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**"Line Effects in Long-distance Transmission of Power
by Electricity."**

By HENRY JAMES SHEDLOCK HEATHER, B.A., Assoc. M. Inst. C.E.

IN practice, the problems to be solved in designing lines for the transmission of power by electricity over long distances differ from those met with in dealing with short lines, not only in the magnitude of the quantities involved, but also in the nature of the considerations that have to be taken into account; and the differences are seen both from the technical and from the commercial point of view.

In the early days of power-transmission, when only continuous currents and what are now regarded as very low voltages were thought of, both efficiency of transmission and facility of regulation were secured by using large quantities of copper, the drop in voltage and the loss in watts on the line both depending solely on its electrical resistance. Interesting calculations were made in order to ascertain the most favourable relation to be observed between the horse-power to be transmitted and the cost of the copper. This relation was modified by the inclusion of a factor representing the cost of the insulation of the copper, and the line problem was solved, the attainment of a fairly high efficiency itself rendering small the range over which regulation would be required. The distance to which transmission could be economically effected was limited by the maximum voltage for which continuous-current machines could be built, for it was of course recognized that line-losses were reduced by increasing the voltage.

The knowledge that by using alternating currents much higher voltages could be rendered practically available was not of much use in extending the limits of transmission of power, owing to the absence of a reliable alternating-current motor. When the development of polyphase motors and polyphase distribution removed this obstacle it was at first thought that the distance to which electrical power could be transmitted would in the future be limited only by the maximum voltage for which machinery

could be built, exactly as had been found to be the case for continuous currents. Difficulties, however, of the same nature as those met with in cable-telegraphy, were soon encountered when underground cables were used; but as, owing to the high cost of cables, long-distance power-transmission had to be carried out by means of overhead lines, on which the effect of the complicating factors was of much less relative importance, there was, perhaps, a tendency to under-estimate these difficulties, which may become very serious when transmission to a distance of 100 miles or more is under consideration.

The fact that it is found advisable in alternating-current power-transmission so to design the generators and motors that the voltage and the current may vary harmonically, their instantaneous values, when plotted as ordinates to a time-base, giving as nearly as possible true sine-curves, has rendered it practicable to calculate accurately instantaneous values for the voltage and current at one end of any given line in terms of the values at the other end. The properties of the line, in addition to the length and the resistance per unit of length, which have to be known in order to do this, are the coefficient of self-induction and the capacity per unit of length. Both of these, as will be seen later, are directly derivable from the geometric data of the line.

Consider an elementary length dx of a pair of parallel wires,
Fig. 1 :—

Let PP^1 be one extremity of the elementary length.

Let QQ^1 be the other extremity,

and let OO^1 represent the ends of the wires at which the power is to be delivered.

Then, denoting OP by x

$$OQ = x + dx,$$

and at the point where the power is delivered $x = 0$.

Let v_x denote the voltage between P and P^1 ;

then v_{x+dx} is the voltage between Q and Q^1 .

If i_x denote the current along the wires at P , i_{x+dx} is the current along the wires at Q .

If l , r and c denote respectively the self-induction, resistance, and capacity per unit of length of the system of two wires, ldx

rdx , and cdx will be the values for the element under consideration.

Then the current along the wire at P, together with the current absorbed between P and Q by the effect of the capacity between PQ and P¹Q¹, must be equal to the current along the wire at Q. The current absorbed by this capacity is equal to the capacity multiplied by the rate of change of the voltage between the wires; that is, to $c dx \frac{dv_x}{dt}$.

$$\text{Therefore} \quad i_{x+dx} = i_x + c dx \frac{dv_x}{dt} \quad . \quad . \quad . \quad (1)$$

