V.—*A Study of Trilinear Co-ordinates: being a Consecutive Series of Seventy-two Propositions in Transversals.* By the REV. HUGH MARTIN, M.A., Free Greyfriars', Edinburgh. Communicated by Professor KELLAND.

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INTRODUCTORY REMARKS.

The following series of theorems is given as an illustration of the modern method of trilinear co-ordinates, having been wrought out after perusal of Mr FERRAR'S very lucid and elegant treatise on that subject. The demonstrations present no difficulty, requiring nothing more complicated than the formation of determinants of two and three places. Accordingly, after exhibiting the method of proof in a few instances, I have merely given the enunciations of the remaining propositions. As the series of theorems advances the manipulation becomes, of course, a little more complicated; but the co-ordinates and co-efficients always appear in such symmetry as very greatly abbreviates the task, and guarantees its accuracy. Two, or perhaps three, of these seventy-two theorems are known mathematical truths; but that so many new consecutive propositions should be so easily found, and so easily proved, is a convincing evidence of the simplicity, fertility, and power of this new and beautiful method.

Treated according to the ancient geometry, the contents of the following pages would constitute a volume of no mean dimensions; and some of the propositions, such as those which affirm that the six points $P_1, P_2, P_3, P_4, P_5, P_6$ range in a straight line, and that the seven straight lines $U_1U_2; R_1R_2, R_3R_4; S_1S_2, S_3S_4; Q_1Q_3, Q_4Q_5, Q_2Q_6$ all meet in a point, would probably have been undiscoverable.

In the admirable treatises of MULCAHY and TOWNSEND a few analogous propositions are demonstrated geometrically. Mr TOWNSEND, in particular, has a chapter in his *first* volume, on concurrent lines and co-linear points, which falls in very closely with the kind of propositions which the following series embraces. His *second* volume I have not been fortunate enough to see; but the subject is only ripening for a systematic gathering-up of the propositions that have been discovered in this line of investigation, and the following pages are presented as a humble contribution towards that desirable result.

A word or two may be permitted in reference to the additions to the terminology which must be made, and generally sanctioned by mathematicians, ere such systematic digest can be successfully accomplished. I have ventured—of course only provisionally—on one or two such additions. When the co-ordinates of two points are respectively the algebraical inverses of each other, I have called these points, in reference to each other, “inverse points;” and it is evident that a very

fine vein of mathematical truth opens up in reference to them, which it needs only a little ingenuity to work advantageously. Thus, at a glance, it is evident that if a point moves in the straight line $l\alpha + m\beta + n\gamma = 0$, its inverse moves in the locus, $\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0$; which is a conic passing through the angular points of the triangle of reference. Since writing the following pages I find Mr TOWNSEND has a chapter entitled "Theory of inverse points with respect to a circle;" and although not treated according to the trilinear method, these points, so called, will be found, I rather think, *if* so treated, to present only a case of what I have called "inverse points" in general. In the same manner I have called two lines "inverse" with respect to each other when the co-efficients of the co-ordinates are respectively the algebraical inverses of each other.

There is another relation between a special point and line which I have not ventured to designate, but to which I would respectfully call attention as requiring designation. When lines from the angular points of a triangle are drawn through any point to intersect the opposite sides, the intersections constitute the angular points of an inscribed triangle, whose sides are known to meet the corresponding sides of the original triangle in points which range in a straight line. Instead of giving a particular designation to this line, I have used the general functional symbol; and, as its position depends exclusively on the point—say P, I have called the line $\phi(P)$, in a few theorems in reference to it (Theorems XXXI.—XXXV.). Of course the inverse functional symbol ϕ^{-1} indicates the point in reference to the line, as the direct symbol indicates the line with reference to the point. This *point* and *line* are, indeed, with respect to each other, a species of pole and polar,—the line being the ordinary polar, not of the point but of its inverse,—to the imaginary conic $\alpha^2 + \beta^2 + \gamma^2 = 0$. Manifestly a special designation is necessary in a case like this, in order to secure that ease of reference and that brevity of treatment without which the pioneering work of farther investigation is brought to a stand.

Theorem LXVI. is the prize question of the "Gentleman's Diary" for 1841; and some long but good geometrical demonstrations of it are given. The proof is perfectly simple according to the trilinear method, and the co-ordinates of the point appear in a form so elegant that one could not help seeing that it must have some singular relations and be worthy of a name. I have accordingly ventured to call it the Anapole of the two given points; and, connecting it with some of the preceding results, I find a few propositions easily deducible, such as that the anapole of two inverse points and the line joining them are pole and polar, to the imaginary conic, $\alpha^2 + \beta^2 + \gamma^2 = 0$. For the three concluding theorems I am indebted to a young mathematical friend—destined, I believe and trust, to scientific eminence—Mr GEORGE M. SMITH, student in the Aberdeen University. On proposing to him the problems of finding the locus of the anapole of a central body and its planet, and

the locus of the anapole of two points which should move away from each other in a straight line with uniform velocities, I was delighted to receive demonstrations, perfectly simple and elegant, to the effect that, in the former case, the anapole moves in a straight line, let the planet move as it may; and that, in the latter case, the locus of the anapole is a conic section, and becomes a straight line if the uniform velocities are equal; and farther, that the anapole of any two points in an ellipse circumscribing the triangle of reference is invariable. Geometers, I am sure, will admire these theorems of a rising young mathematician, and will recognise the vein thus struck as promising to be a fertile one. Mr SMITH added another very beautiful property of the anapole, which turned out, on investigation, to be identical with Theorem LXV. of the following series.

Instead of defining a point by the equations $\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n}$, we shall say the point is— (l, m, n) . Instead of defining a straight line by the equation $l\alpha + m\beta + n\gamma = 0$, we shall say the line is— (l, m, n) .

The straight line joining the points (l_1, m_1, n_1) , (l_2, m_2, n_2) is;—

$$\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}, \quad \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, \quad \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}.$$

The intersection of the two straight lines (l_1, m_1, n_1) , (l_2, m_2, n_2) is defined by the same expression. This identity of form is, *in reality*, the earliest germ of the doctrine of pole and polar; and gives rise to what is *usually* regarded as the first promise of that doctrine, namely, the identity of the condition that the three points (l_1, m_1, n_1) , (l_2, m_2, n_2) , (l_3, m_3, n_3) shall range in a straight line, with the condition that the three straight lines (l_1, m_1, n_1) , (l_2, m_2, n_2) , (l_3, m_3, n_3) shall intersect in a point; which is, in both cases,

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \equiv 0.$$

T H E O R E M S.

THEOREM 1.

On the sides of the triangle ABC , as bases, are constructed three triangles, A_1BC , AB_1C , ABC_1 , similar to each other, and so placed that the angles $A_1BC = A_1B_1C = A_1BC_1$; $B_1CA = B_1C_1A = B_1CA_1$; and $C_1AB = C_1A_1B = C_1AB_1$. Then AA_1 , BB_1 , CC_1 meet in a point.

