

Mr Baron MASERES, indeed, has published a large work, which he entitles, “A Dissertation, on the use of the negative
“ sign in algebra; containing a demonstration of the rules usu-
“ ally given concerning it, and shewing how quadratic and
“ cubic equations may be explained, without the consideration
“ of negative roots.” But this dissertation contributes little to remove the difficulties complained of. For even allowing the author to be right in his notion, that it would be better “if ne-
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“ gative roots had never been admitted into algebra, or were
 “ again discarded from it * ;” yet still he has carried us to the
 length only of cubic equations. The truth is, that the whole
 business of algebra might be carried on without the considera-
 tion of the negative roots. The difference between such a system
 and the present, is precisely this ; that wherever a problem re-
 quired us to consider any of the quantities, as existing in oppo-
 site situations ; wherever, for instance, a line or a point was to be
 considered, as situated first on the right hand, and then on the
 left ; it would be necessary, to find and to resolve a separate equa-
 tion for each of these cases. Thus, in the analysis of any par-
 ticular curve, it would be necessary to have a separate equation
 for each of the four angles of the co-ordinates ; except, indeed,
 the axes were so chosen, as to make us certain that there were
 some of these angles, in which no part of the curve was to be
 found. Since, therefore, the use of negative quantities frees us
 from this inconvenience, which, in many cases, particularly in
 the analysis of curves, would be exceedingly perplexing ; and
 since it evidently affords so great elegance and universality to
 algebraical solutions ; to find our author gravely declaring that
 he can see no advantage in it, is perfectly astonishing : As it is
 to be lamented, that he did not exert his industry and ingenui-
 ty, rather to confirm than to destroy ; rather to demonstrate, how
 far we might rely on the method of negative quantities, than
 to overturn at once so great a part of the labours of the modern
 algebraists.

WHAT follows is an attempt to explain this subject, without
 considering the negative sign in any other light, than as the sign
 of subtraction ; and without proposing any alteration in the
 received system of algebra.

* Differt. on the Neg. Sign, p. 34.

I. *Of the Negative Roots of Equations.*

1. *Of Determinate Equations.*

IN the solution of problems by means of equations, the analyst fixes upon one or more quantities, by the determination of which all that is required may be known or performed. We shall at present suppose that there is only one quantity to be determined. The problem shews the conditions which are required of this quantity; and these conditions, as far as they can be so expressed, are reduced to an equation of the common form,

$$L \dots a + bx + cx^2 + \mathfrak{E}^2c = 0.$$

Here, according to the common method in the general notation of equations, the sign + denotes, at pleasure, either the addition or subtraction of the terms to which it is prefixed.

IN many cases, nothing else is required, but to determine the *magnitude* of the quantity sought. There the positive roots alone can determine the magnitude; so that if the equation has no positive roots, or none which come within the limits of the problem, then the problem is impossible.

BUT let the problem require us to determine, not only the different magnitudes of the line AB (x), but also, with respect to each of these $\overset{B}{\text{---}} \overset{A}{\text{---}} \overset{B}{\text{---}}$ magnitudes, whether it lies on the right or left of the given point A . Here we suppose the problem to be such, that whether we reduce it to an equation, upon the supposition that AB lies to the right, or upon the supposition that it lies to the left, there is no circumstance, except only the opposite situation of AB , to make any difference in these equations.

IN this case, we make either supposition at pleasure; as, for instance, that it lies to the right; and, on this supposition,

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we express the conditions required of AB in an equation, as,

$$L \dots a + bx + cx^2 + \mathfrak{E}c. = 0.$$

Then, in determining the magnitude of AB in this direction, the positive roots are to be employed.

Now, on the other side of A , take MA , which we shall denote by X ; and let it be of any magnitude, not less than any of the negative roots of L . It is evident that L may be reduced to the form

$$M \dots \left\{ \begin{array}{l} + \frac{A}{BX} + Bx \\ + CX^2 + 2CXx + Cx^2 \\ + \mathfrak{E}c. \end{array} \right\} = 0.$$

And this again may be reduced to the form

$$N \dots A + B(X+x) + C(X+x)^2 + \mathfrak{E}c. = 0.$$

THE equation N , therefore, is another expression of the conditions required of x , upon the supposition that it lies to the right of A .

BUT if we make the contrary supposition, that AB lies to the left of A ; then since this is the only alteration in the conditions which the equation N requires of x ; therefore, the conditions required of it, when it lies to the left of A , will be expressed in the equation

$$N' \dots A + B(X-x) + C(X-x)^2 + \mathfrak{E}c. = 0.$$

Now, this equation may be reduced to the form

$$M' \dots \left\{ \begin{array}{l} + \frac{A}{BX} - Bx \\ + CX^2 - 2CXx + Cx^2 \\ + \mathfrak{E}c. \end{array} \right\} = 0.$$

By

By comparing this equation with M and L , it will be evident that it may be reduced to the form

$$L' \dots a - bx + cx^2 - \mathfrak{E}c. = 0.$$

THIS equation L' , therefore, expresses the conditions required of AB , when it lies to the left of A ; and, consequently, the positive roots of L' determine the different magnitudes of AB in that situation.