Also, the decrease in voltage difference along the wires is $v_{x+dx} - v_x$, and this is equal to the sum of the ohmic drop and that due to the electromotive force of self-induction; that is, to

$$ldx \frac{di_x}{dt} + r dx \cdot i_x.$$

So that

$$v_{x+dx} - v_x = l dx \cdot \frac{di_x}{dt} + r dx \cdot i_x \quad . \quad . \quad . \quad (2)$$

$$\text{But} \quad \frac{i_{x+dx} - i_x}{dx} = \frac{di_x}{dx}$$

$$\text{and} \quad \frac{v_{x+dx} - v_x}{dx} = \frac{dv_x}{dx}.$$

$$\text{Therefore} \quad \frac{di_x}{dx} = c \frac{dv_x}{dt} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{and} \quad l \frac{di_x}{dt} + r i_x = \frac{dv_x}{dx} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From these two equations the following can be at once deduced:—

$$cl \frac{d^2 v_x}{dx^2} + cr \frac{dv_x}{dx} = \frac{d^2 v_x}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{and} \quad cl \frac{d^2 i_x}{dx^2} + cr \frac{di_x}{dx} = \frac{d^2 i_x}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These two differential equations have exactly the same form. That for v_x will be first considered.

It is known that the voltage across the points O O¹ (for which $x = 0$) can be represented by an expression of the form

$$V \sin p t$$

where t represents time, p is 2π times the number of complete cycles per second, and V is the maximum voltage at the crest of

the sine-curve wave. If the current at this point lags behind the volts by a fraction of the time of a complete cycle represented by $\frac{\phi}{2\pi}$, the current at this point can, in the same way, be represented by $I \sin (pt - \phi)$, where I is its maximum value.

All expressions of the form

$$e^{at+bx}$$

satisfy equation (5), if a and b are subject to the relation

$$cl a^2 + c r a = b^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The general solution, therefore, is the sum of all terms of such form, each multiplied by an arbitrary constant, the relation expressed by equation (7) being adhered to by the coefficients of t and x in every term. No assumption has to be made as to whether a and b are real or otherwise; it is therefore necessary to consider that each may contain an imaginary as well as a real part.

At the point O^1 , where $x = 0$, the general solution has to assume the form

$$V \sin pt;$$

that is

$$\frac{V}{2\sqrt{-1}} \left(e^{\sqrt{-1}pt} - e^{-\sqrt{-1}pt} \right)$$

This shows that only two values of a , namely, $\sqrt{-1}p$ and $-\sqrt{-1}p$, need be retained. Corresponding to any one value of a , there are two values of b , equal in magnitude, but opposite in sign; that this is so is shown by equation (7).

Consequently, in all, four terms have to be retained in the general solution of equation (5), which accordingly becomes—

$$\left. \begin{aligned} v_x = & A_1 e^{\sqrt{-1}pt + \sqrt{-clp^2 + crp}\sqrt{-1}x} \\ & + A_2 e^{\sqrt{-1}pt - \sqrt{-clp^2 + crp}\sqrt{-1}x} \\ & + A_3 e^{-\sqrt{-1}pt + \sqrt{-clp^2 - crp}\sqrt{-1}x} \\ & + A_4 e^{-\sqrt{-1}pt - \sqrt{-clp^2 - crp}\sqrt{-1}x} \end{aligned} \right\} \quad . \quad . \quad (8)$$

By similar reasoning, remembering that i_x at the point O has to be $I \sin (pt - \phi)$, from equation (6) can be obtained an expression for i_x similar to the foregoing expression for v_x , but differing in that $pt - \phi$ occurs in it in place of the pt in the value for v_x , and that the arbitrary constants are of course different. Comparing these values for v_x and i_x with the relations (3) and (4), and remembering their respective values $V \sin pt$ and

I $\sin (pt - \phi)$ at the point for which $x = 0$, all the constants may be determined in terms of known quantities, and finally the equations are obtained in the forms:—

$$v_x = \left\{ \begin{aligned} & \frac{V}{2} - \frac{I}{2cp} (\beta \cos \phi - \alpha \sin \phi) \Big\} e^{-\alpha x} \sin (pt - \beta x) \\ & + \left\{ \frac{V}{2} + \frac{I}{2cp} (\beta \cos \phi - \alpha \sin \phi) \Big\} e^{\alpha x} \sin (pt + \beta x) \\ & + \frac{I}{2cp} (\alpha \cos \phi + \beta \sin \phi) e^{-\alpha x} \cos (pt - \beta x) \\ & - \frac{I}{2cp} (\alpha \cos \phi + \beta \sin \phi) e^{\alpha x} \cos (pt + \beta x) \end{aligned} \right\} \quad (9)$$

