

On Simultaneous Differential Equations, with special reference to—

1. *The Roots of the Fundamental Determinant.*
2. *The Method of Multipliers.*

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The propositions in this paper were constructed for the purpose of determining the small oscillations of a system which depends on many coordinates. But, as they are of general application, they are here presented in a form which is purely mathematical. No reference is made to any dynamical principle, and when dynamical terms are used, it is only for the sake of explanation.

We begin by taking the equations of the second order with n dependent variables in their most general forms, though such general forms do not occur in dynamics. Two typical equations are then deduced, and from these all the results given in this paper are derived. These are given in Arts. (5) and (7).

The first step usually taken in solving simultaneous equations is to form a certain determinant. The general form of the solution depends on the nature of the roots of this determinant. Two propositions follow immediately from the typical equations. If three functions, here called A, B, C , be one-signed, it is shown—(1) however general the equations may be, the real roots of the determinant cannot be positive; (2) if the equations be of that simpler character which occurs in dynamics, the real part of every imaginary root is negative. The first proposition has been established by Lagrange and Sir W. Thomson when the equations represent the oscillations of a system about a position of equilibrium. The second is to be found in the author's essay on the "Stability of Motion," but expressed in a different form. It is also given in the last edition of Thomson and Tait's "Natural Philosophy."

The same two typical equations lead immediately to four other theorems which constitute the second part of this paper. When a system of dynamical equations has been solved, it becomes necessary to determine the arbitrary constants in terms of the initial values of the coordinates. Perhaps the equations may have been only partially solved, the rest of the solution depending on some roots of the determinant which cannot be found. In such a case, we may yet require to determine the arbitrary constants which occur in that part of the solution which has been found. The method of Multipliers is used to effect this purpose, and the process is explained in Arts. 11 and 12.

The method of Multipliers is in general use when we express a function by Fourier's rule in a series of sines or cosines, when we expand a function in Laplace's coefficients, and in a great variety of physical problems. The difficulty is, of course, to find the multipliers. In all the cases just alluded to, the rule to find the multipliers is the same as that used in Dynamics when a system oscillates about a position of equilibrium, and the vis viva can be expressed as the sum of the squares of the velocities of the coordinates. The rule in this case is very simple, and is continually used by Lagrange. He also gives, but less distinctly, a rule for the more general case in which the vis viva is any quadric function of the velocities, provided no resistances act on the system.

I do not think that it has been noticed that the multipliers given by Lagrange are not unique. There is another set which may be used when those founded on the vis viva are inconvenient.

Passing over these cases as being sufficiently well known, we shall turn our attention to finding the proper multipliers in the two following cases:—(1) When the equations are such as occur when a system oscillates about a position of equilibrium under forces of resistance which vary as the velocities; (2) when the equations are such as occur when a system oscillates about a state of steady motion, as when a hoop rolls on the ground, and oscillates about the plane of motion. I am not aware that any rule has been given to determine the multipliers in these two cases.

The roots of the Fundamental Determinant.

1. Let there be any number of dependent variables x, y, z , &c., to be found in terms of t , by means of as many differential equations of the second order with constant coefficients. Whatever these equations may be, they may be very conveniently written in the form

$$\begin{aligned}
 &A_{11}\delta^2 + B_{11}\delta + C_{11})x + \left(\begin{array}{c} A_{12}\delta^2 + B_{12}\delta + C_{12} \\ + D_{12}\delta^2 + E_{12}\delta + F_{12} \end{array} \right)y + \left(\begin{array}{c} A_{13}\delta^2 + B_{13}\delta + C_{13} \\ + D_{13}\delta^2 + E_{13}\delta + F_{13} \end{array} \right)z \\
 &\qquad\qquad\qquad + \&c. = 0, \\
 &\left(\begin{array}{c} A_{12}\delta^2 + B_{12}\delta + C_{12} \\ - D_{12}\delta^2 - E_{12}\delta - F_{12} \end{array} \right)x + (A_{22}\delta^2 + B_{22}\delta + C_{22})y + \left(\begin{array}{c} A_{23}\delta^2 + B_{23}\delta + C_{23} \\ + D_{23}\delta^2 + E_{23}\delta + F_{23} \end{array} \right)z \\
 &\qquad\qquad\qquad + \&c. = 0, \\
 &\left(\begin{array}{c} A_{13}\delta^2 + B_{13}\delta + C_{13} \\ - D_{13}\delta^2 - E_{13}\delta - F_{13} \end{array} \right)x + \left(\begin{array}{c} A_{23}\delta^2 + B_{23}\delta + C_{23} \\ - D_{23}\delta^2 - E_{23}\delta - F_{23} \end{array} \right)y + (A_{33}\delta^2 + B_{33}\delta + C_{33})z \\
 &\qquad\qquad\qquad \&c. \qquad\qquad + \qquad\qquad \&c. \qquad\qquad + \&c. = 0,
 \end{aligned}$$

where the symbol δ represents differentiation with regard to t , and the order of suffixes is immaterial, so that $A_{12} = A_{21}$, and so on.

We see here two sets of terms, (1) those which depend on the letters A, B, C , and which by themselves constitute a symmetrical determinant; (2) those which depend on the letters D, E, F , and which by themselves constitute a skew determinant.

2. For the reasons given in Chapter ix. of the fourth edition of the Author's "Rigid Dynamics," we may call the terms which depend on the letter A the *effective forces*, those which depend on the letter B the *forces of resistance*, those on C the *forces of restitution*. It will generally happen that the terms which depend on the letters D and F are absent. The terms which depend on the letter E will occur when we consider the oscillations about a state of motion.* These we shall call the *centrifugal forces*.

If we write A, B, C for the three functions

$$\begin{aligned} A &= \frac{1}{2}A_{11}x^2 + A_{12}xy + \frac{1}{2}A_{22}y^2 + \dots, \\ B &= \frac{1}{2}B_{11}x^2 + B_{12}xy + \frac{1}{2}B_{22}y^2 + \dots, \\ C &= \frac{1}{2}C_{11}x^2 + C_{12}xy + \frac{1}{2}C_{22}y^2 + \dots, \end{aligned}$$

we see that the terms in the several equations which arise from A, B, C may be written

$$\begin{array}{ccc} \delta^2 \frac{dA}{dx} + \delta \frac{dB}{dx} + \frac{dC}{dx} \\ \text{\&c.} & & \text{\&c.} \end{array}$$

Hence A, B, C may be called respectively the potentials of the effective forces, the forces of resistance, and the forces of restitution.

