THE DIVISION OF AN ALTERNATING CURRENT IN PARALLEL
CIRCUITS WITH MUTUAL INDUCTION.¹

By FREDERICK BEDELL.

A DIVIDED circuit with mutual induction between the two branches is shown in Fig. 1, being the same as a transformer with the primary and secondary circuits connected in parallel. The electromotive force equations for the two circuits are similar, the internal electromotive forces in each being equal to the same impressed electromotive force. The electromotive force of mutual induction will be positive or negative according to the sense or direction in which the coils are connected. If the coils are connected as in Fig. 2, so that the ampere turns of the two coils assist each other, the electromotive forces of self and mutual induction will be of the same sign, and the coefficient of mutual induction will be positive. If the coils are connected so as to oppose each other, as in Fig. 3, the electromotive force of mutual induction will be opposite in sign to that of self-

Fig. 1.

Divided circuit with mutual induction.

Fig. 2.

Coils additive.

Fig. 3.

Coils opposed.

induction. The coefficient of mutual induction may accordingly be $+M$ or $-M$. Writing the electromotive force as a function of the time, we have the following equations:—

$$e = f(t) = R_1 i_1 + L_1 D i_1 \pm M D i_2; \quad (1)$$

$$e = f(t) = R_2 i_2 + L_2 D i_2 \pm M D i_1. \quad (2)$$

By differentiation, the above equations become

$$f'(t) = R_1 D i_1 + L_1 D^2 i_1 \pm M D^2 i_2;$$

$$f'(t) = R_2 D i_2 + L_2 D^2 i_2 \pm M D^2 i_1.$$

¹ A paper read at the Liverpool meeting of the British Association for the Advancement of Science, 1896.

Eliminating i_2 by determinants, we have

$$\begin{array}{cccc} i_2 & Di_2 & D^2i_2 & \text{OTHER TERMS.} \\ \circ & \pm M & \circ & R_1i_1 + L_1Di_1 - f(t) \\ \circ & \circ & \pm M & R_1Di_1 + L_1D^2i_1 - f'(t) \\ R_2 & L_2 & \circ & \pm MDi_1 - f(t) \\ \circ & R_2 & L_2 & \pm MD^2i_1 - f'(t) \end{array} = 0.$$

Canceling out the first column and third row, as indicated by fine type, we obtain

$$\begin{aligned} M^2[\pm MD^2i_1 - f'(t)] \mp MR_2[R_1i_1 + L_1Di_1 - f(t)] \\ \mp ML_2[R_1Di_1 + L_1D^2i_1 - f'(t)] = 0. \end{aligned} \quad (3)$$

From this equation the value for the current i_1 is obtained:—

$$i_1 = \frac{R_2f(t) + (L_2 \mp M)f'(t)}{(L_1L_2 - M^2)D^2 + (R_1L_2 + R_2L_1)D + R_1R_2}. \quad (4)$$

A similar expression is found for i_2 .

For an harmonic electromotive force

$$\begin{aligned} e = f(t) &= E \sin \omega t; \\ f'(t) &= E\omega \cos \omega t. \end{aligned}$$

Since $D^2 \sin \omega t = -\omega^2 \sin \omega t$, we may write $-\omega^2$ for D^2 .

In equation (4) we may make these substitutions for $f(t)$, $f'(t)$, and D^2 . By combining the sine and cosine terms into one term, and multiplying numerator and denominator by D , we have

$$i_1 = \frac{DE\sqrt{R_2^2 + (L_2 \mp M)^2\omega^2} \sin\left(\omega t + \arctan \frac{(L_2 \mp M)\omega}{R_2}\right)}{\alpha D - \omega\beta}; \quad (5)$$

where

$$\begin{aligned} \alpha &= R_1R_2 - \omega^2(L_1L_2 - M^2), \\ \beta &= \omega(R_1L_2 + R_2L_1). \end{aligned}$$

To free equation (5) from the operator D , perform the operation D as indicated in the numerator (by differentiating the numerator), and multiply numerator and denominator by $\alpha D + \omega\beta$. Substitute $-\omega^2$ for D^2 in the denominator and again perform the operation D in the numerator. We thus obtain an expression for i_1 which is free from D ; this may be written in the following form:—

$$i_1 = I_1 \sin(\omega t + \phi_1); \quad (6)$$

where
$$I_1 = \frac{E}{\frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{R_2^2 + (L_2 \mp M)^2 \omega^2}}}. \quad (7)$$

$$\phi_1 = -\arctan \frac{\beta}{\alpha} + \arctan \frac{(L_2 \mp M)\omega}{R_2}. \quad (8)$$

Similarly, for the current in the second branch,

$$i_2 = I_2 \sin(\omega t + \phi_2); \quad (9)$$

where
$$I_2 = \frac{E}{\frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{R_1^2 + (L_1 \mp M)^2 \omega^2}}}. \quad (10)$$

$$\phi_2 = -\arctan \frac{\beta}{\alpha} + \arctan \frac{(L_1 \mp M)\omega}{R_1}. \quad (11)$$

The ratio of the two currents is

$$\frac{I_1}{I_2} = \frac{\sqrt{R_2^2 + (L_2 \mp M)^2 \omega^2}}{\sqrt{R_1^2 + (L_1 \mp M)^2 \omega^2}}. \quad (12)$$

Their phase difference is

$$\phi_1 - \phi_2 = \arctan \frac{(L_2 \mp M)\omega}{R_2} - \arctan \frac{(L_1 \mp M)\omega}{R_1}. \quad (13)$$

Where the coils are opposed and nearly similar the angle of phase difference between the currents depends largely upon the amount of magnetic leakage.

Fig. 4.