The sides of ABC , taken as the triangle of reference, being a, b, c , the perpendiculars from A_1 on a, b, c are, respectively,

$$\frac{a \cdot \sin B_1 \cdot \sin C_1}{\sin A_1}, \quad \frac{a \cdot \sin B_1 \cdot \sin (C + C_1)}{\sin A_1}, \quad \frac{a \cdot \sin C_1 \cdot \sin (B + B_1)}{\sin A_1}.$$

Hence the trilinear co-ordinates of A_1 are $-1, \frac{\sin (C+C_1)}{\sin C_1}, \frac{\sin (B+B_1)}{\sin B_1}.$

„ „ of B_1 are $\frac{\sin (C+C_1)}{\sin C_1}, -1, \frac{\sin (A+A_1)}{\sin A_1}.$

„ „ of C_1 are $\frac{\sin (B+B_1)}{\sin B_1}, \frac{\sin (C+C_1)}{\sin C_1}, -1.$

For $\frac{\sin (A+A_1)}{\sin A_1}, \frac{\sin (B+B_1)}{\sin B_1}, \frac{\sin (C+C_1)}{\sin C_1},$ put $f, g, h.$ Then ;—

$$\left. \begin{array}{l} A_1 \text{ is } -1, \quad h, \quad g. \\ B_1 \text{ is } \quad h, -1, \quad f. \\ C_1 \text{ is } \quad g, \quad f, -1. \end{array} \right\} \text{ Also, } \left\{ \begin{array}{l} A \text{ is } 1, 0, 0. \\ B \text{ is } 0, 1, 0. \\ C \text{ is } 0, 0, 1. \end{array} \right\} \text{ Hence ;—}$$

$$\left. \begin{array}{l} A A_1 \text{ is } \\ B B_1 \text{ is } \\ C C_1 \text{ is } \end{array} \right| \begin{array}{l} 0, \quad g, \quad -h. \\ -f, \quad 0, \quad h. \\ f, \quad -g, \quad 0. \end{array} \equiv 0. \text{ Therefore } A A_1, B B_1, C C_1 \text{ intersect in a point—say } P_1.$$

$$P_1 \text{ is } gh, hf, fg, \text{ or } f^{-1}, g^{-1}, h^{-1}.$$

THEOREM 2.

$$\left. \begin{array}{l} B_1 C_1 \text{ is } 1-f^2, \quad h+fg, \quad g+hf \\ C_1 A_1 \text{ is } h+fg, \quad 1-g^2, \quad f+gh \\ A_1 B_1 \text{ is } g+hf, \quad f+gh, \quad 1-h^2 \end{array} \right\} \text{ Also, } \left\{ \begin{array}{l} B C \text{ is } 1, 0, 0. \\ C A \text{ is } 0, 1, 0. \\ A B \text{ is } 0, 0, 1. \end{array} \right\} \text{ Hence ;—}$$

$$\left. \begin{array}{l} B C, B_1 C_1 \text{ meet in } \mathfrak{A}_1 \\ C A, C_1 A_1 \text{ „ } \mathfrak{B}_1 \\ A B, A_1 B_1 \text{ „ } \mathfrak{C}_1 \end{array} \right\} \text{ which is } \left| \begin{array}{l} 0, \quad g+hf, \quad -(h+fg). \\ -(f+gh), \quad 0, \quad h+fg. \\ f+gh, \quad -(g+hf), \quad 0. \end{array} \right| \equiv 0. \text{ Therefore ;—}$$

$$\mathfrak{A}_1 \mathfrak{B}_1 \mathfrak{C}_1 \text{ is a straight line, viz., } (f+gh)^{-1}, (g+hf)^{-1}, (h+fg)^{-1}.$$

THEOREM 3.

$$\left. \begin{array}{l} B C_1 \text{ is } 1, 0, g \\ C A_1 \text{ is } h, 1, 0 \\ A B_1 \text{ is } 0, f, 1 \end{array} \right\} \text{ Also } \left\{ \begin{array}{l} B_1 C \text{ is } 1, h, 0 \\ C_1 A \text{ is } 0, 1, f \\ A_1 B \text{ is } g, 0, 1 \end{array} \right\} \text{ Hence } \left\{ \begin{array}{l} B C_1, B_1 C \text{ meet in } A_2, \text{ viz. } \\ C A_1, C_1 A \text{ „ } B_2, \text{ „ } \\ A B_1, A_1 B \text{ „ } C_2, \text{ „ } \end{array} \right| \begin{array}{l} -hg, \quad g, \quad h \\ f, \quad -hf, \quad h \\ f, \quad g, \quad -fg \end{array}$$

$$\left. \begin{array}{l} A A_2 \text{ is } \\ B B_2 \text{ is } \\ C C_2 \text{ is } \end{array} \right| \begin{array}{l} 0, \quad -h, \quad g \\ h, \quad 0, \quad -f \\ -g, \quad f, \quad 0 \end{array} \equiv 0. \text{ Therefore } A A_2, B B_2, C C_2 \text{ intersect in a point—say } P_2.$$

$$P_2 \text{ is } f, g, h.$$

P_1 and P_2 may be called inverse or reciprocal points.

THEOREM 4.

$$\left. \begin{array}{l} B C \text{ is } \\ B_1 C_1 \text{ is } \\ B_2 C_2 \text{ is } \end{array} \right| \begin{array}{l} 1, \quad 0, \quad 0. \\ 1-f^2, \quad h+fg, \quad g+hf. \\ -gh(1-f^2), \quad f(h+fg), \quad f(g+hf). \end{array} \equiv 0. \text{ Therefore ;—}$$

$$\left. \begin{array}{l} B C, B_1 C_1, B_2 C_2 \text{ intersect in } \mathfrak{A}_1. \\ * C A, C_1 A_1, C_2 A_2 \text{ „ } \mathfrak{B}_1. \\ A B, A_1 B_1, A_2 B_2 \text{ „ } \mathfrak{C}_1. \end{array} \right\}$$

* By symmetry.

THEOREM 5.

$$\begin{array}{l|l} A_1 A_2 \text{ is} & g^2 - h^2, \quad -h(1-g^2), \quad g(1-h^2) \\ B_1 B_2 \text{ is} & h(1-f^2), \quad h^2 - f^2, \quad -f(1-h^2) \\ C_1 C_2 \text{ is} & -g(1-f^2), \quad f(1-g^2), \quad f^2 - g^2 \end{array} \equiv 0. \text{ Therefore ;—}$$

$A_1 A_2, B_1 B_2, C_1 C_2$ intersect in a point,—say P_3 , viz., $f - gh, g - hf, h - fg$.

THEOREM 6.

$$\begin{array}{l|l} P_1 \text{ is} & gh, \quad hf, \quad fg. \\ P_2 \text{ is} & f, \quad g, \quad h. \\ P_3 \text{ is} & f - gh, \quad g - hf, \quad h - fg. \end{array} \equiv 0. \text{ Therefore } P_1, P_2, P_3 \text{ range in a straight line.}$$

$$P_1 P_2 P_3 \text{ is } f(g^2 - h^2), g(h^2 - f^2), h(f^2 - g^2).$$

THEOREM 7.

