The Mathematical Law of Mortality. By DR. THEODOR WITTSTEIN of Hanover. Translated by D. A. BUMSTED, F.I.A.

[A translation of an essay by DR. WITTSTEIN on the Mathematical Law of Mortality appeared in this *Journal* (vol. xxiv, p. 153). DR. WITTSTEIN afterwards published a pamphlet in continuation of the subject, in which he used the method of Least Squares for the determination of two of the constants in his formula, and applied the formula to the graduation of the H^M Table of Mortality. The following is an abridged translation]:

A COROLLARY TO THE METHOD OF LEAST SQUARES.

§ 18. IN applying the method of Least Squares a peculiar difficulty is often experienced whenever the function, of which we have a series of observed values, is not linear with respect to the unknown constants to be determined and is therefore either of a higher algebraic order or is transcendental.

§ 19. We can scarcely conceive of this difficulty ever being entirely obviated, but we have succeeded in proving that it is possible in respect of a certain class of functions, namely when we can transform the given function, by ordinary operations, into another which is linear, as for example,

$$y = e^{a+bx} \text{ into } \log y = a \log e + bx \log e$$

$$y = \sin(a + bx) \text{ into } \arcsin y = a + bx$$

where the new functions have become linear in regard to the unknown constants a and b . In cases of this kind it is always possible to determine exact values of these constants from the new functions, provided we bear in mind the following proposition.

§ 20. Theorem. If a series of observed values A &c. of a function y is given, having the weights w &c. we may, instead of making the sum of the squares of the errors of y a minimum, with perfect accuracy determine the unknown constants contained in y by making the sum of the squares of the errors of an

arbitrary function of y , $f(y)$ a minimum, ascribing at the same time to each observation a weight w' , such that,

$$\frac{w}{w'} = K \left(\frac{df(A)}{dA} \right)^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where K denotes an arbitrary positive constant which is the same for all the observations.

The proof is very simple. If the sum $\Sigma w(y-A)^2$ and the sum $\Sigma w'[f(y)-f(A)]^2$ (where Σ includes all the observations) are each to be a minimum for the same values of the constants contained in $f(y)$, this is only possible when each term of the one sum is either equal to the corresponding term of the other sum or differs from it by a positive factor which is the same for every term. Denoting this factor by K , we must therefore have

$$\frac{w(y-A)^2}{w'[f(y)-f(A)]^2} = K$$

or

$$\frac{w}{w'} = K \left(\frac{f(y)-f(A)}{y-A} \right)^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which the above expression (1) follows at once, as we may correctly substitute for the ratio of the small differences $f(y)-f(A)$ and $(y-A)$ the ratio of their differentials. For it is a fundamental principle in applying the method of Least Squares that the errors of observation must be nearly infinitesimal in comparison with the observed quantities themselves.

§ 21. Since the weights have only a relative significance we may, for the purpose of simplifying the calculations, so assume the arbitrary constant K that if the expression

$$\left(\frac{df(A)}{dA} \right)^2$$

contains a constant factor which has the same value for all the observations, we have the product of K into this factor = 1. If there be no such factor we take $K=1$.

Thus if, in the above example of § 19 the function

$$y = e^{a+bx} \quad \text{weight} = w$$

is transformed into

$$\log y = a \log e + bx \log e, \quad \text{weight} = w',$$

we get

$$w' = wA^2$$

with which weight the constants of the new function $a \log e$ and $b \log e$, must be calculated.

Similarly if in the second of the above examples the function
 $y = \sin (a + bx), \quad \text{weight} = w,$
 is transformed into

$$\text{arc sin } y = a + bx, \quad \text{weight} = w'$$

we get $w' = w(1 - A^2).$

APPLICATION OF THE FORMULA TO A SECOND EXAMPLE.

§24. The example given in the first part of this paper (see *J.I.A.* Vol. xxiv) was incomplete inasmuch as the method of Least Squares was not applied to the solution, and that example cannot therefore be regarded as an entirely suitable test of our formula. We therefore give another example in which the method of Least Squares is fully applied.

Table D contains the figures from which the H^M Mortality Table was calculated and we have added Col. (5) which gives the weights of the probabilities of dying calculated according to the formula of § 4 $w = \frac{l}{q(1-q)}$ the maximum weights being taken as 100 and only whole numbers being used.

Table E contains Woolhouse's graduation, which we use for comparison, and Table F shows the results of our calculation which we will now explain.

§25. In order to apply our formula (No. 17, § 13)

$$q_x = a^{-(M-x)^n} + \frac{1}{m} a^{-(mx)^n} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

to the observations in Table D it is necessary to determine 4 constants, of which we will first consider M and m .