Coils additive; M positive.

The diagram for the electromotive forces in a divided circuit in which the effects of self and mutual induction are additive is shown in Fig. 4. The two currents are I_1 and I_2 , respectively, the main current being their vector sum I .

In branch one, OH is the power electromotive force (in the absence of branch two); HJ and JK are the electromotive forces of self and mutual induction, being at right angles to the currents I_1 and I_2 , respectively. The corresponding electromotive forces for the second circuit are shown in the same diagram. The sum of the several electromotive forces in either circuit add up to E , the impressed electromotive force.

Ordinarily the branch current and the main current would lag behind the impressed electromotive force by a large angle approaching 90° .

Fig. 5.

Coils opposed; M negative.

When the two circuits are identical, the branch currents I_1 and I_2 would have the same magnitude and direction; the points H' and J' would coincide with the points H and J , respectively. An inspection of the figure will show that for this case, in the absence of magnetic leakage, the effect of the mutual induction would be to double the tangent of the angle of lag.

Figure 5 shows the corresponding diagram for the case in which the electromotive forces of self and mutual induction are opposed. The current I_1 produces an electromotive force HJ in the first branch, and J'K in the second branch; these electromotive forces being at right angles to the current I_1 , and opposite to each other in direction, on account of the sense in which the two coils are connected. Similarly, H'J' is opposite in direction to JK. The effects of self and mutual induction consequently tend to neutralize each other. The electromotive force OK, required to cause the currents I_1 and I_2 to flow, is accordingly made less on account of the mutual induction, and would be reduced to the value of the ohmic electromotive force in the case of two similar coils wound oppositely. In

this case the angle of lag would be reduced to zero, and the coil would be non-inductive.

The equivalent resistance and self-induction for the two coils together, whether they are additive or opposed, may be found by resolving E into two components, one in the direction of the main current, and the other at right angles to it. These components will be equal to $R'I$ and $L'\omega I$, respectively. Their magnitudes may be obtained graphically, and from them the values of the equivalent resistance R' and the equivalent self-induction L' . The equivalent resistance and self-induction of either branch may be obtained in the same manner.

The conclusions relating to the special case of two identical coils wound without magnetic leakage, stated above in connection with the graphical construction, may be reached by substituting in the formulæ the values $R_1 = R_2 = R$; and $L_1 = L_2 = \pm M = L$. Also, $\beta + \alpha = 2L\omega + R$. The two cases are as follows:—

When M is positive.—When M is positive, the coils being wound in the same direction,

$$\phi = -\arctan \frac{2L\omega}{R} + \arctan 0 = -\arctan \frac{2L\omega}{R}. \quad (14)$$

The current in each branch is

$$I_1 = I_2 = \frac{E}{\sqrt{R^2 + (2L\omega)^2}}. \quad (15)$$

For this case, therefore, the effect produced is the same as though the mutual induction were zero and the self-induction of each circuit doubled, the tangent of the angle of lag being doubled and each current being diminished accordingly.

The main current is

$$I = I_1 + I_2 = \frac{2E}{\sqrt{R^2 + (2L\omega)^2}} = \frac{E}{\sqrt{\left(\frac{R}{2}\right)^2 + L^2\omega^2}}, \quad (16)$$

which is the same that would flow in a single coil, with the same self-induction and half the resistance as either branch.

When M is negative.—When the coils are identical and opposed,

$$\phi = -\arctan \frac{2L\omega}{R} + \arctan \frac{2L\omega}{R} = 0; \quad (17)$$

$$I_1 = I_2 = \frac{E}{R}. \quad (18)$$

Each coil acts in this case as though non-inductive, the magnitude of the current being in accordance with Ohm's law, and the angle of lag being zero.

Cases where the two coils are not identical may be treated by substituting particular values for the resistance and coefficients of induction in the general equations.

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ON THE SPECIFIC GRAVITY AND ELECTRICAL CONDUCTIVITY OF
THE NORMAL SOLUTIONS OF SODIUM AND POTASSIUM HY-
DROXIDES, AND HYDROCHLORIC, SULPHURIC, NITRIC, AND
OXALIC ACIDS.

By E. H. LOOMIS.

THE various tables of specific gravity of these common alkaline and acid solutions differ in some cases so considerably that the attempt to compute the specific gravity of a particular solution, of known strength, by use of the tables, results in much uncertainty.

This uncertainty has made it desirable to measure the specific gravity of certain normal solutions about which the most doubt exists. I have accordingly undertaken to prepare such solutions of NaOH, KOH, HCl, HNO₃, H₂SO₄, and (COOH)₂, for the purpose of measuring their specific gravity. I have also taken this opportunity to measure their electrical conductivity.

The various solutions were prepared by titration. The titration standards were normal solutions of sodium carbonate and oxalic acid whose preparation will be first described.

Normal Sodium Carbonate. — Sodium acid carbonate (NaHCO₃, Trommsdorff, c.p.) was glowd, and thus converted into Na₂CO₃. A saturated solution of this product was then treated with a current of pure CO₂ until the precipitation of NaHCO₃ was complete. This precipitated NaHCO₃ was then carefully heated at a temperature slightly under red glow until its weight became constant. The Na₂CO₃ thus obtained was deemed sufficiently pure for the given purpose.

The molecular weight of this salt (O = 16) is 106.12. Thus a normal solution requires 53.06 g. per liter of the solution.

The liter flask employed at 18° C. contained 1000 c.c.

A simple calculation shows that such a flask should hold apparently 997.667 g. of water whose temperature is 18° C. when the weighing is made in the air with brass weights. The mark on the flask was fixed in

Fig. 2.
Coils additive.

Fig. 3.
Coils opposed.