$$\begin{array}{l} A_1 B, B C \text{ meet in } \mathfrak{A}_2, \text{ viz., } \left\{ \begin{array}{ccc} 0, & 1, & -f \\ -g, & 0, & 1 \\ 1, & -h, & 0 \end{array} \right\}. \text{ Also } \left\{ \begin{array}{l} C_1 A, B C \text{ meet in } \mathfrak{A}_3 \text{ viz., } \left\{ \begin{array}{ccc} 0, & f, & -1 \\ -1, & 0, & g \\ h, & -1, & 0 \end{array} \right\}. \\ A_1 B, C A \text{ ,, } \mathfrak{B}_2, \text{ ,, } \left\{ \begin{array}{ccc} 0, & 1, & -f \\ -g, & 0, & 1 \\ 1, & -h, & 0 \end{array} \right\}. \\ B_1 C, A B \text{ ,, } \mathfrak{C}_3, \text{ ,, } \left\{ \begin{array}{ccc} 0, & f, & -1 \\ -1, & 0, & g \\ h, & -1, & 0 \end{array} \right\}. \end{array} \right.$$

$$\text{Hence ;—} \quad \begin{array}{l} \mathfrak{B}_2 \mathfrak{C}_2 \text{ is } \left\{ \begin{array}{ccc} h, & 1, & gh \\ hf, & f, & 1 \\ 1, & fg, & g \end{array} \right\}; \quad \mathfrak{B}_3 \mathfrak{C}_3 \text{ is } \left\{ \begin{array}{ccc} g, & gh, & 1 \\ 1, & h, & hf \\ fg, & 1, & f \end{array} \right\}; \\ \mathfrak{C}_2 \mathfrak{A}_2 \text{ is } \left\{ \begin{array}{ccc} h, & 1, & gh \\ hf, & f, & 1 \\ 1, & fg, & g \end{array} \right\}; \quad \mathfrak{C}_3 \mathfrak{A}_3 \text{ is } \left\{ \begin{array}{ccc} g, & gh, & 1 \\ 1, & h, & hf \\ fg, & 1, & f \end{array} \right\}; \\ \mathfrak{A}_2 \mathfrak{B}_2 \text{ is } \left\{ \begin{array}{ccc} h, & 1, & gh \\ hf, & f, & 1 \\ 1, & fg, & g \end{array} \right\}; \quad \mathfrak{A}_3 \mathfrak{B}_3 \text{ is } \left\{ \begin{array}{ccc} g, & gh, & 1 \\ 1, & h, & hf \\ fg, & 1, & f \end{array} \right\}; \end{array}$$

$$\text{Also ;—} \quad \begin{array}{l} \mathfrak{B}_2 \mathfrak{C}_2, \mathfrak{B}_3 \mathfrak{C}_3 \text{ meet in } \mathfrak{A}_3, \text{ viz., } \left\{ \begin{array}{ccc} 1 - g^2 h^2, & h(g^2 - 1), & g(h^2 - 1) \\ h(f^2 - 1), & 1 - h^2 f^2, & f(h^2 - 1) \\ g(f^2 - 1), & f(g^2 - 1), & 1 - f^2 g^2 \end{array} \right\}. \\ \mathfrak{C}_2 \mathfrak{A}_2, \mathfrak{C}_3 \mathfrak{A}_3 \text{ ,, } \mathfrak{B}_3, \text{ ,, } \left\{ \begin{array}{ccc} 1 - g^2 h^2, & h(g^2 - 1), & g(h^2 - 1) \\ h(f^2 - 1), & 1 - h^2 f^2, & f(h^2 - 1) \\ g(f^2 - 1), & f(g^2 - 1), & 1 - f^2 g^2 \end{array} \right\}. \\ \mathfrak{A}_2 \mathfrak{B}_2, \mathfrak{A}_3 \mathfrak{B}_3 \text{ ,, } \mathfrak{C}_3, \text{ ,, } \left\{ \begin{array}{ccc} 1 - g^2 h^2, & h(g^2 - 1), & g(h^2 - 1) \\ h(f^2 - 1), & 1 - h^2 f^2, & f(h^2 - 1) \\ g(f^2 - 1), & f(g^2 - 1), & 1 - f^2 g^2 \end{array} \right\}. \end{array}$$

$$\text{Farther ;—} \quad \begin{array}{l} \mathfrak{B}_2 \mathfrak{C}_3 \text{ is } \left\{ \begin{array}{ccc} 1, & h, & g \\ h, & 1, & f \\ g, & f, & 1 \end{array} \right\}; \quad \mathfrak{B}_3 \mathfrak{C}_2 \text{ is } \left\{ \begin{array}{ccc} gh, & g, & h \\ f, & hf, & h \\ f, & g, & fg \end{array} \right\}; \\ \mathfrak{C}_2 \mathfrak{A}_3 \text{ is } \left\{ \begin{array}{ccc} 1, & h, & g \\ h, & 1, & f \\ g, & f, & 1 \end{array} \right\}; \quad \mathfrak{C}_3 \mathfrak{A}_2 \text{ is } \left\{ \begin{array}{ccc} gh, & g, & h \\ f, & hf, & h \\ f, & g, & fg \end{array} \right\}; \\ \mathfrak{A}_2 \mathfrak{B}_3 \text{ is } \left\{ \begin{array}{ccc} 1, & h, & g \\ h, & 1, & f \\ g, & f, & 1 \end{array} \right\}; \quad \mathfrak{A}_3 \mathfrak{B}_2 \text{ is } \left\{ \begin{array}{ccc} gh, & g, & h \\ f, & hf, & h \\ f, & g, & fg \end{array} \right\}; \end{array}$$

$$\text{Hence ;—} \quad \begin{array}{l} \mathfrak{B}_2 \mathfrak{C}_3, \mathfrak{B}_3 \mathfrak{C}_2 \text{ meet in } A_4, \text{ viz., } \left\{ \begin{array}{ccc} h^2 - g^2, & h(g^2 - 1), & g(h^2 - 1) \\ h(f^2 - 1), & f^2 - h^2, & f(h^2 - 1) \\ g(f^2 - 1), & f(g^2 - 1), & g^2 - f^2 \end{array} \right\}. \\ \mathfrak{C}_2 \mathfrak{A}_3, \mathfrak{C}_3 \mathfrak{A}_2 \text{ ,, } B_4, \text{ ,, } \left\{ \begin{array}{ccc} h^2 - g^2, & h(g^2 - 1), & g(h^2 - 1) \\ h(f^2 - 1), & f^2 - h^2, & f(h^2 - 1) \\ g(f^2 - 1), & f(g^2 - 1), & g^2 - f^2 \end{array} \right\}. \\ \mathfrak{A}_2 \mathfrak{B}_3, \mathfrak{A}_3 \mathfrak{B}_2 \text{ ,, } C_4, \text{ ,, } \left\{ \begin{array}{ccc} h^2 - g^2, & h(g^2 - 1), & g(h^2 - 1) \\ h(f^2 - 1), & f^2 - h^2, & f(h^2 - 1) \\ g(f^2 - 1), & f(g^2 - 1), & g^2 - f^2 \end{array} \right\}. \end{array}$$

$$\text{Now ;—} \quad \begin{array}{l|l} A \text{ being} & 1, \quad 0, \quad 0 \\ A_3 \text{ ,,} & 1 - g^2 h^2, \quad h(g^2 - 1), \quad g(h^2 - 1) \\ A_4 \text{ ,,} & h^2 - g^2, \quad h(g^2 - 1), \quad g(h^2 - 1) \end{array} \equiv 0. \text{ Therefore ;—}$$

$A A_3 A_4$ is a straight line.
 $B B_3 B_4$ " "
 $C C_3 C_4$ " "

THEOREM 8.

$$\begin{array}{l|l} A A_3 A_4 \text{ is} & 0, \quad g(1-h^2), \quad -h(1-g^2) \\ B B_3 B_4 \text{ is} & -f(1-h^2), \quad 0, \quad h(1-f^2) \\ C C_3 C_4 \text{ is} & f(1-g^2), \quad -g(1-f^2), \quad 0 \end{array} \equiv 0. \text{ Therefore ;—}$$

$$A A_3 A_4, B B_3 B_4, C C_3 C_4 \text{ meet in a point } P_4, \text{ viz., } \frac{1-f^2}{f}, \frac{1-g^2}{g}, \frac{1-h^2}{h}.$$

THEOREM 9.