$$i_x = \left\{ \begin{aligned} & \frac{I}{2} - \frac{V}{2\sqrt{r^2 + p^2 l^2}} (\beta \cos \phi - \alpha \sin \phi) \Big\} e^{-\alpha x} \sin (pt - \beta x - \phi) \\ & + \left\{ \frac{I}{2} + \frac{V}{2\sqrt{r^2 + p^2 l^2}} (\beta \cos \phi - \alpha \sin \phi) \Big\} e^{\alpha x} \sin (pt + \beta x - \phi) \\ & - \frac{V}{2\sqrt{r^2 + p^2 l^2}} (\alpha \cos \phi + \beta \sin \phi) e^{-\alpha x} \cos (pt - \beta x - \phi) \\ & + \frac{V}{2\sqrt{r^2 + p^2 l^2}} (\alpha \cos \phi + \beta \sin \phi) e^{\alpha x} \cos (pt + \beta x - \phi) \end{aligned} \right\} \quad (10)$$

in which

$$\alpha = \sqrt{\frac{cp}{2} (\sqrt{r^2 + p^2 l^2} - pl)} \quad . \quad . \quad . \quad (11)^1$$

$$\beta = \sqrt{\frac{cp}{2} (\sqrt{r^2 + p^2 l^2} + pl)} \quad . \quad . \quad . \quad (12)^1$$

These equations give full particulars of the voltage and current at any point of the line; that is to say, they give magnitudes and phases of both in terms of those at the origin, *i.e.* of those required at the point where the power is to be delivered. The phase-difference between the volts and the amperes at the power-station is also therefore deducible for any given load. It is shown by the form of the equations that the volts and amperes follow a wave law with respect to distance, x , as well as with respect to time, t , although further investigation reveals that, even in the case of transmission to a distance of 200 or 300 miles, only a small portion of the first wave-length is covered. This is shown by a quantitative

¹ Since this Paper was written the Author has learned that these and the preceding general results are not novel; similar expressions may be found in Bedell and Crehore's "Alternating Currents" (Whittaker & Co., 1895), pp. 182-3 and 190-1.

determination of the values of α and β in general practice, which now becomes necessary.

As already stated, p is 2π times the number of complete cycles per second, which can theoretically be given any value whatever; the lower the value selected, the higher become the cost and weight of the transforming and generating plant, and the lower becomes their efficiency at light loads. The latter point is seldom important in long-distance transmission-schemes, as in such projects a cheap source of power is usually implied. A more important consideration than increased cost of the generating-plant, which after all only represents a small fraction of the cost of the line, is the fact that at very low frequencies the regulation of the generators becomes exceedingly bad. As an ordinary case, a frequency of 24 cycles per second may be taken, so that p is about 150.

The resistance per unit length of the line, r , depends on the material and cross-sectional area of the conductor. The increase in resistance due to the effect of self-induction internal to the wire (investigated by Lord Rayleigh and Lord Kelvin), which is large enough to demand attention in very heavy conductors at high frequencies, is negligible in long-distance transmissions, where low frequencies and comparatively small conductors must be used. If considered at all in these cases it should be regarded as a simple but very slight increase in resistance. In copper of ordinary sizes it will not amount to one-half of 1 per cent. It would be much increased if aluminium were used, but this is not likely to be the case in long-distance work owing to the important increase in the capacity-effects of the line which the necessity of using a wire of larger diameter would involve. A copper wire of, say, 0.425 inch in diameter has a resistance of 0.3 ohm per mile, giving for the resistance of a mile of a system of two wires 0.6 ohm.

There remain to be fixed the capacity and the self-induction for the unit of length. For cylindrical parallel wires these are easily calculated, as each depends on a simple integration. The results obtained are for the capacity per unit of length of two parallel cylindrical wires of radius a , and distance apart b .

$$c = \frac{1}{4 \log. \frac{b}{a}} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For the self-induction of the same two wires per unit of length the result is

$$l = 4 \log. \frac{b}{a} + 1 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It is noteworthy that both of these expressions are completely fixed when the ratio between the radius of, and the distance between, the wires is fixed, so that a proportionate increase or decrease of both the radius and the distance leaves the two results unaltered. Further, any change in this ratio which decreases the capacity increases the self-induction, and *vice versa*; also, as only the logarithm of the ratio appears, it is clear that an alteration of either the radius or the distance apart does not produce by any means a proportionate alteration in the capacity and self-induction. Consequently no very great variation is produced in these, even when the geometry of the system is very largely changed.