With respect to M we can draw the same conclusion as in § 11. Table D ends with two living at age 96, both of whom died in the following year. It is possible that with a greater number of observations there might be some survivors at age 97, and we will therefore take the limiting age at 98, or make $M=97$. It will be observed that this agrees with Woolhouse's Table E.

To determine the second constant m , we seek firstly the minimum probability of dying. This is found at age 16 in Table D whence it would follow according to § 14 that $m=5$. But this deduction is very unsafe in view of the small numbers observed at this age; as appears also from the fact that $q_{16}=0$; which is a value of which no use whatever can be made. It is

further to be noticed that the observations under consideration deal with persons who were accepted as healthy and that this quality must be assumed in respect of the children also. We therefore think it would be safer to put $m=6$ by which the minimum is brought down to age 14. In this also we agree with Woolhouse's Table E.

Inserting these two constants in equation (3) it becomes

$$q_x = a^{-(97-x)^n} + \frac{1}{6} a^{-(6x)^n} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

§ 26. The other constants a and n must be determined by the method of Least Squares, and we must first decide how and to what extent the observations in Table D can be used.

The calculation is based as before, not upon the observed deaths, but upon the observed probabilities of dying, so that we make direct use of our formula for q_x . We will therefore determine the two unknowns a and n so that they may reduce the sum of the squares of the errors of these probabilities, each multiplied by its weight, to a minimum. The weights can be calculated directly by the formula in § 4, without using the values in Col. 5 of Table D.

Now we know that in getting this formula it is assumed that the observed numbers of living and dying are large, and that if it be applied to numbers which are not large only approximate values of the weights are obtained. This is of itself no disadvantage for in this case, as in many others, the final results are scarcely affected when approximate values of the weights are used instead of the true weights. But when the deaths = 0, the formula is not applicable and we must limit our calculations to the observations for ages 17 to 93 inclusive. Our task therefore consists in determining two unknown constants from 77 observations and their weights.

§ 27. The following is the course usually prescribed for solving this problem, as the function q_x is transcendental in respect of a and n .

We must first, as in § 11, choose arbitrarily from the observations two ages, and insert in our function the corresponding pairs of values of x and q_x . Two equations are thus obtained from which we find a and n . In order that the solution may be possible we must be careful to choose the ages sufficiently high to secure that the second part of our function which pertains to infant mortality may vanish; then the calculation can be proceeded with exactly as in § 11.

§ 28. Since the second part of our formula, which relates to infant mortality, gives a value which is small in comparison with that given by the first part, we may in the meantime neglect that second part and deal only with the function $q = a^{-(M-x)^n}$ to which we attribute the weights given by the formula of § 4.

We have already in § 11 (14) deduced from the above equation the following:

$$\log \log \frac{1}{q} = n \log (M-x) + \log \log a$$

which is linear with respect to the constants n and $\log \log a$; the weights corresponding to it may be denoted by w' .

Denoting in future the observed values of the probability of dying by q , we get by means of the proposition in § 20

$$\frac{df(A)}{dA} = \frac{d \log \log \frac{1}{q}}{dq} = \frac{(\log e)^2}{q \log q} \text{ and } \frac{w}{w'} = \frac{K(\log e)^4}{q^2(\log q)^2}$$

and finally taking $K(\log e)^4 = 1$ (§ 21): and substituting for w its value, we have

$$w' = \frac{lq}{1-q} \left(\log \frac{1}{q} \right)^2$$

Since according to this new function the expression for the error of observation takes the form

$$n \log (M-x) + \log \log a - \log \log \frac{1}{q}$$

we must now make

$$\Sigma w' \left\{ n \log (M-x) + \log \log a - \log \log \frac{1}{q} \right\}^2$$

a minimum, where Σ extends to all the 77 observations: Then differentiation with respect to n and $\log \log a$ gives us the two equations

$$\begin{aligned} n \Sigma w' \log (M-x) + \log \log a \Sigma w' &= \Sigma w' \log \log \frac{1}{q} \\ n \Sigma w' \left(\log (M-x) \right)^2 + \log \log a \Sigma w' \log (M-x) \\ &= \Sigma w' \log \log \frac{1}{q} \log (M-x) \end{aligned}$$

and the solution of these gives the values of n and $\log \log a$. We thus get

$$a = 1.41863 \quad n = .63507$$

Substituting these in the formula for q (No. 4) the probabilities of dying are obtained. These however are not the final probabilities given in Table F, since agreeably with what was said at the beginning of this Section they require another correction with which we will now deal.