3. When we compare the equations of motion with those given by Lagrange for the oscillations about a position of equilibrium, we see that the function A cannot be otherwise than positive. So also these oscillations are stable if the function C be always positive.

Thus, it will frequently occur that the three functions A, B, C , or some of them, are such that they *keep one sign, and do not become zero whatever real quantities we write for x, y, z , &c.* Such functions will be referred to as *one-signed quadrics*.

4. The method of solving the differential equations in Art. (1) is well known. Let m_1, m_2 , &c., be the roots of the fundamental determinant, which we need not here write down. Let us suppose that these roots are unequal, the case of equal roots being regarded as a limiting case of unequal roots. The solution may be written thus:—

$$\left. \begin{aligned} x &= x_1 e^{m_1 t} + x_2 e^{m_2 t} + \dots \\ y &= y_1 e^{m_1 t} + y_2 e^{m_2 t} + \dots \\ z &= \text{\&c.} \end{aligned} \right\}.$$

* See a Paper contributed in 1875 by the author to the Society (*Proceedings*, Vol. vi., p. 96).

If $x'_1 = x_1 m_1$, $y'_1 = y_1 m_1$, &c., $x'_2 = x_2 m_2$, &c., we have

$$\left. \begin{aligned} \frac{dx}{dt} &= x'_1 e^{m_1 t} + x'_2 e^{m_2 t} + \dots \\ \frac{dy}{dt} &= y'_1 e^{m_1 t} + y'_2 e^{m_2 t} + \dots \\ \&c. &= \&c. \end{aligned} \right\}$$

Here x_1, y_1, z_1 , &c. contain as a common factor one constant of integration, x_2, y_2 , &c. another constant, and so on. The forms of these constants are not wanted here. It is enough that we should remember that the *coefficients which belong to a real exponential are themselves real*. On the other hand, if m_1, m_2 be a pair of imaginary roots, the coefficients (x_1, x_2) , &c., take the form $P \pm Q\sqrt{-1}$.

5. If we substitute the first terms of each of these values of x, y, z , &c., in the equations of Art. (1), we obtain a set of equations which differs from those only in having m_1 written for δ , and x_1, y_1 , &c. for x, y , &c.

Multiply these respectively by x_1, y_1 , &c., and add the results together; we have

$$(A_{11}x_1^2 + 2A_{12}x_1y_1 + \&c.)m_1^2 + (B_{11}x_1^2 + 2B_{12}x_1y_1 + \&c.)m_1 + (C_{11}x_1^2 + 2C_{12}x_1y_1 + \&c.) = 0.$$

It should be noticed that the terms which depend on the letters D, E, F have altogether disappeared from this equation.

It should also be noticed that the coefficients of the powers of m are the functions A, B, C with x_1, y_1 , &c. written for x, y , &c.

6. PROP. I.—ON REAL ROOTS.—We may immediately deduce the three following theorems:—

(1) *If the potentials A, B, C be either zero or one-signed functions, and if all three have the same sign, the fundamental determinant cannot have a real positive root.*

For if m_1 were real, the coefficients x_1, y_1 , &c. would be real. We should thus have the sum of three positive quantities equal to zero.

(2) *If there be no forces of resistance, i.e. the term B be absent, and if the potentials A and C be one-signed and have the same sign, the fundamental determinant cannot have a real root, positive or negative.*

(3) *If A, B, C be one-signed functions, but if the sign of B be opposite to that of A and C , the fundamental determinant cannot have a negative root.*

These three propositions, are true, whether there be any terms which depend on the functions D, E, F or not.

7. Exactly as in Art. 5, let us again substitute the first term of each

of the values of $x, y, \&c.$ in the equations of motion. But let us now multiply these by $x_2, y_2, \&c.$, and add the results. We thus obtain

$$\begin{aligned} & [A_{11}x_1x_2 + A_{12}(x_1y_2 + x_2y_1) + A_{23}(y_1z_2 + y_2z_1) + \&c.] m_1^2 \\ & + [B_{11}x_1x_2 + \&c.] m_1 + [C_{11}x_1x_2 + \&c.] \\ & = [D_{12}(x_1y_2 - x_2y_1) + D_{23}(y_1z_2 - y_2z_1) + \&c.] m_1^2 \\ & + [E_{12}(x_1y_2 - x_2y_1) + \&c.] m_1 + [F_{12}(x_1y_2 - x_2y_1) + \&c.] \end{aligned}$$

To bring this equation within bounds, we must use some notation to shorten the coefficients. Let us represent the halves of these series by their first terms, omitting suffixes to $A, B, \&c.$ We may therefore write the equation in the form

$$A(x_1x_2) m_1^2 + B(x_1x_2) m_1 + C(x_1x_2) = D(x_1y_2) m_1^2 + E(x_1y_2) m_1 + F(x_1y_2).$$

In the same way we have

$$A(x_1x_2) m_2^2 + B(x_1x_2) m_2 + C(x_1x_2) = -D(x_1y_2) m_2^2 - E(x_1y_2) m_2 - F(x_1y_2).$$

Also we deduce from these the two equations

$$\left. \begin{aligned} A(x_1x_1) m_1^2 + B(x_1x_1) m_1 + C(x_1x_1) &= 0 \\ A(x_2x_2) m_2^2 + B(x_2x_2) m_2 + C(x_2x_2) &= 0 \end{aligned} \right\}.$$

The first of these is the same as that already found in Art. (5).

Here we may notice that the functions $A(xx), B(xx), C(xx)$ are really the same as those we have already more simply denoted by A, B, C . We also notice that $D(x_1y_1) = 0, E(x_1y_1) = 0$, and $F(x_1y_1) = 0$.