$$\begin{array}{l} A_1 \text{ is} \\ A_2 \text{ is} \\ A_3 \text{ is} \end{array} \left| \begin{array}{ccc} -1, & h, & g \\ -gh, & g, & h \\ 1-g^2h^2, & h(g^2-1), & g(h^2-1) \end{array} \right| \equiv 0. \quad \text{Therefore ;—}$$

$$\begin{array}{l} A_1 A_2 A_3 \text{ is a straight line.} \\ B_1 B_2 B_3 \quad \quad \quad \text{,,} \quad \text{,,} \\ C_1 C_2 C_3 \quad \quad \quad \text{,,} \quad \text{,,} \end{array}$$

THEOREM 10.

$$\left. \begin{array}{lll} B_3 C_3 \text{ is } 1+fg h, & h+fg, & g+h f; \text{ meeting } B C \text{ in } \mathfrak{A}_1 \\ C_3 A_3 \text{ is } h+fg, & 1+fg h, & f+g h; \quad \quad \quad \text{,, } C A \quad \text{,, } \mathfrak{B}_1 \\ A_3 B_3 \text{ is } g+h f, & f+g h, & 1+fg h; \quad \quad \quad \text{,, } A B \quad \text{,, } \mathfrak{C}_1 \end{array} \right\}. \quad \text{Therefore ;—}$$

$$\begin{array}{l} B C, B_1 C_1, B_2 C_2, B_3 C_3 \text{ meet in } \mathfrak{A}_1. \\ C A, C_1 A_1, C_2 A_2, C_3 A_3 \quad \quad \quad \text{,,} \quad \mathfrak{B}_1. \\ A B, A_1 B_1, A_2 B_2, A_3 B_3 \quad \quad \quad \text{,,} \quad \mathfrak{C}_1. \end{array}$$

THEOREM 11.

$$\begin{array}{lll} B C, B_4 C_4 \text{ meet in } \mathfrak{A}_4, \text{ viz.,} & 0, & (f-gh) \cdot (h+fg), \quad -(f+gh) \cdot (g-hf) \\ C A, C_4 A_4 \quad \quad \quad \text{,,} \quad \mathfrak{B}_4, \quad \quad \quad \text{,,} & -(g+hf) \cdot (h-fg), & 0, \quad (g-hf) \cdot (f+gh) \\ A B, A_4 B_4 \quad \quad \quad \text{,,} \quad \mathfrak{C}_4, \quad \quad \quad \text{,,} & (h-fg) \cdot (g+hf), & -(h+fg) \cdot (f-gh), \quad 0 \end{array} \Bigg| \equiv 0.$$

$$\text{Therefore } \mathfrak{A}_4 \mathfrak{B}_4 \mathfrak{C}_4 \text{ is a straight line, viz., } \frac{1}{(g+hf) \cdot (h-fg)}, \frac{1}{(h+fg) \cdot (f-gh)}, \frac{1}{(f+gh) \cdot (g-hf)}.$$

THEOREM 12.

$$\begin{array}{lll} B_1 C, A_1 B \text{ meet in } A_5, \text{ viz.,} & \left\{ \begin{array}{ccc} -h, & 1, & gh \\ hf, & -f, & 1 \\ 1, & fg, & -g \end{array} \right\}. & B C_1, C A_1 \text{ meet in } A_6, \text{ viz.,} & \left\{ \begin{array}{ccc} -g, & gh, & 1 \\ 1, & -h, & hf \\ fg, & 1, & -f \end{array} \right\}. \\ C_1 A, B_1 C \quad \quad \quad \text{,,} \quad B_5, \quad \quad \quad \text{,,} & & C A_1, A B_1 \quad \quad \quad \text{,,} \quad B_6, \quad \quad \quad \text{,,} \\ A_1 B, C_1 A \quad \quad \quad \text{,,} \quad C_5, \quad \quad \quad \text{,,} & & A B_1, B C_1 \quad \quad \quad \text{,,} \quad C_6, \quad \quad \quad \text{,,} \end{array}$$

$$\begin{array}{lll} A_5 A_6, B C \text{ meet in } \mathfrak{A}_5, \text{ viz.,} & 0, & g(1-h^2), \quad -h(1-g^2) \\ B_5 B_6, C A \quad \quad \quad \text{,,} \quad \mathfrak{B}_5, \quad \quad \quad \text{,,} & -f(1-h^2), & 0, \quad h(1-f^2) \\ C_5 C_6, A B \quad \quad \quad \text{,,} \quad \mathfrak{C}_5, \quad \quad \quad \text{,,} & f(1-g^2), & -g(1-f^2), \quad 0 \end{array} \Bigg| \equiv 0. \quad \text{Therefore ;—}$$

$$\mathfrak{A}_5 \mathfrak{B}_5 \mathfrak{C}_5 \text{ is a straight line, viz., } \frac{1-f^2}{f}, \frac{1-g^2}{g}, \frac{1-h^2}{h}.$$

THEOREM 13.

$$\begin{array}{l} B C \text{ is} \\ B_5 C_6 \text{ is} \\ B_6 C_5 \text{ is} \end{array} \left| \begin{array}{ccc} 1, & 0, & 0 \\ f^2-1, & f(g+hf), & f(h+fg) \\ hg(1-f^2), & g+hf, & h+fg \end{array} \right| \equiv 0. \quad \text{Therefore ;—}$$

$$\begin{array}{l} B C, B_5 C_6, B_6 C_5 \text{ meet in a point: } \mathfrak{A}_6 \\ C A, C_5 A_6, C_6 A_5 \quad \quad \quad \text{,,} \quad \text{,,} \quad \mathfrak{B}_6 \\ A B, A_5 B_6, A_6 B_5 \quad \quad \quad \text{,,} \quad \text{,,} \quad \mathfrak{C}_6 \end{array}$$

THEOREM 14.

$$\begin{array}{l} \mathfrak{A}_6 \text{ is} \\ \mathfrak{B}_6 \text{ is} \\ \mathfrak{C}_6 \text{ is} \end{array} \left| \begin{array}{ccc} 0, & h+fg, & -(g+hf) \\ -(h+fg), & 0, & f+gh \\ g+hf, & -(f+gh), & 0 \end{array} \right| \equiv 0. \quad \text{Therefore ;—}$$

$\mathfrak{A}_6 \mathfrak{B}_6 \mathfrak{C}_6$ is a straight line, viz., $f+gh$, $g+hf$, $h+fg$.

$\mathfrak{A}_1 \mathfrak{B}_1 \mathfrak{C}_1$ and $\mathfrak{A}_6 \mathfrak{B}_6 \mathfrak{C}_6$ are inverse or reciprocal lines.

THEOREM 15.

$$\begin{array}{l} \mathfrak{B}_1 \mathfrak{C}_6 \text{ is} \\ \mathfrak{C}_1 \mathfrak{A}_6 \text{ is} \\ \mathfrak{A}_1 \mathfrak{B}_6 \text{ is} \end{array} \left| \begin{array}{ccc} (h+fg) \cdot (f+gh), & (g+hf) \cdot (h+fg), & (f+gh)^2 \\ (g+hf)^2, & (f+gh) \cdot (g+hf), & (h+fg) \cdot (f+gh) \\ (f+gh) \cdot (g+hf), & (h+fg)^2, & (g+hf) \cdot (h+fg) \end{array} \right| \begin{array}{l} \text{and meets BC in } \mathfrak{A}_7, \\ \text{,, CA in } \mathfrak{B}_7, \\ \text{,, AB in } \mathfrak{C}_7 \end{array}$$

$$\begin{array}{l} \mathfrak{A}_7 \text{ is} \\ \mathfrak{B}_7 \text{ is} \\ \mathfrak{C}_7 \text{ is} \end{array} \left| \begin{array}{ccc} 0, & (f+gh)^2, & -(g+hf) \cdot (h+fg) \\ -(h+fg) \cdot (f+gh), & 0, & (g+hf)^2 \\ (h+fg)^2, & -(f+gh) \cdot (g+hf), & 0 \end{array} \right| \equiv 0. \quad \text{Therefore ;—}$$

$\mathfrak{A}_7 \mathfrak{B}_7 \mathfrak{C}_7$ is a straight line, viz., $\frac{g+hf}{h+fg}$, $\frac{h+fg}{f+gh}$, $\frac{f+gh}{g+hf}$.