Of the two expressions (13) and (14), (13) gives the capacity in absolute electrostatic units, and (14) gives the self-induction in absolute electro-magnetic units. Taking the diameter of the wire, as before, at 0.425 inch, and the distance between the two wires as about 18 inches, c becomes, when converted to farads per mile, approximately

$$1 \times 10^{-8}.$$

and l similarly converted to henrys per mile becomes

$$3 \times 10^{-3}.$$

Since r , c , and l are determined in ohms, farads, and henrys, respectively, per mile of the system, x will be in miles, and v_x and i_x will give instantaneous values in volts and amperes respectively when V and I are the maximum values in volts and amperes respectively at the point of delivery.

Putting these values for p , r , c , and l in equations (11) and (12), the values found for α and β are

$$\alpha = \frac{4.7}{10^4}$$

$$\beta = \frac{9.5}{10^4}$$

α and β are thus very small quantities, and even for such values of x as 400 miles, αx and βx are proper fractions. Use may therefore be made of the expansions in powers of αx and βx for the expressions $e^{\alpha x}$, $e^{-\alpha x}$, $\sin \beta x$ and $\cos \beta x$. If these substitutions be made in the general equations (9) and (10) and special values be given to certain of the constants, a check may be obtained on the accuracy of the expressions. For instance, if c be given the value 0, which makes α and β also equal to 0, the two equations becomes

$$\begin{aligned} v_x &= V \sin pt + plIx \cos(pt - \phi) \\ &\quad + rIx \sin(pt - \phi) \\ i_x &= I \sin(pt - \phi) \end{aligned}$$

which are the ordinary equations in the case of conductors considered to have resistance and self-induction only.

Again, if I is 0, the equation for i_x becomes

$$i_x = \frac{V \sqrt{cp}}{\sqrt{r^2 + p^2 l^2}} \cdot \sqrt{e^{2ax} + e^{-2ax} - 2 \cos 2\beta x} \cdot \sin(pt + \theta)$$

showing that the maximum value (or the virtual value, i.e., the square root of the mean square if V is given as a virtual value) of the current that flows into the unloaded line is

$$\frac{V \sqrt{cp}}{\sqrt{r^2 + p^2 l^2}} \cdot \sqrt{e^{2ax} + e^{-2ax} - 2 \cos 2\beta x}.$$

If both r and l are zero, a supposition which makes α and β also zero, this simplifies to

$$V c p x.$$

an expression commonly used for the charging-current of a line. From the foregoing reasoning, however, it is evident that it would only be strictly accurate if r and l were both zero, which, of course, is never the case, although the error introduced by neglecting higher powers of αx and βx is not in most cases large enough to be important.

By a common trigonometrical transformation the expression for v_x can be put into the form

$$v_x = \sqrt{L.V^2 + M.V.I. + N.I^2} \cdot \sin(pt + \theta). \quad (15)$$

where $L = \frac{1}{4}(e^{2ax} + e^{-2ax} + 2 \cos 2\beta x)$

$$M = \frac{1}{2cp} \{ (\beta \cos \phi - \alpha \sin \phi)(e^{2ax} - e^{-2ax}) + 2(\alpha \cos \phi + \beta \sin \phi)(\sin 2\beta x) \}$$

and $N = \frac{1}{4c^2 p^2}(e^{2ax} + e^{-2ax} - 2 \cos 2\beta x)(\alpha^2 + \beta^2)$

and θ is the angle of which the tangent is the fraction which has for numerator the coefficient of $\cos pt$ in equation (9) and for denominator the coefficient of $\sin pt$ in the same equation. These coefficients are very cumbrous in form, and need not be given here in full. θ then represents the angle by which the voltage at the point x leads that at the point O . The expression for i can be put into a corresponding form, and the tangent of the angle by which the amperes at the point x lead those at the point O can be similarly determined. The angle between the phases of the volts and the amperes at the point O being, by supposition, ϕ , the angle between the volts and the amperes at the point x can be determined.