8. Let us now suppose that there is a pair of imaginary roots in the fundamental determinant of the form $m_1 = r + p\sqrt{-1}, m_2 = r - p\sqrt{-1}$. The values of $x, y, \&c.$, given in Art. 4, become

$$\begin{aligned} x &= (x_1 + x_2) e^{rt} \cos pt + (x_1 - x_2) \sqrt{-1} e^{rt} \sin pt + \&c., \\ y &= (y_1 + y_2) e^{rt} \cos pt + (y_1 - y_2) \sqrt{-1} e^{rt} \sin pt + \&c., \end{aligned}$$

which may be conveniently abbreviated into

$$\left. \begin{aligned} x &= X_1 e^{rt} \cos pt + X_2 e^{rt} \sin pt + x_3 e^{m_3 t} + \dots \\ y &= Y_1 e^{rt} \cos pt + Y_2 e^{rt} \sin pt + y_3 e^{m_3 t} + \dots \\ z &= \&c. \end{aligned} \right\}.$$

If $X'_1 = rX_1 + pX_2$ and $X'_2 = -pX_1 + rX_2, \&c.$,

$$\left. \begin{aligned} \frac{dx}{dt} &= X'_1 e^{rt} \cos pt + X'_2 e^{rt} \sin pt + x'_3 e^{m_3 t} + \dots \\ \frac{dy}{dt} &= Y'_1 e^{rt} \cos pt + Y'_2 e^{rt} \sin pt + y'_3 e^{m_3 t} + \dots \\ \&c. &= \&c. \end{aligned} \right\}.$$

9. Returning now to the two first equations of Art. 7, let us divide them by m_1 and m_2 respectively. If we first add and then subtract the results, we have

$$\begin{aligned} A(x_1, x_2)r + B(x_1, x_2) + C(x_1, x_2) \frac{r}{r^2 + p^2} \\ = \left\{ D(x_1, y_2)p - F(x_1, y_2) \frac{p}{r^2 + p^2} \right\} \sqrt{-1}, \\ A(x_1, x_2)p - C(x_1, x_2) \frac{p}{r^2 + p^2} \\ = \left\{ D(x_1, y_2)r + E(x_1, y_2) + F(x_1, y_2) \frac{r}{r^2 + p^2} \right\} \frac{1}{\sqrt{-1}}. \end{aligned}$$

By substitution, we find that

$$\begin{aligned} 4A(x_1, x_2) &= A(X_1, X_1) + A(X_2, X_2) \} \\ -2D(x_1, y_2) \sqrt{-1} &= D(X_1, Y_2), \end{aligned}$$

with similar results for the other letters. We also infer from these equations that if A be a one-signed function, $A(x_1, x_2)$ is not only real, but has always the same sign as A . Similar remarks apply to the functions B and C .

If the functions D, E, F be absent, the two first equations of this Article reduce to

$$\begin{aligned} A(x_1, x_2)2r + B(x_1, x_2) &= 0 \\ -A(x_1, x_2)(r^2 + p^2) + C(x_1, x_2) &= 0 \end{aligned} \},$$

except when $p = 0$, i.e., except when the roots (which we have supposed imaginary) are real.

10. PROP. II. ON IMAGINARY ROOTS.—We may immediately deduce the following theorem from the equation of Art. 9.

(1) Let the fundamental determinant be symmetrical, i.e., let the functions D, E, F be all absent. Let the potentials A and B be one-signed and have the same sign (whether C be a one-signed function or not). Then the real part r of every imaginary root must be negative and not zero. But if the potential B be zero, then the real part of every imaginary root is zero.

If the potentials A and C be one-signed and have opposite signs, there can be no imaginary roots.

These results follow by simply looking at the two last equations of Art. 9.

(2) If the terms depending on D and F be absent from the equations, whether the terms depending on E be present or not, and if the three potential functions A, B, C be all one-signed and have the same sign, then the real part r of every imaginary root is negative, and not zero. But if the forces of resistance, i.e. B , be also absent, then the real part of every imaginary root is zero.

(3) If the terms depending on D and E be absent, but not necessarily those depending on F , and if A, B, C be all one-signed and have the

same sign, then the real part r of every imaginary root must be negative, or, if positive, must be less than p .

The Method of Multipliers.

11. We shall now make another application of the equations given in Art. (7).

If we examine the form of the solution of the equations given in Art. (4) or (8), we see that the columns are arranged according to the roots of the fundamental determinant. Each column contains one arbitrary constant which has to be determined from the initial values of the variables x, y, z , &c. If the whole solution be known, we may therefore find the constants by common algebra; though if there be many unknown constants, the process may be very long. But if the whole solution be not known, the processes of common algebra fail.

Thus, suppose we have found only one root of the fundamental determinant, then we know the terms which occur in one column only. The other columns depend on the other roots which have not yet been investigated. We may yet wish to find the value of the constant which occurs in this column in terms of the initial values of the variables. We should thus be able to find the magnitude of any one oscillation without finding the others.

12. To effect this, we use the *method of multipliers*. Our object is to find some multipliers for the equations which express the values of x, y , &c., $\frac{dx}{dt}, \frac{dy}{dt}$, &c., such that, on adding together the products, all the columns will disappear, except the one we wish to retain. Supposing this done, we have one equation containing the constant to be found and the initial values of x, y , &c. This equation is sufficient to determine the value of the constant.

13. The two most important problems which occur in Dynamics are those in which we have—

(1) Oscillations about a position of equilibrium, whether with forces of resistance or not.

(2) Oscillations about a state of steady motion.

In the first of these cases, the terms depending on D, E, F in Art. (1) are absent from the equations, and the fundamental determinant is therefore symmetrical. In the second, the terms depending on D, F are absent, but those depending on E are present. The fundamental determinant is therefore unsymmetrical. In general, the forces of resistance, B , are also absent.

We shall now investigate the forms of the multipliers in these two cases.