THEOREM 16.

Similarly, $\left\{ \begin{array}{l} \mathfrak{B}_6 \mathfrak{C}_1 \text{ meeting BC in } \mathfrak{A}_8 \\ \mathfrak{C}_6 \mathfrak{A}_1 \text{ ,, CA in } \mathfrak{B}_8 \\ \mathfrak{A}_6 \mathfrak{B}_1 \text{ ,, AB in } \mathfrak{C}_8 \end{array} \right\}, \mathfrak{A}_8 \mathfrak{B}_8 \mathfrak{C}_8 \text{ is a straight line, viz., } \frac{h+fg}{g+hf}, \frac{f+gh}{h+fg}, \frac{g+hf}{f+gh}.$

$\mathfrak{A}_7 \mathfrak{B}_7 \mathfrak{C}_7$ and $\mathfrak{A}_8 \mathfrak{B}_8 \mathfrak{C}_8$ are reciprocal lines.

THEOREM 17.

$$\begin{array}{l} \mathfrak{A} \mathfrak{A}_1 \text{ is} \\ \mathfrak{B} \mathfrak{B}_5 \text{ is} \\ \mathfrak{C} \mathfrak{C}_6 \text{ is} \end{array} \left| \begin{array}{ccc} 0, & g, & -h \\ -1, & 0, & hf \\ 1, & -fg, & 0 \end{array} \right| \equiv 0. \quad \text{Also } \left\{ \begin{array}{l} \mathfrak{A} \mathfrak{A}_2 \text{ is} \\ \mathfrak{B} \mathfrak{B}_6 \text{ is} \\ \mathfrak{C} \mathfrak{C}_5 \text{ is} \end{array} \left| \begin{array}{ccc} 0, & h, & -g \\ -hf, & 0, & 1 \\ fg, & -1, & 0 \end{array} \right| \equiv 0. \quad \text{Therefore ;—}$$

$\mathfrak{A} \mathfrak{A}_1, \mathfrak{B} \mathfrak{B}_5, \mathfrak{C} \mathfrak{C}_6$ meet in a point $\mathfrak{A}_7$ viz., fgh, h, g .
 $\mathfrak{B} \mathfrak{B}_1, \mathfrak{C} \mathfrak{C}_5, \mathfrak{A} \mathfrak{A}_6$,, ,, $\mathfrak{B}_7$,, h, fgh, f .
 $\mathfrak{C} \mathfrak{C}_1, \mathfrak{A} \mathfrak{A}_5, \mathfrak{B} \mathfrak{B}_6$,, ,, $\mathfrak{C}_7$,, g, f, fgh .

THEOREM 18.

$\mathfrak{A} \mathfrak{A}_2, \mathfrak{B} \mathfrak{B}_6, \mathfrak{C} \mathfrak{C}_5$ meet in a point $\mathfrak{A}_8$ viz., $1, fg, hf$.
 $\mathfrak{B} \mathfrak{B}_2, \mathfrak{C} \mathfrak{C}_6, \mathfrak{A} \mathfrak{A}_5$,, ,, $\mathfrak{B}_8$,, $fg, 1, gh$.
 $\mathfrak{C} \mathfrak{C}_2, \mathfrak{A} \mathfrak{A}_6, \mathfrak{B} \mathfrak{B}_5$,, ,, $\mathfrak{C}_8$,, $hf, gh, 1$.

But instead of continuing the manipulation, we shall gather up these results, and continue the series of propositions.

THEOREM I.

$A A_1, B B_1, C C_1$ intersect, in a point, P_1 .

THEOREM II.

Let $B C, B_1 C_1$ meet in $\mathfrak{A}_1$
 $C A, C_1 A_1$ „ $\mathfrak{B}_1$
 $A B, A_1 B_1$ „ $\mathfrak{C}_1$ } . Then;— $\mathfrak{A}_1 \mathfrak{B}_1 \mathfrak{C}_1$ is a straight line.

THEOREM III.

Let $B C_1, B_1 C$ meet in A_2
 $C A_1, C_1 A$ „ B_2
 $A B_1, A_1 B$ „ C_2 } . Then;— $A A_2, B B_2, C C_2$ meet in a point, P_2 .

P_1 and P_2 are reciprocal points.

THEOREM IV.

$B C, B_1 C_1, B_2 C_2$ intersect in a point, and that point is $\mathfrak{A}_1$
 $C A, C_1 A_1, C_2 A_2$ „ „ „ $\mathfrak{B}_1$
 $A B, A_1 B_1, A_2 B_2$ „ „ „ $\mathfrak{C}_1$

THEOREM V.

$A_1 A_2, B_1 B_2, C_1 C_2$ intersect in a point, P_3 .

THEOREM VI.

$P_1 P_2 P_3$ is a straight line.

THEOREM VII.

Let $A_1 B, B C$ meet in $\mathfrak{A}_2$
 $B_1 C, C A$ „ $\mathfrak{B}_2$
 $C A, A B$ „ $\mathfrak{C}_2$ } ; And let $C_1 A, B C$ meet in $\mathfrak{A}_3$
 $A_1 B, C A$ „ $\mathfrak{B}_3$
 $B_1 C, A B$ „ $\mathfrak{C}_3$ } .
 Also let $\mathfrak{B}_2 \mathfrak{C}_2, \mathfrak{B}_3 \mathfrak{C}_3$ meet in A_3
 $\mathfrak{C}_2 \mathfrak{A}_2, \mathfrak{C}_3 \mathfrak{A}_3$ „ B_3
 $\mathfrak{A}_2 \mathfrak{B}_2, \mathfrak{A}_3 \mathfrak{B}_3$ „ C_3 } ; And let $\mathfrak{B}_2 \mathfrak{C}_3, \mathfrak{B}_3 \mathfrak{C}_2$ meet in A_4
 $\mathfrak{C}_2 \mathfrak{A}_3, \mathfrak{C}_3 \mathfrak{A}_2$ „ B_4
 $\mathfrak{A}_2 \mathfrak{B}_3, \mathfrak{A}_3 \mathfrak{B}_2$ „ C_4 } .

Then,— $A A_3 A_4$ is a straight line.

$B B_3 B_4$ „ „
 $C C_3 C_4$ „ „

THEOREM VIII.

$A A_3 A_4, B B_3 B_4, C C_3 C_4$ meet in a point, P_4 .

THEOREM IX.

$A, A_2 A_3$ is a straight line.

$B_1 B_2 B_3$ „ „
 $C_1 C_2 C_3$ „ „

THEOREM X.

$BC, B_1C_1, B_2C_2, B_3C_3$ all meet in $\mathfrak{A}_1$.
 $CA, C_1A_1, C_2A_2, C_3A_3$ „ $\mathfrak{B}_1$.
 $AB, A_1B_1, A_2B_2, A_3B_3$ „ $\mathfrak{C}_1$.

THEOREM XI.