§ 29. The values of the constants a and n in the last § were calculated from an incomplete formula for q : the complete one being:

$$q = a^{-(M-x)^n} + c$$

$$\text{where } c = \frac{1}{m} a^{-(mx)^n}$$

This may be written

$$q - c = a^{-(M-x)^n}$$

where c indicates a given constant for each observation.

The weights w , which belong to this new function $q - c$, are identical with those of the function q in the last § as can easily be seen from the theorem in § 20.

The calculation on the basis of this new function is as follows:

§ 30. Proceeding as in § 28, we deduce from the above expression for $q - c$

$$\log \log \frac{1}{q-c} = n \log (M-x) + \log \log a,$$

which is linear in respect to n and $\log \log a$.

The weights w' corresponding to this new function are not strictly speaking identical with those of § 28. For applying § 20 it is seen that

$$\frac{dfA}{dA} = \frac{d \log \log \frac{1}{q-c}}{d(q-c)} = \frac{(\log e)^2}{(q-c) \log (q-c)}$$

$$\text{and } \frac{w}{w'} = \frac{K (\log e)^4}{(q-c)^2 [\log (q-c)]^2}.$$

Substituting for w its known value, and taking $K (\log e)^4 = 1$, we obtain for w' a different value from that in § 28. But the effect of the difference must be very insignificant on our values and moreover as it only concerns weights which themselves are more or less approximations, we may quite disregard it. We retain therefore the values of w' in § 28 unaltered.

The quantity to be reduced to a minimum will therefore be the following,

$$\Sigma w' \left(n \log (M-x) + \log \log a - \log \log \frac{1}{q-c} \right)^2.$$

Differentiating this with respect to n and $\log \log a$ we get the two equations,

$$n \Sigma w' \log (M-x) + \log \log a \Sigma w' = \Sigma w' \log \log \frac{1}{q-c}$$

$$n \Sigma w' [\log (M-x)]^2 + \log \log a \Sigma w' \log (M-x) = \Sigma w' \log \log \frac{1}{q-c} \times \log (M-x),$$

solution of which gives the final values of the two unknown quantities, n and $\log \log a$.

On comparing these two equations with those at the end of § 28, we notice that the terms on the left side are identical and therefore will not require recalculation. On the right side those terms only need be recalculated in which the quantity c has a significant value; as our calculations extend only to five decimal places, this was found to occur at the ages 17 to 34; the remaining terms can be taken unaltered from § 28. We see therefore that the correction of § 28 given by the calculations of this § does not greatly affect the result.

The solution of these equations gives

$$a = 1.41790 \quad n = .63549$$

and the probability of dying at any age x is,

$$q = 1.41790^{-(97-x) \cdot 63549} + \frac{1}{6} \times 1.41790^{-(6x) \cdot 63549}$$

By this formula we have calculated the numbers given in Columns 4, 5, 6 of Table F. We may observe that the numbers in Col. 5 calculated to 5 decimal places were identical with the values of the quantity c , except in a few isolated cases in which the difference was unity in the 5th decimal place. This proves that we were justified in adopting the value of c found in § 28.

Finally, from the probabilities of dying (Table F, Col. 6) we have calculated in the usual manner the numbers living (Col. 2) and the deaths (Col. 3); using as a radix the arbitrary number of 10,000 living at age 10, which is the basis of Tables D and E.

§ 31. It must strike anyone that the values of the constants a and n , which form the basis of Tables C and F, namely

$$\begin{array}{ll} a=1.42423 & n=.63033 \\ \text{and} & \\ a=1.41790 & n=.63549 \end{array}$$

differ from each other by very small amounts and that their differences sometimes act in opposite directions and neutralize one another. Table C is founded on observations of Officials, who were not subject to medical examination at entry; while Table F is founded on the observation of men who were proved to be healthy at entry. Nevertheless the differences between the two values of a and of n do not amount to so much as 1 per-cent and the greatest part of the difference in the tables is in the value of the constant M , that is the limiting age.

We may hence conjecture that possibly the quantities a and n are constants in the absolute sense of the word; like, for example, the accelerating force of gravity—the mechanical equivalent of heat, &c., and that the above are only approximate values. The consequence of this would be that in all tables of mortality—the instance before us being that of males after the exclusion of the disturbing influence of infant mortality—the probabilities of dying would be absolutely identical and the tables would differ only in the limit of age; so that the probability of dying would always be the same when $\omega - x$ is the same. Or in other words, one of these tables of mortality would be immediately altered to another when we shift Col. 4 along Col. 1 in the proper direction for as many years as are equal to the difference between the limits of age in the tables. Columns 2 and 3 then follow in the usual way.