14. PROP. A.—*To determine the multipliers when the fundamental determinant is symmetrical and the forces of resistance not absent.*

Let m_1, m_2 be any two roots of this determinant. Then, by Art. (7),

$$\left. \begin{aligned} A(x_1, x_2) m_1^2 + B(x_1, x_2) m_1 + C(x_1, x_2) &= 0 \\ A(x_1, x_2) m_2^2 + B(x_1, x_2) m_2 + C(x_1, x_2) &= 0 \end{aligned} \right\} \dots\dots\dots (1).$$

Eliminating B and C in turn from these equations, we have

$$\left. \begin{aligned} A(x_1, x_2) m_1 m_2 &= C(x_1, x_2) \\ -A(x_1, x_2) (m_1 + m_2) &= B(x_1, x_2) \end{aligned} \right\} \dots\dots\dots (2),$$

except when m_1 and m_2 are the same root.

Either of these equations may be used to find the required multipliers. We thus find two sets of multipliers. We shall choose the first equation, as giving the simpler results.

If there be a pair of imaginary roots in the fundamental determinant, say $m_1 = r + p\sqrt{-1}$, $m_2 = r - p\sqrt{-1}$, and if m_3 be any other root, the first of equations (2) gives

$$\left. \begin{aligned} A(x_1, x_3) (r + p\sqrt{-1}) m_3 &= C(x_1, x_3) \\ A(x_2, x_3) (r - p\sqrt{-1}) m_3 &= C(x_2, x_3) \end{aligned} \right\} \dots\dots\dots (3).$$

Remembering that A and C are linear functions, we see that these give by addition and subtraction

$$\left. \begin{aligned} A(X'_1, x_3) m_3 &= C(X_1, x_3) \\ A(X'_2, x_3) m_3 &= C(X_2, x_3) \end{aligned} \right\} \dots\dots\dots (4),$$

where X_1, X'_1, X_2, X'_2 have the meaning given to them in Art. (8).

The function $A(x_1, x_2)$ may obviously be deduced from the potential $A(x_1, x_1)$ by the process

$$2A(x_1, x_2) = x_2 \frac{dA(x_1, x_1)}{dx_1} + y_2 \frac{dA(x_1, x_1)}{dy_1} + \dots,$$

where of course $A(x_1, x_1)$ (see Art. 7) represents the value of $A(x)$, or A when x_1, y_1 , &c. have been written for x, y , &c. The functions B and C may be treated in a similar manner.

We may now immediately deduce the proper multipliers.

Taking the solutions written down in Art. (4) (not reproduced here, to save space), let us multiply the expressions for x, y , &c. by $-\frac{dC}{dx}$, $-\frac{dC}{dy}$, &c., after writing x_1, y_1 , &c. in these multipliers for x, y , &c.; also let us multiply the expressions for $\frac{dx}{dt}$, &c. by $\frac{dA}{dx}$, &c.,

after writing x'_1, y'_1 , &c., for x, y , &c., in these multipliers. Finally, let us add the products; then, by virtue of the first of equations (2), the sum of every column except the first is zero.

If we have imaginary roots in the fundamental determinant, we take the solution given in Art. (8). Treating it in the same way, we see by equations (4) that all the columns disappear except the two first. Repeating the process for the second column, we again find that all the columns except the two first disappear.

The rule may be summed up as follows:—

Let the fundamental determinant be symmetrical, and the forces of resistance not absent. Let it be required to separate by the method of multipliers any given column from the others. *The proper multipliers for the coordinates are the values of $\frac{dO}{dx}, \frac{dC}{dy}$, &c., after we have substituted for x, y , &c., in these multipliers the corresponding coefficients in the column we wish to preserve. The proper multipliers for the velocities are the values of $-\frac{dA}{dx}, -\frac{dA}{dy}$, &c., after we have substituted for x, y , &c., in these multipliers the corresponding coefficients in the column of velocities we wish to preserve. Finally, we add the products together.*

When the equations are given, the following rule to find A, B, C will be useful:—*Multiply the equations by x, y, z , &c., and add the products, treating the operator δ as an algebraical factor. The halves of the coefficients of the powers of δ are the functions A, B, C .*

In this way we can find an equation connecting the initial values of the coordinates with the constant which accompanies any one column. Since these initial values are arbitrary, neither side of this equation can wholly vanish unless all the multipliers themselves vanish. Hence the coefficient of the exponential on the right-hand side cannot be zero, except in this one case.

The multipliers cannot all vanish unless the quadric functions O and A also vanish for some *finite values* of the coordinates. In Dynamics the function A is such a function of the coordinates as the vis viva is of the velocities. It is therefore impossible that A could vanish for any finite values of the coordinates.

15. *Example.*—Let us consider the equations

$$\left. \begin{aligned} (\delta^2 + \delta + 1)x + \frac{1}{2}(\delta - \frac{3}{2})y &= 0 \\ \frac{1}{2}(\delta - \frac{3}{2})x + (\delta^2 - \delta + \frac{1}{4})y &= 0 \end{aligned} \right\}.$$

It is easily seen that the determinant of the solution reduces to

$$m^4 - \frac{5}{16} = 0.$$

We therefore have, if m now stands for $\frac{1}{2}\sqrt{5}$,

$$\left. \begin{aligned} x &= x_1 e^{mt} + x_2 e^{-mt} + X_3 \cos mt + X_4 \sin mt \\ y &= y_1 e^{mt} + y_2 e^{-mt} + Y_3 \cos mt + Y_4 \sin mt \\ \frac{dx}{dt} &= mx_1 e^{mt} - mx_2 e^{-mt} + mX_4 \cos mt - mX_3 \sin mt \\ \frac{dy}{dt} &= my_1 e^{mt} - my_2 e^{-mt} + mY_4 \cos mt - mY_3 \sin mt \end{aligned} \right\}.$$

Also multiplying the equations by x and y , and taking the halves of the coefficients of the powers of δ , we have

$$\left. \begin{aligned} A &= \frac{1}{2} (x^2 + y^2) \\ C &= \frac{1}{2} x^2 - \frac{3}{2} xy + \frac{1}{2} y^2 \end{aligned} \right\}.$$