BUT L' differs from L , only by having changed the signs of the terms involving the odd powers of x . Therefore L and L' have the same roots, except only that the positive roots of the one are the negative roots of the other.

THEREFORE, the negative roots of the equation L , to which the problem is reduced, upon the supposition that AB lies to the right of A , are the positive roots of the equation to which it would be reduced, upon the supposition that AB lies to the left of A . Therefore, these negative roots are the determinations of AB in this last situation.

AGAIN: Let it be required to determine, not only the *distance* of time between a certain event and a given instant; but also, whether the event happened *after* or *before* the given instant. Then reduce the problem to an equation, L , upon either supposition, as, for instance, that it happened after; and by assuming X , a period of time immediately before the given instant, we can shew, as above, that the negative roots of L will be the determination of the epoch, when it happened before the given instant.

IN like manner, let it be required to determine, not only the *momentum* of the force which acts in the right line AB , but also, whether it acts in the *direction* AB , $\overset{A}{\rule{1.5cm}{0.4pt}}\overset{B}$ or in the direction BA . Then, if we reduce the problem to an equation, upon the supposition that it acts in the direction AB , we are to assume X , an additional force acting in the same direction.

IN like manner, let it be required to determine, not only the *value* of a certain sum of money, but also, whether it is part of a certain person's *stock*, or whether it is part of his *debts*. Then, if we reduce the problem to an equation, upon the supposition that it is part of his stock; we are to assume X a supposed additional quantity of stock.

THERE are cases, therefore, where the quantity sought is to be considered in two different situations, which may be represented by addition and subtraction from another assignable quantity of the same kind; and where there is nothing in the problem, except only this opposite situation, which can produce any difference in the equations to which it is reducible upon each supposition. From the preceding observations it will appear what may be understood by the negative roots, and how mathematicians are justified in the conclusions which they draw from them in such cases.

THE negative roots are sometimes also useful in the solution of problems relating to abstract quantities. For the equation

$$L \dots a + bx + cx^2 + \mathcal{C}c. = 0$$

has the same roots with the equation

$$L' \dots a - bx + cx^2 - \mathcal{C}c. = 0,$$

except only that the negative roots of the one are the positive roots of the other. Wherever, therefore, any problem producing the equation L , is so connected with any other problem producing the equation L' , that they may be considered as different cases of the same problem, or that the consideration of the one suggests the other; there it is evident, that the negative roots will be useful, by affording, from either of these equations, the solution of both problems. Thus, the equation $x^2 + x = a$, gives us, not only the number, which, added to its square, makes a sum equal to a ; but also, by the negative root, the

the number which, subtracted from its square, leaves a remainder equal to a .

WHAT has been said is easily applicable to those problems, where more than one quantity is to be determined.

2. Of Indeterminate Equations.

IN problems which are solved by indeterminate equations, there is often nothing else required, but to determine the different *magnitudes* of one or more of the variable quantities, corresponding to any given magnitudes of the remaining variable quantities. There the positive roots alone can determine these corresponding magnitudes; so that if there be no positive roots, or none which come within the limits of the problem, then the problem is impossible.

BUT let the problem require us to determine, not only the different corresponding magnitudes of the lines AB (x) and BC (y); but also, with respect to each of the values of x , whether it lies to the *right* or *left* of the given point A ; and with respect to each of the corresponding values of y , whether it is to be taken *above* or *below* the line AB . Here we suppose that, whether AB lies to the right or left, and whether the corresponding line BC is to be taken

upwards or downwards, there is nothing, except only these opposite situations of AB and BC , to make any difference in the equations, to which the problem would be reduced upon each of these suppositions. Then, we make any of these suppositions at pleasure; as for instance, that AB lies to the right, and that BC lies above AB ; and on this supposition, we express the conditions

ditions required of these quantities in an equation of the common form, as,

$$L \dots a + bx + cy + dx^2 + exy + fy^2 + \mathfrak{E}c. = 0.$$

Then, in determining the corresponding magnitudes of AB and BC in these directions, the positive roots are to be employed.