Let BC, B_4C_4 meet in $\mathfrak{A}_4$ }
 CA, C_4A_4 „ $\mathfrak{B}_4$ }
 AB, A_4B_4 „ $\mathfrak{C}_4$ } . Then ;— $\mathfrak{A}_4\mathfrak{B}_4\mathfrak{C}_4$ is a straight line.

THEOREM XII.

Let B_1C, A_1B meet in $\mathfrak{A}_5$ }
 C_1A, B_1C „ B_5 }
 A_1B, C_1A „ C_5 } ;— And let BC_1, CA_1 meet in $\mathfrak{A}_6$ }
 CA_1, AB_1 „ B_6 }
 AB_1, BC_1 „ C_6 } .
 Also let BC, A_5A_6 meet in $\mathfrak{A}_5$ }
 CA, B_5B_6 „ $\mathfrak{B}_5$ }
 AB, C_5C_6 „ $\mathfrak{C}_5$ } . Then ;— $\mathfrak{A}_5\mathfrak{B}_5\mathfrak{C}_5$ is a straight line.

THEOREM XIII.

BC, B_5C_6, B_6C_5 meet in a point, $\mathfrak{A}_6$.
 CA, C_5A_6, C_6A_5 „ „ $\mathfrak{B}_6$.
 AB, A_5B_6, A_6B_5 „ „ $\mathfrak{C}_6$.

THEOREM XIV.

$\mathfrak{A}_6\mathfrak{B}_6\mathfrak{C}_6$ is a straight line; reciprocal to $\mathfrak{A}_1\mathfrak{B}_1\mathfrak{C}_1$.

THEOREM XV.

Let $BC, \mathfrak{B}_1\mathfrak{C}_6$ meet in $\mathfrak{A}_7$ }
 $CA, \mathfrak{C}_1\mathfrak{A}_6$ „ $\mathfrak{B}_7$ }
 $AB, \mathfrak{A}_1\mathfrak{A}_6$ „ $\mathfrak{C}_7$ } . Then ;— $\mathfrak{A}_7\mathfrak{B}_7\mathfrak{C}_7$ is a straight line.

THEOREM XVI.

Let $BC, \mathfrak{B}_6\mathfrak{C}_1$ meet in $\mathfrak{A}_8$ }
 $CA, \mathfrak{C}_6\mathfrak{A}_1$ „ $\mathfrak{B}_8$ }
 $AB, \mathfrak{A}_6\mathfrak{B}_1$ „ $\mathfrak{C}_8$ } . Then ;— $\mathfrak{A}_8\mathfrak{B}_8\mathfrak{C}_8$ is a straight line; reciprocal to $\mathfrak{A}_7\mathfrak{B}_7\mathfrak{C}_7$.

THEOREM XVII.

AA_1, BB_5, CC_6 meet in a point, A_7 .
 BB_1, CC_5, AA_6 „ „ B_7 .
 CC_1, AA_5, BB_6 „ „ C_7 .

THEOREM XVIII.

AA_2, BB_6, CC_5 meet in a point, A_8 .
 BB_2, CC_6, AA_5 „ „ B_8 .
 CC_2, AA_6, BB_5 „ „ C_8 .

THEOREM XIX.

A_7A_8, B_7B_8, C_7C_8 meet in a point P_5 .

THEOREM XX.

BC, B_7C_7, B_8C_8 meet in a point, $\mathfrak{A}_9$.
 CA, C_7A_7, C_8A_8 „ „ $\mathfrak{B}_9$.
 AB, A_7B_7, A_8B_8 „ „ $\mathfrak{C}_9$.

THEOREM XXI.

$\mathfrak{A}_9\mathfrak{B}_9\mathfrak{C}_9$ is a straight line.

THEOREM XXII.

BC, AA_2, A_8P_1, A_7P_2 all meet in a point, $\mathfrak{A}_{10}$.
 CA, BB_2, B_8P_1, B_7P_2 „ „ $\mathfrak{B}_{10}$.
 AB, CC_2, C_8P_1, C_7P_2 „ „ $\mathfrak{C}_{10}$.

THEOREM XXIII.

$A\mathfrak{A}_9, AC, A\mathfrak{A}_{10}, AB$ is a harmonic pencil.
 $B\mathfrak{B}_9, BA, B\mathfrak{B}_{10}, BC$ „ „
 $C\mathfrak{C}_9, CB, C\mathfrak{C}_{10}, CA$ „ „

THEOREM XXIV.

BC, A_1P_2, A_2P_1 meet in a point, $\mathfrak{A}_{11}$.
 CA, B_1P_2, B_2P_1 „ „ $\mathfrak{B}_{11}$.
 AB, C_1P_2, C_2P_1 „ „ $\mathfrak{C}_{11}$.

THEOREM XXV.

$A\mathfrak{A}_{11}, B\mathfrak{B}_{11}, C\mathfrak{C}_{11}$ intersect in a point, P_6 .

THEOREM XXVI.

BC, A_1A_8, A_2A_7 meet in a point, $\mathfrak{A}_{12}$.
 CA, B_1B_8, B_2B_7 „ „ $\mathfrak{B}_{12}$.
 AB, C_1C_8, C_2C_7 „ „ $\mathfrak{C}_{12}$.

THEOREM XXVII.

$A\mathfrak{A}_{12}, B\mathfrak{B}_{12}, C\mathfrak{C}_{12}$ intersect in a point, Q_1 .

THEOREM XXVIII.

Let BC, A_7A_8 meet in $\mathfrak{A}_{13}$ }
 CA, B_7B_8 „ $\mathfrak{B}_{13}$ }
 AB, C_7C_8 „ $\mathfrak{C}_{13}$ }

—Then, $A\mathfrak{A}_{13}, B\mathfrak{B}_{13}, C\mathfrak{C}_{13}$ intersect in a point, Q_2 .

THEOREM XXIX.

Let B_5B_6, C_5C_6 meet in A_9 }
 C_5C_6, A_5A_6 „ B_9 }
 A_5A_6, B_5B_6 „ C_9 }

—Then, $A\mathfrak{A}_9, B\mathfrak{B}_9, C\mathfrak{C}_9$ intersect in a point, Q_3 .

THEOREM XXX.

Let A_1A_2, A_5A_6 meet in A_{10} }
 B_1B_2, B_5B_6 „ B_{10} }
 C_1C_2, C_5C_6 „ C_{10} }

—Then, $A\mathfrak{A}_{10}, B\mathfrak{B}_{10}, C\mathfrak{C}_{10}$ intersect in a point, Q_4 .

THEOREM XXXI.

Let $\left. \begin{array}{l} \text{B C, A A}_7 \text{ meet in } l \\ \text{C A, B A}_7 \text{ ,, } m \\ \text{A B, C A}_7 \text{ ,, } n \end{array} \right\};$ And let $\left. \begin{array}{l} \text{B C, } m n \text{ meet in } l_1 \\ \text{C A, } n l \text{ ,, } m_1 \\ \text{A B, } l m \text{ ,, } n_1 \end{array} \right\}.$

Then $l_1 m_1 n_1$ is a straight line, as is well known. As it depends entirely on the position of A_7 let it be called $\phi(A_7)$. Then ;—

$\phi(A_7)$ is the polar of A_8 to the imaginary conic, $\alpha^2 + \beta^2 + \gamma^2 = 0$.
 $\phi(B_7)$,, B_8 ,, ,, ,,
 $\phi(C_7)$,, C_8 ,, ,, ,,

THEOREM XXXII.

$\phi(A_8)$ is the polar of A_7 to the imaginary conic, $\alpha^2 + \beta^2 + \gamma^2 = 0$.
 $\phi(B_8)$,, B_7 ,, ,, ,,
 $\phi(C_8)$,, C_7 ,, ,, ,,

THEOREM XXXIII.