Suppose we wish to find the coefficients x_1, y_1 in terms of the initial conditions. Following the rule, we multiply x and y by the differential coefficients of C after we have written x_1, y_1 for x, y in the multipliers. We multiply the velocities by minus the differential coefficients of A , writing in the multipliers mx_1 and my_1 for x and y . Finally, we add the results. Thus we have

$$\left. \begin{aligned} x(x_1 - \frac{3}{2}y_1) + y(-\frac{3}{2}x_1 + \frac{1}{2}y_1) \\ - \frac{dx}{dt} mx_1 - \frac{dy}{dt} my_1 \end{aligned} \right\} = \left\{ \begin{aligned} x_1^2 - \frac{3}{2}x_1y_1 + \frac{1}{2}y_1^2 \\ -m^2(x_1^2 + y_1^2) \end{aligned} \right\} e^{mt}.$$

Putting $t = 0$, and giving x, y and their velocities their known initial values, we have one equation to find the constants x_1, y_1 . Their

$$\text{ratio, } \frac{y_1}{x_1} = -\frac{m^2 + m + 1}{\frac{1}{2}(m - \frac{3}{2})}, \quad m = \frac{1}{2}\sqrt{5},$$

being known from the first equation, we easily find both x_1 and y_1 .

If we wish to find the coefficients of the trigonometrical terms, we use two sets of multipliers, because the two imaginary exponentials have become mixed up together in the trigonometrical term; or we may replace them by their imaginary exponentials, and find the coefficients of either by one set of multipliers. Taking the first alternative, one set of multipliers will be respectively

$$X_3 - \frac{3}{2}Y_3, \quad -\frac{3}{2}X_3 + \frac{1}{2}Y_3, \quad -mX_4, \quad -mY_4.$$

The other set will be

$$X_4 - \frac{3}{2}Y_4, \quad -\frac{3}{2}X_4 + \frac{1}{2}Y_4, \quad +mX_3, \quad +mY_3.$$

16. PROP. B.—To determine the multipliers when the fundamental determinant is symmetrical and the forces of resistance absent.

This proposition is really included in the last. But as the absence of the function B introduces great simplification, it is worth while to consider this case separately.

Since the forces of resistance are absent, none but even powers of δ enter into the equations. Hence for every root of the fundamental determinant there is another equal in magnitude but contrary in sign. If A and O are one-signed functions, and have same sign, these roots are of the form $\pm p\sqrt{-1}$. Choosing this as the type, we may write the equations of Art. (8) in the form

$$\begin{aligned}x &= X_1 \cos pt + X_2 \sin pt + x_3 e^{m_3 t} + \dots \\&\&c. = \&c., \\ \frac{dx}{dt} &= X'_1 \cos pt + X'_2 \sin pt + x'_3 e^{m_3 t} + \dots \\&\&c. = \&c.\end{aligned}$$

Here, unless there be equal roots, we have

$$\frac{X_2}{X_1} = \frac{Y_2}{Y_1} = \&c. = \frac{X'_1}{-X'_2} = \frac{Y'_1}{-Y'_2} = \&c. = H,$$

because the ratios of the coefficients of any exponential are expressed by the minors of the fundamental determinant, and these, containing only even powers of m , are the same when the exponents are equal in magnitude but contrary in sign.

Here H will stand for the constant in the second column on the right-hand side of the equations, the constant in the first column being included as a factor in X_1 , Y_1 , &c., X'_2 , Y'_2 , &c.

Since the function B is zero, the equations (2) of Art. (14) reduce to

$$\begin{aligned}A(x_1, x_2) &= 0, \\ O(x_1, x_2) &= 0,\end{aligned}$$

except when $m_1 = \pm m_2$. For a pair of imaginary roots $m_1 = r + p\sqrt{-1}$, $m_2 = r - p\sqrt{-1}$, combined with a third root m_3 , we have (exactly as in that article)

$$\left. \begin{aligned}A(X_1, x_3) &= 0 \\ A(X_2, x_3) &= 0\end{aligned} \right\}, \quad \left. \begin{aligned}O(X_1, x_3) &= 0 \\ O(X_2, x_3) &= 0\end{aligned} \right\}.$$

We may use either the function A or the function O to supply the proper multipliers. We thus find two sets of multipliers. Which we should choose depends on the forms of A and O .

If either of these functions contain only the squares of the co-ordinates, i.e., if it be of the form

$$ax^2 + by^2 + cz^2 + \dots,$$

it is clear that its differential coefficients will be much simpler than if the terms containing the products of the coordinates were also present. The multipliers are indicated by these differential coefficients, and will therefore also be simpler. That function is therefore to be chosen which has the fewest terms containing the products of the coordinates.

Choosing the function A , we have the following rule to find the multipliers. Let it be required to separate from the others any particular oscillation—say the two columns containing the phase pt . The proper multipliers for the coordinates x, y, z , &c. are the values of $\frac{dA}{dx}, \frac{dA}{dy}$, &c., after we have substituted for x, y , &c. in these multipliers the coefficients of either of the columns containing the phase pt . Adding these products, we have one equation from which all the oscillations except the one to be preserved have disappeared. The same multipliers may now be used for the velocities, and thus by a second addition we obtain another equation of the same kind.

The two equations thus obtained may be written thus:—

$$x \frac{dA(X_1 X_1)}{dX_1} + \&c. = 2A(X_1 X_1) \{\cos pt + H \sin pt\},$$

$$\frac{dx}{dt} \frac{dA(X_1 X_1)}{dX_1} + \&c. = 2A(X_1 X_1) \{Hp \cos pt - p \sin pt\}.$$

Putting $t = 0$ either before or after using the multipliers, we have two equations to determine H and the other constant included in X_1, Y_1 , &c.

17. PROP. C.—To determine the multipliers when the forces of resistance are absent but the determinant is skewed by the centrifugal forces.

Referring to the equations in Art. (1), we omit the terms which depend on the letters B, D, F , but retain those which depend on E (Art. 13). Eliminating x, y , &c., we form the determinant which we have called the fundamental determinant. It is unnecessary to write this determinant, as its form is evident from the merest inspection of the equations.