Now, on the other side of A , in the line AB , take MA , which we shall denote by X , not less than any of the negative values of x ; and in the line AC , below AB , take AL , which we shall denote by Y , not less than any of the negative values of y . It is evident, that the equation L may be reduced to the form

$$M \dots \left\{ \begin{array}{l} A \\ + BX + Bx \\ + CY + \dots + Cy \\ + DX^2 + 2DXx + \dots + Dx^2 \\ + EXY + EYx + EXy + \dots + Exy \\ + FY^2 + \dots + 2FYy + \dots + \dots + Fy^2 \\ + \mathfrak{E}c. \end{array} \right\} = 0.$$

And this equation may be reduced to the form

$$N \dots A + B(X+x) + C(Y+y) + D(X+x)^2 + E(X+x)(Y+y) + F(Y+y)^2 + \mathfrak{E}c. = 0.$$

THIS equation N , therefore, is another expression of the conditions which are required of the variable quantities, upon the supposition that x lies to the right of A , and that y lies above AB .

BUT if we make the supposition that x lies still to the right of A , but that y lies below AB ; then, since this is the only alteration in the conditions which the equation N requires of x and y ; therefore the conditions required of them, upon this new supposition, will be expressed in the equation

$$N' \dots A + B(X+x) + C(Y-y) + D(X+x)^2 + E(X+x)(Y-y) + F(Y-y)^2 + \mathfrak{E}c. = 0.$$

Now,

Now, by reducing this equation to the form of M , and comparing it with L , it will be evident, that it is the same with the following equation ;

$$L' \dots a + bx - cy + dx^2 - exy + fy^2 + \mathfrak{E}c = 0.$$

THIS equation L' , therefore, expresses the conditions required of the variable quantities, when x lies to the right of A , and when y lies below AB . And, consequently, the corresponding positive roots of L' are the determinations of the corresponding magnitudes of AB and BC in this situation.

BUT L' differs from L , only by having changed the signs of the terms involving the odd powers of y . Therefore, L' and L have the same roots, except only that the positive values of y in the one, are its negative values in the other.

THEREFORE, in the equation L , the positive values of x , and the corresponding negative values of y , are the determinations of the corresponding magnitudes of AB and BC , when AB lies to the right of A , and when BC lies below AB .

IN the same manner, it may be shewn, that the negative values of x , and the corresponding positive values of y , are the determinations of AB , BC , when AB lies to the left, and BC lies above AB .

AND, *lastly*, That the corresponding negative values of x and y , are the determinations of AB , BC , when AB lies to the left, and BC lies below AB .

THE observations which have been made under the head of determinate equations, are equally applicable to those which are indeterminate. And the foregoing demonstration may be easily extended to any number of indeterminate quantities.

II. *Of the Negative Quantities which are not the Roots of the Equation.*

BESIDES the roots of the equation, or the quantities to be determined, problems frequently occur, which require us to consider some of the *given* quantities, as capable, consistently with the conditions of the problem, of existing in one or other of two opposite situations, which may be represented by addition and subtraction.

SUPPOSE, for instance, that the given lines $AB = a$, $AC = b$, may be taken either on the right or left of the given point A ; and let the problem be reduced to an equation, upon the supposition that they are each taken to the right. This equation, when ordered according to the powers of a and b , will be represented by the following general formula:

$$L \dots p + qa + rb + sa^2 + tab + vb^2 + \mathcal{C}c. = 0.$$

IN the lines AB , AC , and on the left of the point A , take $AM = A$, $AN = B$, of any magnitudes, not less respectively than a , b .

THEN the equation L may be reduced to the form,

$$M \dots \left\{ \begin{array}{l} P \\ + 2A + 2a \\ + RB + \dots + Rb \\ + SA^2 + 2SAa + \dots + Sa^2 \\ + TAB + TBa + TAb + \dots + Tab \\ + VB^2 + \dots + 2VBb + \dots + Vb^2 \\ + \mathcal{C}c. \end{array} \right\} = 0.$$

And

And this again is reducible to the form,

$$N \dots P + Q(A+a) + R(B+b) + S(A+a)^2 + T(A+a)(B+b) + V(B+b)^2 + Ec. = 0.$$

THIS equation, therefore, is another expression of the conditions of the problem, upon the supposition that AB , AC are taken to the right of A .

Now, by reasoning in the very same manner as before, it is evident, that when AB or AC is taken to the left, the conditions of the problem will still be expressed by an equation, which will, in all respects, be the same with L , except only, that the quantity which has changed its situation, will change its sign.

FROM the whole, the two following conclusions seem to be demonstrated.

1. WHERE the problem allows us to consider x , any of the unknown or indeterminate quantities, as capable of existing in two opposite situations, which may be represented by addition and subtraction; then the equation, which expresses the conditions required of x in one of these situations, and whose positive roots determine the magnitudes of x in that situation; the same equation, by its negative roots, will determine the magnitudes of x in the opposite situation.

2. WHERE the problem allows us to consider a , any of the given quantities, as capable of existing in two such opposite situations; then the equation which expresses the conditions of the problem, upon the supposition that a is in one of these situations, will be reduced to the equation expressing the conditions of the problem on the contrary supposition; by simply changing the sign of a , or, in other words, the sign of the terms involving the odd powers of a .