$\phi(P_1)$ is the polar of P_2 to the imaginary conic, $\alpha^2 + \beta^2 + \gamma^2 = 0$.
 $\phi(P_2)$,, P_1 ,, ,, ,,

THEOREM XXXIV.

$\phi^{-1}(\mathfrak{A}_1 \mathfrak{A}_1 \mathfrak{C}_1)$ is the pole of $\mathfrak{A}_6 \mathfrak{B}_6 \mathfrak{C}_6$ to the conic, $\alpha^2 + \beta^2 + \gamma^2 = 0$.
 $\phi^{-1}(\mathfrak{A}_6 \mathfrak{B}_6 \mathfrak{C}_6)$,, $\mathfrak{A}_1 \mathfrak{B}_1 \mathfrak{C}_1$,, ,,

THEOREM XXXV.

$\phi^{-1}(\mathfrak{A}_7 \mathfrak{B}_7 \mathfrak{C}_7)$ is the pole of $\mathfrak{A}_8 \mathfrak{B}_8 \mathfrak{C}_8$ to the conic, $\alpha^2 + \beta^2 + \gamma^2 = 0$.
 $\phi^{-1}(\mathfrak{A}_8 \mathfrak{B}_8 \mathfrak{C}_8)$,, $\mathfrak{A}_7 \mathfrak{B}_7 \mathfrak{C}_7$,, ,,

The principle of reciprocation would introduce here a number of Propositions which it is unnecessary to enunciate.

THEOREM XXXVI.

Let $\left. \begin{array}{l} \text{A A}_1, \text{ B}_7 \text{C}_7 \text{ meet in } l_2 \\ \text{B B}_1, \text{ C}_7 \text{A}_7 \text{ ,, } m_2 \\ \text{C C}_1, \text{ A}_7 \text{B}_7 \text{ ,, } n_2 \end{array} \right\};$ And let $\left. \begin{array}{l} \text{B C, } m_2 n_2 \text{ meet in } \alpha_1 \\ \text{C A, } n_2 l_2 \text{ ,, } \beta_1 \\ \text{A B, } l_2 m_2 \text{ ,, } \gamma_1 \end{array} \right\}.$

Then ;— $\alpha_1 \beta_1 \gamma_1$ is a straight line.

THEOREM XXXVII.

Let $\left. \begin{array}{l} \text{A A}_2, \text{ B}_8 \text{C}_8 \text{ meet in } l_3 \\ \text{B B}_2, \text{ C}_8 \text{A}_8 \text{ ,, } m_3 \\ \text{C C}_2, \text{ A}_8 \text{B}_8 \text{ ,, } n_3 \end{array} \right\};$ And let $\left. \begin{array}{l} \text{B C, } m_3 n_3 \text{ meet in } \alpha_2 \\ \text{C A, } n_3 l_3 \text{ ,, } \beta_2 \\ \text{A B, } l_3 m_3 \text{ ,, } \gamma_2 \end{array} \right\}.$

Then ;— $\alpha_2 \beta_2 \gamma_2$ is a straight line.

THEOREM XXXVIII.

Let $\left. \begin{array}{l} \beta_1 \gamma_2, \beta_2 \gamma_1 \text{ meet in } \alpha_3 \\ \gamma_1 \alpha_2, \gamma_2 \alpha_1 \text{ ,, } \beta_3 \\ \alpha_1 \beta_2, \alpha_2 \beta_1 \text{ ,, } \gamma_3 \end{array} \right\};$ —Then, $\alpha_3 \beta_3 \gamma_3$ is a straight line.

THEOREM XXXIX.

Let $\begin{matrix} \alpha_1\beta_2, \gamma_1\alpha_2 \\ \beta_1\gamma_2, \alpha_1\beta_2 \\ \gamma_1\alpha_2, \beta_1\gamma_2 \end{matrix}$ meet in $\begin{matrix} \alpha_4 \\ \beta_4 \\ \gamma_4 \end{matrix}$; And let $\begin{matrix} \alpha_2\beta_1, \gamma_2\alpha_1 \\ \beta_2\gamma_1, \alpha_2\beta_1 \\ \gamma_2\alpha_1, \beta_2\gamma_1 \end{matrix}$ meet in $\begin{matrix} \alpha_5 \\ \beta_5 \\ \gamma_5 \end{matrix}$ }.

Then ;— $\begin{matrix} A \alpha_4\alpha_5 \\ B \beta_4\beta_5 \\ C \gamma_4\gamma_5 \end{matrix}$ is a straight line.
" "
" "

THEOREM XL.

A $\alpha_4\alpha_5$, B $\beta_4\beta_5$, C $\gamma_4\gamma_5$ meet in a point.

THEOREM XLI.

Postulating again, similarly as in Theorems XXXVI. and XXXVII.,—

Let $\begin{matrix} A A_2, B_7C_7 \\ A A_1, B_8C_8 \end{matrix}$ meet in $\begin{matrix} \bar{l}_2 \\ \bar{l}_3 \end{matrix}$; and similarly } Then ;—
" "
 $\bar{\alpha}_1 \bar{\beta}_1 \bar{\gamma}_1$ is a straight line.

THEOREM XLII.

$\bar{\alpha}_2 \bar{\beta}_2 \bar{\gamma}_2$ is a straight line.

THEOREM XLIII.

$\bar{\alpha}_3 \bar{\beta}_3 \bar{\gamma}_3$ is a straight line.

THEOREM XLIV.

A $\bar{\alpha}_4 \bar{\alpha}_5$ is a straight line.
 B $\bar{\beta}_4 \bar{\beta}_5$ " "
 C $\bar{\gamma}_4 \bar{\gamma}_5$ " "

THEOREM XLV.

A $\bar{\alpha}_4 \bar{\alpha}_5$, B $\bar{\beta}_4 \bar{\beta}_5$, C $\bar{\gamma}_4 \bar{\gamma}_5$ meet in a point.

THEOREM XLVI.

Let $\begin{matrix} \alpha_1\alpha_2, \bar{\alpha}_1\bar{\alpha}_2 \\ \beta_1\beta_2, \bar{\beta}_1\bar{\beta}_2 \\ \gamma_1\gamma_2, \bar{\gamma}_1\bar{\gamma}_2 \end{matrix}$ meet in $\begin{matrix} \alpha_6 \\ \beta_6 \\ \gamma_6 \end{matrix}$ Then ;—
 A α_6 , B β_6 , C γ_6 ; A₇A₈, B₇B₈, C₇C₈ all intersect in P₅.

Let P₃Q₃ and P₆Q₁ meet in Q₅ ; and let P₅Q₅ and P₄Q₄ meet in Q₆.
 Then the points P₁, P₂, P₃, P₄, P₅, P₆ ; Q₁, Q₂, Q₃, Q₄, Q₅, Q₆ have very remarkable relations.

THEOREM XLVII.

P₁, P₂, P₃, P₄, P₅, P₆ all range in a straight line.

THEOREM XLVIII.

Q_1Q_4 and Q_3Q_5 intersect in P_3 .

THEOREM XLIX.

Q_2Q_5 and Q_4Q_6 intersect in P_4 .

THEOREM L.

Q_1Q_3 and P_1P_2 intersect in P_5 .

THEOREM LI.

Q_1Q_5 and Q_3Q_4 intersect in P_6 .

THEOREM LII.

P_4Q_5 and P_5Q_4 intersect in Q_2 .

THEOREM LIII.