If in this determinant we write $-\delta$ for δ , the rows of the new determinant are the same as the columns of the old, so that the determinant is unaltered. When expanded, the determinant will contain only even powers of δ , and therefore its roots enter in pairs. We shall therefore take as our standard form of solution, instead of that in Art. (8), the expressions

$$\left. \begin{aligned} x &= X_1 \cos pt + X_2 \sin pt + x_3 e^{m_3 t} + \dots \\ y &= Y_1 \cos pt + Y_2 \sin pt + y_3 e^{m_3 t} + \dots \\ \&c. &= \&c. \end{aligned} \right\} \dots\dots\dots(1);$$

$$\left. \begin{aligned} \frac{dx}{dt} &= X'_1 \cos pt + X'_2 \sin pt + x'_3 e^{m_3 t} + \dots \\ \frac{dy}{dt} &= Y'_1 \cos pt + Y'_2 \sin pt + y'_3 e^{m_3 t} + \dots \\ \&c. &= \&c. \end{aligned} \right\} \dots\dots\dots(2);$$

Here the first two columns represent the most common form of a principal oscillation, and the third column represents any other form. When the centrifugal forces (*i.e.* the terms depending on E) are present, the minors of the fundamental determinant do not contain only even powers of δ . It follows that the coefficients in the second column do not necessarily bear a uniform ratio to those in the first column.

We have by Art. (7) the equations

$$\left. \begin{aligned} A(x_1 x_2) m_1 + O(x_1 x_2) \frac{1}{m_1} &= E(x_1 y_2) \\ A(x_1 x_2) m_2 + O(x_1 x_2) \frac{1}{m_2} &= -E(x_1 y_2) \end{aligned} \right\} \dots\dots\dots (3).$$

Adding these to eliminate the functional symbol E , we find

$$A(x_1 x_2) m_1 m_2 + O(x_1 x_2) = 0 \dots\dots\dots (4),$$

except when $m_1 = -m_2$.

We notice also, that by Art. (7),

$$\left. \begin{aligned} A(x_1 x_1) m_1^2 + O(x_1 x_1) &= 0 \\ A(x_2 x_2) m_2^2 + O(x_2 x_2) &= 0 \end{aligned} \right\} \dots\dots\dots (5).$$

We might also eliminate the function A or O from the equations (3) instead of the function E , and in each case we may deduce a rule to find the multipliers; but the simplest rule is found by eliminating the function E .

The formulæ (4) resemble that used in Art. (14), and there called (2), except in the sign of A . Proceeding therefore exactly as in that article, we shall deduce the corresponding rule for the multipliers.

Instead of equations (3) of Art. (14), we now have (since $r = 0$)

$$\left. \begin{aligned} A(x_1 x_2) p \sqrt{-1} m_2 + O(x_1 x_2) &= 0 \\ -A(x_1 x_2) p \sqrt{-1} m_1 + O(x_1 x_2) &= 0 \end{aligned} \right\} \dots\dots\dots (6).$$

Remembering that A and O are linear functions of the letters of any one suffix, these give by addition and subtraction

$$\left. \begin{aligned} A(X'_1 x_2) m_2 + O(X_1 x_2) &= 0 \\ A(X'_2 x_2) m_2 + O(X_1 x_2) &= 0 \end{aligned} \right\} \dots\dots\dots (7),$$

where as before $X_1 = x_1 + x_2$, $X_2 = (x_1 - x_2) \sqrt{-1}$,

and

$$X'_1 = p X_2, \quad X'_2 = -p X_1.$$

Also writing $m_1 = p \sqrt{-1}$, $m_2 = -p \sqrt{-1}$ in equations (5), we find by subtraction

$$A(X'_1 X'_2) + O(X_1 X_2) = 0 \dots\dots\dots (8).$$

From these formulæ we now deduce the following rule to find the multipliers.

Let the forces of resistance be absent, and let the fundamental determinant be skewed by the centrifugal forces only. Let it be required to separate any principal oscillation from the others. *Selecting one of the two columns which form the oscillation, the proper multipliers for the coordinates x, y , &c. are the values of $\frac{dC}{dx}, \frac{dC}{dy}$, &c., after we have substituted for x, y , &c. in these multipliers the corresponding coefficients in the column selected. The proper multipliers for the velocities are the values of $\frac{dA}{dx}, \frac{dA}{dy}$, &c., after we have substituted for x, y , &c. in these multipliers the coefficients corresponding to these velocities in the column selected. Finally, we add all these products together. We then repeat the process with the coefficients of the other of the two columns which form the oscillation.*

By virtue of equations (5) and (8) it will be found that in each of these processes every column *except one* will disappear from the final summation. But we may notice a curious difference between the columns which contain real exponentials and those which contain trigonometrical expressions. If we operate with the coefficients of one of the former introduced into the multipliers, it is the *companion column which does not disappear*; but if we operate with the coefficients of one of the latter, it is the *column whose coefficients we have used which does not disappear*.

18. *Example.*—Let us consider the equations

$$\left. \begin{aligned} (\delta^2 - 8)x + \sqrt{6}\delta y &= 0 \\ -\sqrt{6}\delta x + (\delta^2 + 2)y &= 0 \end{aligned} \right\}.$$

It is easily seen that the fundamental determinant reduces to

$$m^4 - 16 = 0.$$

Hence we have

$$\left. \begin{aligned} x &= X_1 \cos 2t + X_2 \sin 2t + x_3 e^{2t} + x_4 e^{-2t} \\ y &= Y_1 \cos 2t + Y_2 \sin 2t + y_3 e^{2t} + y_4 e^{-2t} \\ \frac{dx}{dt} &= 2X_1 \cos 2t - 2X_2 \sin 2t + 2x_3 e^{2t} - 2x_4 e^{-2t} \\ \frac{dy}{dt} &= 2Y_1 \cos 2t - 2Y_2 \sin 2t + 2y_3 e^{2t} - 2y_4 e^{-2t} \end{aligned} \right\};$$

$$\text{where } \left. \begin{aligned} 2x_3 &= \sqrt{6}y_3 \\ 2x_4 &= -\sqrt{6}y_4 \end{aligned} \right\}, \quad \left. \begin{aligned} Y_1 &= -\sqrt{6}X_1 \\ Y_2 &= \sqrt{6}X_1 \end{aligned} \right\}.$$