So far as the direct effects of the line alone are concerned, the amount of regulation required in the voltage on account of a change in value of the current I will obviously depend on the magnitude of the coefficients M and N in equation (15), and the range of regulation is made small by making M and N small. By expanding the exponential and trigonometrical functions, the following approximate values are obtained:—

$$M = 2x(r \cos \phi + pl \sin \phi) \quad . \quad . \quad . \quad (16)$$

$$N = x^2(r^2 + p^2 l^2) \quad . \quad . \quad . \quad . \quad . \quad (17)$$

so that the condition that M and N are to be small entails primarily that r and pl , that is the resistance and the self-induction, shall be small; and these would be the conditions to be aimed at if a source of electric supply constant at the line voltage at all loads and angles of lag could be secured. In practice, however, this is not the case, as the capacity and self-induction of the line, by changing the angle of lag or lead as the load changes, have an indirect effect on the pressure-drop in the transformers and generators, which is of such magnitude as, in most cases, altogether to outweigh the direct effects. It becomes therefore of more importance to reduce the angle of lag or lead than to reduce the line-resistance.

As already explained, the angle between the phases of the volts and the amperes at the point x can be determined from the equation (15) for v_x and the corresponding equation for i_x . When the expression which is obtained for it, which in its original form is somewhat complicated, is simplified by expanding the logarithmic and trigonometrical functions and subsequently rejecting powers of αx and βx higher than the first, the approximate formula for the angle of lag that is obtained is

$$\phi + \tan^{-1} \left(\frac{plx I^2 \cos \phi - rx I^2 \sin \phi - cpx V^2}{rx I^2 \cos \phi + plx I^2 \sin \phi + V.I.} \right) \quad (18)$$

which may be reduced to

$$\tan^{-1} \left(\frac{plx I^2 - cpx V^2 \cos \phi + V.I. \sin \phi}{rx I^2 + V.I. \cos \phi + cpx V^2 \sin \phi} \right) \quad (19)$$

As this angle has to be kept small, its tangent

$$\frac{plx I^2 - cpx V^2 \cos \phi + V.I. \sin \phi}{rx I^2 + V.I. \cos \phi + cpx V^2 \sin \phi} \quad (20)$$

has also to be kept as small as possible, whatever the variations in the current I may be.

In considering this expression, the point that first attracts

attention is that any reduction of r , the resistance of the line, does not improve the regulation at all, but actually serves to make it worse. Several other interesting deductions may be made when the ratio between the normal current and the normal voltage is known, especially when it is also kept in mind that the actual energy received is $\frac{V.I.}{2} \cos \phi$, where V and I are maximum values, or $V.I. \cos \phi$ when they are the square roots of the mean square values.

In general, it is clearly an advantage to make p small; c and l should also be diminished as much as possible, the reduction of the former being of more importance when I is small, and of the latter when I is comparatively large. Whenever I is large enough to make the numerator of the fraction positive it is advisable to reduce $V.I. \sin \phi$ by making ϕ small; that is to say, the current received should if possible not be allowed to lag much when the load is heavy. On the other hand, when I is very small the terms $-cp\alpha V^2 \cos \phi$ and $cp\alpha V^2 \sin \phi$, in the numerator and denominator respectively, become of more importance than the others, so that under these circumstances the difference in phase between the volts and the amperes at the power-house (where the current now leads the voltage) can only be reduced by diminishing the ratio $\frac{\cos \phi}{\sin \phi}$; that is by making ϕ , the angle of lag at the receiving-station, large. This, as it happens, actually occurs when, as is usually the case in long-distance transmissions, step-down transformers are used at the receiving-station, for these, when not loaded, take a current that has a heavy lag.

The main requirements for good regulation, therefore, are ordinarily the following:—

1. High voltage, which, by reducing the load-current for a given power, minimizes both the indirect and direct effects of change of load; it, however, increases the indirect effects at light loads.