P_1P_2 , Q_1Q_2 , Q_3Q_6 intersect in a point.

THEOREM LIV.

Q_1Q_3 , Q_4Q_5 , Q_2Q_6 intersect in a point,—say $\mathcal{P}$.

THEOREM LV.

Let $\left. \begin{array}{l} Q_3Q_5 \text{ and } Q_5P_5 \text{ meet in } S_1 \\ Q_1Q_5 \text{ and } Q_4P_5 \text{ „ } S_2 \end{array} \right\}$; And let $\left. \begin{array}{l} Q_3Q_5 \text{ and } Q_4Q_6 \text{ meet in } S_3 \\ Q_1Q_4 \text{ and } Q_2Q_5 \text{ „ } S_4 \end{array} \right\}$. Then ;—

$S_1S_2\mathcal{P}$ is a straight line.

THEOREM LVI.

$S_3S_4\mathcal{P}$ is a straight line.

THEOREM LVII.

Let $\left. \begin{array}{l} P_1Q_1 \text{ and } P_2Q_3 \text{ meet in } R_1 \\ P_1Q_3 \text{ and } P_2Q_1 \text{ „ } R_2 \end{array} \right\}$; And let $\left. \begin{array}{l} P_1Q_4 \text{ and } P_2Q_5 \text{ meet in } R_3 \\ P_1Q_5 \text{ and } P_2Q_4 \text{ „ } R_4 \end{array} \right\}$. Then ;—

$R_1R_2\mathcal{P}$ is a straight line.

THEOREM LVIII.

$R_3R_4\mathcal{P}$ is a straight line.

THEOREM LIX.

Let $\begin{matrix} P_1Q_6 \text{ and } P_2Q_2 \text{ meet in } U_1 \\ P_1Q_2 \text{ and } P_2Q_6 \text{ „ } U_2 \end{matrix} \}$ Then ;—

$U_1U_2\mathfrak{P}$ is a straight line.

THEOREM LX.

U_1U_2 ; R_1R_2 , R_3R_4 ; S_1S_2 , S_3S_4 ; Q_1Q_3 , Q_4Q_5 , Q_2Q_6 ; all intersect in $\mathfrak{P}$.

THEOREM LXI.

$\mathfrak{P}$ is the pole of $P_1P_2P_3P_4P_5P_6$ to the imaginary conic, $\alpha^2 + \beta^2 + \gamma^2 = 0$.

THEOREM LXII.

The lines AA_1 , BB_1 , CC_1 ; AA_2 , BB_2 , CC_2 cut the sides of the triangle ABC in six points which lie in the conic ;—

$$\alpha^2 + \beta^2 + \gamma^2 - \left(\frac{g}{h} + \frac{h}{g}\right)\beta\gamma - \left(\frac{h}{f} + \frac{f}{h}\right)\gamma\alpha - \left(\frac{f}{g} + \frac{g}{f}\right)\alpha\beta = 0.$$

For the six points are inverse or reciprocal points. Substituting the co-ordinates of five of them (the third of the *second* set being omitted) in the general equation of the second degree, and eliminating the arbitrary constants, gives the conic as above. To substitute the co-ordinates of five of the points, omitting now the third of the *first* set, amounts evidently to inverting the separate terms in the constants of the above equation ; and as this leaves it unaltered, the proof of the theorem is obvious.

THEOREM LXIII.

The lines AA_5 , BB_5 , CC_5 ; AA_6 , BB_6 , CC_6 cut the sides of the triangle ABC in six points which lie in the conic ;—

$$\alpha^2 + \beta^2 + \gamma^2 - \left(g h + \frac{1}{gh}\right)\beta\gamma - \left(h f + \frac{1}{hf}\right)\gamma\alpha - \left(f g + \frac{1}{fg}\right)\alpha\beta = 0.$$

THEOREM LXIV.

The points $\mathfrak{A}_1\mathfrak{B}_1\mathfrak{C}_1$; $\mathfrak{A}_2\mathfrak{B}_2\mathfrak{C}_2$ lie in the conic ;—

$$\alpha^2 + \beta^2 + \gamma^2 + \left(f + \frac{1}{f}\right)\beta\gamma + \left(g + \frac{1}{g}\right)\gamma\alpha + \left(h + \frac{1}{h}\right)\alpha\beta = 0.$$

THEOREM LXV.

If the point P_1 move in a straight line, the line $\phi(P_1)$ will always touch a conic which touches the three sides of the triangle ABC.

Let the equation of the line in which P_1 moves be $la + m\beta + n\gamma = 0$; or substituting the co-ordinates of P_1 now supposed variable (see Theorem 1),

$$\frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0 \quad (1).$$

$$\phi(P_i) \text{ is } fa + g\beta + h\gamma = 0 = u \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now h may be considered as constant, since it is the ratios only that are concerned. Take f as the independent variable, then g varies with f by (1); and by the theory of envelopes we have

$$\frac{du}{df} = 0 = \left(\frac{du}{df} \right) + \left(\frac{du}{dg} \right) \cdot \frac{dg}{df} \quad (3).$$

Elimination gives $\pm (la)^{\frac{1}{2}} \pm (m\beta)^{\frac{1}{2}} \pm (n\gamma)^{\frac{1}{2}} = 0$;—

the equation of a conic touching the sides of the triangle of reference.—*Q. E. D.*

THEOREM LXVI.

Let lines be drawn from the angles of a triangle through any two points and terminating in the opposite sides ; by joining the extremities of each set of lines so drawn, two other triangles will be formed. The three lines joining the intersections of corresponding sides of these two triangles with the corresponding angle of the original triangle meet in a point.

If the co-ordinates of the two assumed points be a_1, b_1, c_1 and a_2, b_2, c_2 those of the third point are

$$a_1 a_2 \begin{vmatrix} b_1, b_2 \\ c_1, c_2 \end{vmatrix}, \quad b_1 b_2 \begin{vmatrix} c_1, c_2 \\ a_1, a_2 \end{vmatrix}, \quad c_1 c_2 \begin{vmatrix} a_1, a_2 \\ b_1, b_2 \end{vmatrix}.$$

Let this point be called the anapole of the two assumed points.

THEOREM LXVII.

The anapoles of A_7, A_8 ; of B_7, B_8 ; and of C_7, C_8 are in a straight line,—say $A_1 B_1 C_1$.

THEOREM LXVIII.

The straight line joining the anapoles of P_1, A_8 and P_2, A_7 passes through A ; cutting BC (say) in A_2 .

"	"	"	P_1, B_8	"	P_2, B_7	"	$B;$	"	CA	"	B_2
"	"	"	P_1, C_8	"	P_2, C_7	"	$C;$	"	AB	"	C_2

$A_2 B_2 C_2$ is a straight line, identical with $A_5 B_5 C_5$ of THEOREM XII.

THEOREM LXIX.

The lines $A_1 B_1 C_1$, $A_2 B_2 C_2$ intersect in the anapole of P_1, P_2 , which is also the pole of the line $P_1 P_2$ to the imaginary conic, $\alpha^2 + \beta^2 + \gamma^2 = 0$. So that the

anapole of inverse points is the pole of the line joining them—to the same imaginary conic.

THEOREM LXX.

If one of two points remain fixed and the other move in any manner whatever, the locus of the anapoles is a straight line.

THEOREM LXXI.

If two points move away from each other along a straight line with uniform velocities, their anapole will describe a conic section ; and if the uniform velocities be equal, a straight line.

THEOREM LXXII.

The anapole of any two points in a conic section passing through the angles of the triangle of reference is invariable, and its co-ordinates are proportional to the sides of the triangle.