Also multiplying the equations (see Art. 14) by x, y , adding and

taking the halves of the coefficients of the powers of δ ,

$$\left. \begin{aligned} A &= \frac{1}{2} (x^2 + y^2) \\ O &= \frac{1}{2} (-8x^2 + 2y^2) \end{aligned} \right\}.$$

The proper multipliers are indicated (Art. 17) by the formula

$$x \frac{dO}{dx} + y \frac{dO}{dy} + \frac{dx}{dt} \frac{dA}{dx} + \frac{dy}{dt} \frac{dA}{dy}.$$

Now $\frac{dO}{dx} = -8x, \quad \frac{dO}{dy} = 2y, \quad \frac{dA}{dx} = x, \quad \frac{dA}{dy} = y.$

Having chosen the column whose coefficients are to be used in the multipliers, we see by Art. (16) that the proper multiplier for the first equation is minus eight times the coefficient of the column in that equation; the proper multiplier for the second equation is twice the coefficient in that equation; the proper multipliers for the third and fourth equations are the coefficients themselves in those equations.

Suppose first we wish to find x_4 and y_4 , then, because the fourth column contains a *real* exponential, we operate with the coefficients of the companion column. The multipliers are therefore

$$\frac{dO}{dx} = -8x_3, \quad \frac{dO}{dy} = 2y_3, \quad \frac{dA}{dx} = 2x_3, \quad \frac{dA}{dy} = 2y_3.$$

Hence we find

$$-8x_3x + 2y_3y + 2x_3 \frac{dx}{dt} + 2y_3 \frac{dy}{dt} = 16y_3y_4e^{-2t};$$

substituting for x_3 in terms of y_3 and putting $t = 0$, we find

$$-4\sqrt{6}x + 2y + \sqrt{6} \frac{dx}{dt} + 2 \frac{dy}{dt} = 16y_4,$$

which determines y_4 in terms of the initial values of the coordinates and their velocities.

Suppose next we wish to find X_1, X_2 . Taking the coefficients of the first column, the multipliers are

$$\frac{dO}{dx} = -8X_1, \quad \frac{dO}{dy} = 2Y_1, \quad \frac{dA}{dx} = 2X_2, \quad \frac{dA}{dy} = 2Y_2.$$

Since these columns contain trigonometrical expressions, we know that when we operate with the coefficients of either column in the multipliers, the other column disappears. Hence, paying no attention to any column except the first, we have

$$-8X_1x + 2Y_1y + 2X_2 \frac{dx}{dt} + 2Y_2 \frac{dy}{dt} = 16(X_1^2 + X_2^2) \cos 2t;$$

substituting for Y_1 and Y_2 and putting $t = 0$, we find

$$-8X_1x - 2\sqrt{6}X_2y + 2X_2 \frac{dx}{dt} + 2\sqrt{6}X_1 \frac{dy}{dt} = 16(X_1^2 + X_2^2).$$

Operating in the same way with the coefficients of the second column, we have

$$-8X_2x + 2Y_2y - 2X_1\frac{dx}{dt} - 2Y_1\frac{dy}{dt} = 16(X_1^2 + X_2^2)\sin 2t;$$

substituting as before, we have

$$-8X_2x + 2\sqrt{6}X_1y - 2X_1\frac{dx}{dt} + 2\sqrt{6}X_2\frac{dy}{dt} = 0.$$

These equations determine X_1 and X_2 in terms of the initial values of x, y , and their differential coefficients.

19. PROP. D.—*To consider the effect of equal roots on the rules already given.*

When there are equal roots in the fundamental determinant, we require only some slight modification of our rules. Referring to the general solution exhibited in Art. (4), let us suppose, for example, that there are three roots equal to m_1 . Regarding these as the limits of the unequal roots, m_1, m_1+h, m_1+k , we may write that solution in the form

$$x = x_1 e^{m_1 t} + G \frac{d}{dm_1} (x_1 e^{m_1 t}) + H \frac{d^2}{dm_1^2} (x_1 e^{m_1 t}) + x_4 e^{m_4 t} + \dots$$

$$y = y_1 e^{m_1 t} + G \frac{d}{dm_1} (y_1 e^{m_1 t}) + H \frac{d^2}{dm_1^2} (y_1 e^{m_1 t}) + y_4 e^{m_4 t} + \dots$$

&c. = &c.,

$$\frac{dx}{dt} = x'_1 e^{m_1 t} + \frac{d}{dm_1} (x'_1 e^{m_1 t}) + \frac{d^2}{dm_1^2} (x'_1 e^{m_1 t}) + x'_4 e^{m_4 t} + \dots$$

&c. = &c.;

where $x'_1 = x_1 m_1$, $x'_4 = x_4 m_4$, &c., and G, H are the two constants in addition to the one included in x_1, y_1 , &c.

Two questions now present themselves:—(1) When we use certain multipliers to separate a column which depends on a solitary root such as m_4 , will the columns which depend on other equal roots such as m_1 (and therefore contain powers of t as factors) still disappear?

(2) What multipliers must we use to separate the three columns which depend on the three equal roots from the remaining columns?

20. Taking the first of these questions, suppose we wish to separate the fourth column of the equations of Art. (19) from the others. Let us use the same multipliers as if there were no equal roots. It is obvious that, since the three first columns disappear in the general case in which h and k have any values, these columns must also disappear when h and k are indefinitely small. We therefore infer that any column which depends on a solitary root may be separated by the same rules as before.

As an example, take the rule given in Prop. A, Art. (14). To separate the fourth column, we multiply the equations by

$$\frac{dC(x_4x_4)}{dx_4} \&c. - \frac{dA(x_4x_4)}{dx_4'} \&c.,$$

and add the products. Since the three first columns must disappear, we have

$$\left. \begin{aligned} C(x_1x_4) - A(x_1'x_4') &= 0 \\ C\left(\frac{dx_1}{dm_1}x_4\right) - A\left(\frac{dx_1'}{dm_1}x_4'\right) &= 0 \\ C\left(\frac{d^2x_1}{dm_1^2}x_4\right) - A\left(\frac{d^2x_1'}{dm_1^2}x_4'\right) &= 0 \end{aligned} \right\}.$$

The last two of these equations also follow from the first by an evident process.