2. Low frequency, which reduces both the indirect and direct effects of change of load.

3. Small capacity, which reduces the indirect effects of change of load when the load is light, but does not do so when it is heavy.

4. Small self-induction, which reduces the direct effects of change of load at all loads, and the indirect effects when the load is large, but increases the indirect effects at very light loads.

5. Low resistance, which reduces the direct effects, but increases the indirect effects at all loads.

It will be seen that it is impossible to lay down very definitely any general rules that should be adhered to in all cases, as so much depends on the magnitude and nature of the loads to be expected. All that can be done is to investigate the various effects in any specific case, and then select the arrangement that is best suited on the whole to the particular conditions. The extent to which variations of some of the quantities involved can be carried has already been referred to in ascribing to them practical values for the purpose of investigating the general equations (9) and (10). It was then mentioned that the limit to the reduction of the frequency was brought about by the difficulty of designing good generators to work at very low frequencies. The connection between the values of the capacity and self-induction of an overhead line was given, and it was pointed out that an increase in the one caused a decrease in the other, and *vice versa*; also, that no very great alteration in either is possible when the distance between the wires is limited in practice by the necessity of carrying them on one line of poles. In connection with this point the effect of using duplicate lines in parallel is worth considering. This is an arrangement that is usually proposed in order that a breakdown on the line shall not necessarily entail a stoppage of power-supply during the time when repairs are being carried on. Its adoption has an important bearing on the financial side of any transmission-scheme, as it still further increases the cost of that portion of the line which is practically independent of the weight of copper used. Even on a single line this cost often, especially in climates where iron poles have to be used, exceeds the cost of the copper, so that it is very evident that the capital expended on a line cannot even approximately be considered as the cost of the wire on it. Thus, doubling the line makes the current in each portion half its original value; the resistance of each portion is twice the original value, whilst the self-induction and the capacity are only altered to a very slight extent. The result is to reduce the direct line-effects at all loads. The indirect effects may be either reduced or increased according to the ratio that the original current bears to the voltage. At light loads, when the current leads, the lead is increased by halving the current, and regulation is worse. At heavy loads, when the current lags, the lag is decreased and regulation is improved. On the whole, taking both direct and indirect effects into consideration, the usual result is an improvement in regulation from doubling the line.

The limits to the voltage that may be adopted have not yet

been referred to. The first difficulties that are met with in increasing the voltage are usually in connection with the insulators. As has been pointed out by the Author,¹ the use of high voltages necessitates large insulators, and the only materials that are electrically satisfactory—porcelain and glass—give trouble through mechanical weakness when they are made in pieces of large size. Already, however, insulators are procurable that will withstand 60,000 volts, and there appears to be no reason why a certain amount of further improvement should not be attainable if necessary. It is unlikely, however, that any great advance in this respect will ever be called for, owing to the probability of a more serious difficulty imposing a maximum limit to the voltage that may be employed. This difficulty lies in the continuous brush-discharge that occurs from wire to wire with very high voltages. In experiments in statical electricity it is found that when the surface-density of the charge on a conductor reaches 20 electrostatic units per square centimetre a brush-discharge in dry air is always maintained. The voltage between two wires of course produces a surface-density of charge, the value of which depends on the capacity between them and also on the extent of their surface, the formula connecting the two being

$$V = 2,400 \pi \rho a \log_e \frac{b}{a}.$$

where V denotes the voltage, ρ the surface-density, a the radius of each wire and b the distance between the two wires. With wires of 0.425 inch diameter at a distance of 18 inches apart this critical value would be reached at a pressure of about 110,000 volts. This pressure is attained at the crest of the wave of an alternating voltage of about 80,000 volts, assuming that the sine law is obeyed.

Consequently it would appear that for wires of this size and distance apart, a voltage of 80,000 volts is a limit which it will never be possible to exceed. Some interesting experiments showing the amount of power consumed by these brush-discharges have been carried out at Telluride, Colorado, by Mr. Mershore.

The Paper is accompanied by a diagram from which the figure in the text has been prepared.

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¹ "Electric Transmission Plants at Moodie's, De Kaap Goldfields, Transvaal." Minutes of Proceedings Inst. C.E., vol. cxli. p. 269.