21. Taking the second question, we wish to find what multipliers will separate the three first columns from the others. But these are supplied by the equations just written down. Since m_4 is any other root, and

$$2C(x_1x_4) = \frac{dC}{dx_1}x_4 + \frac{dC}{dy_1}y_4 + \dots,$$

we have merely to use the multipliers indicated by the coefficients of x_4 , y_4 , &c. in these equations. The rule may be enunciated as follows:—

Multiply the equations by the proper factors for the first column, treating x_1 , y_1 , &c., x_1' , y_1' , &c. as the coefficients, and add the products. We thus have one of the three required equations. Multiply the equations by the proper factors for the second column as if $\frac{dx_1}{dm_1}$, $\frac{dy_1}{dm_1}$, &c., $\frac{dx_1'}{dm_1}$, $\frac{dy_1'}{dm_1}$, &c. were the coefficients, and add the products. We thus obtain the second equation. Lastly, multiply the equation by the proper factors for the third column as if $\frac{d^2x_1}{dm_1^2}$, &c., $\frac{d^2x_1'}{dm_1^2}$, &c. were the coefficients, and add the products. We thus have, on the whole, three equations to find the three constants which enter into the three first columns.

The proper factors just mentioned are those calculated from the coefficients by the rules of Prop. A or Prop. C.

22.* In some cases of equal roots it is known that some of the terms with t as a factor fail to introduce themselves into the solution. The number of constants is then made up by a greater indeterminateness in the coefficients which accompany the exponential. Regarding these equal roots as the limits of unequal roots, as in Art. 20, it

* This and the next article were added after the paper was read.

follows that we can still use the same rules to find the multipliers. We arrange our solution in columns with one constant in each column. Then, using the proper multipliers, as described above, we can separate any solitary root at once. To determine the constants which accompany the equal roots, we shall require as many sets of multipliers as there are columns with that root or its companion root.

23. *Example.*—Let us consider the equations

$$\left. \begin{aligned} (\delta^2 - 1)x + y + z &= 0 \\ x + (\delta^2 - 1)y + z &= 0 \\ x + y + (\delta^2 - 1)z &= 0 \end{aligned} \right\}.$$

It is easily seen that the fundamental determinant reduces to

$$(m^2 - 2)^2 (m^2 + 1) = 0.$$

Putting $\alpha = \sqrt{2}$, we write the solution in the form

$$\left. \begin{aligned} x &= Ee^{\alpha t} + Ge^{-\alpha t} + K \sin t + L \cos t \\ y &= Fe^{\alpha t} + He^{-\alpha t} + K \sin t + L \cos t \\ z &= -Ee^{\alpha t} - Fe^{\alpha t} - Ge^{-\alpha t} - He^{-\alpha t} + K \sin t + L \cos t \end{aligned} \right\},$$

where E, F, G, H, K, L are the six constants to be determined.

Looking at the equations to be solved, we see that the potential functions A and C are given by

$$\left. \begin{aligned} 2C &= -x^2 - y^2 - z^2 + 2xy + 2yz + 2zx \\ 2A &= x^2 + y^2 + z^2 \end{aligned} \right\}.$$

Following the rule indicated in Art. 16, we choose the function A to operate with, because this function will supply the simplest multipliers. The proper multipliers will therefore be

$$\frac{dA}{dx} = x, \quad \frac{dA}{dy} = y, \quad \frac{dA}{dz} = z,$$

where we write for x, y, z the coefficients of the column under consideration. The proper multipliers are therefore the coefficients of the columns in succession.

Suppose we wish to find K and L . The coefficients in either of these two columns are all equal. The multipliers are therefore equal. We therefore obtain, by adding the equations and putting $t = 0$,

$$x + y + z = 3L.$$

Treating the differential coefficients in the same way (Art. 16), we have

$$\delta x + \delta y + \delta z = 3K.$$

If we wish to find the four constants E, F, G, H which are all connected with the companion roots $\pm \alpha$, we must find four equations.

According to the rule, the multipliers are the coefficients of the several columns. We thus obtain, when $t = 0$,

$$\begin{aligned} Ex + 0y - Ez &= E(2E + 2G + F + H) \\ Gx + Fy - Fz &= F(E + G + 2F + 2H) \\ E\delta x + 0\delta y - E\delta z &= E\alpha(2E - 2G + F - H) \\ 0\delta x + F\delta y - F\delta z &= F\alpha(E - G + 2F - 2H) \end{aligned} \left. \vphantom{\begin{aligned} Ex + 0y - Ez \\ Gx + Fy - Fz \\ E\delta x + 0\delta y - E\delta z \\ 0\delta x + F\delta y - F\delta z \end{aligned}} \right\}.$$

This simple and obvious example sufficiently illustrates the method of proceeding when the proper multipliers could not be otherwise found.

Some Remarks on those Solutions of Linear Differential Equations which depend on Multiple Types. By E. J. ROUTH, F.R.S.

[Added July 20th, 1883.]

When a set of linear differential equations is given to find x, y , &c. in terms of an independent variable t , the solution in general consists of a set of exponentials real or imaginary, the exponents depending on the roots of a certain determinant. If some of these roots be equal, the exponent is sometimes accompanied by powers of t as factors; and sometimes, instead of these powers of t , we have merely a greater indeterminateness in the constants which accompany the exponential. Occasionally both these occur at once.

We now propose to consider under what circumstances this last event can happen. We begin with a statement of the ordinary method of solution as the shortest way of explaining the notation. We first find that each term of the form $t^n e^{mt}$ is accompanied by one constant. We then discover that this solution becomes indeterminate in a certain case. Obtaining another solution, we find that each term of the form just given is now accompanied by two constants. If the solution again become indeterminate, the term reappears with three constants, and so on. Sometimes we find that some of the powers of t which multiply a given exponential have two, while other powers have three, independent constants. Our object is to construct a rule to determine beforehand the number of independent constants which accompany any term of the solution.

The problem to be solved is enunciated in Art. (30), and the result is stated in Art. (37).

24. The equations which occur in Dynamics are, in general, all of the second order; but, as this restriction is not necessary in what follows, we shall suppose the equations to contain differential coefficients of any order.