If this property, which is only approximately true in the cases we have considered, were found to be generally true, it is evident that the construction of mortality tables would be facilitated and simplified to an extraordinary degree. In this connection it may be noted that as soon as a and n are accurately determined, the value of the constant M and therefore the limiting age, can be found by the formula (obtained from (13) in § 11),

$$M = x + \left(\frac{\log \frac{1}{q}}{\log a} \right)^{\frac{1}{n}}$$

The following is an example of the use of this formula.

If we assume the accurate value of the constants a and n to be the arithmetical means of the above say,

$$a = 1.421$$

$$n = .633$$

we can then, without knowing M , calculate all the probabilities of dying for the successive values of $M-x$, and thereby form a normal table wanting only the ages. In order to apply this normal table to the observations in Table A we select from the latter the probability of dying which has the greatest weight, namely

$$x = 36$$

$$q = .009384;$$

and then inserting these numbers in the above formula we have $M = 95.5$, whereas we had fixed $M = 95$ as a basis for Table C.

Similarly if we select from Table D the probability of dying which has the greatest weight, namely

$$x = 35$$

$$q = .008236$$

and substitute these numbers in the same formula we have $M = 97.2$; while for Table F the value $M = 97$ had been assumed. The Mortality tables derived from constants thus determined will obviously differ very little from our Tables C and F, and will correspond almost as closely to the actual observations.

We will not pursue this further, as we are proceeding upon entirely hypothetical ground. More extensive observations are needed before we can decide whether, and how far, the above-mentioned conjecture holds good.

D. H^M *Table of Mortality.*

(1) Age	(2) Numbers exposed to Risk of Death	(3) Numbers Dying	(4) Probability of Dying in a Year = $(3) \div (2)$	(5) Relative Weights	(6) Unadjusted Numbers Living	(7) Numbers Dying
0	21	0	0·0000	...	...	...
1	47·5	0	0·0000	...	...	...
2	65	1	0·0154	...	...	...
3	79	0	0·0000	...	...	...
4	93·5	0	0·0000	...	...	...
5	119	1	0·0084	...	...	...
6	147·5	1	0·0068	...	...	...
7	186·5	0	0·0000	...	...	...
8	239·5	0	0·0000	...	...	...
9	309·5	1	0·0032	...	...	...
10	379	3	0·0079	...	10,000	79
11	434	0	0·0000	...	9,921	0
12	491·5	2	0·0041	...	9,921	40
13	578	2	0·0035	...	9,881	35
14	731	3	0·0041	...	9,846	40
15	908	2	0·0022	...	9,806	22
16	1,129	0	0·0000	...	9,784	0
17	1,421	6	0·0042	8	9,784	41
18	1,810·5	11	0·0061	7	9,743	59
19	2,414	17	0·0070	8	9,684	68
20	3,293·5	19	0·0058	13	9,616	56
21	4,578·5	32	0·0070	15	9,560	67
22	6,397	40	0·0063	23	9,493	59
23	8,534	66	0·0077	25	9,434	73
24	10,936	75	0·0069	37	9,361	64
25	13,622·5	70	0·0051	61	9,297	48
26	16,339	113	0·0069	54	9,249	64
27	19,170·5	124	0·0065	68	9,185	60
28	21,837	171	0·0078	64	9,125	71
29	24,588	181	0·0074	77	9,054	67
30	27,112·5	224	0·0083	75	8,987	74
31	29,213	215	0·0074	91	8,913	65
32	31,232	260	0·0083	86	8,848	74
33	32,969	274	0·0083	91	8,774	73
34	34,535·5	300	0·0087	91	8,701	76
35	35,818·5	295	0·0082	100	8,625	71
36	36,840·5	326	0·0089	96	8,554	75
37	37,360	357	0·0096	90	8,479	81
38	37,804·5	389	0·0108	85	8,398	87
39	38,112·5	405	0·0106	83	8,311	88
40	38,195	377	0·0099	89	8,223	81
41	37,838	396	0·0105	83	8,142	85
42	37,258·5	399	0·0107	80	8,057	87
43	36,534·5	387	0·0106	79	7,970	84
44	35,693	421	0·0118	90	7,886	93
45	34,735·5	429	0·0124	65	7,793	97
46	33,660·5	421	0·0125	62	7,696	96
47	32,502	460	0·0142	53	7,600	107
48	31,228	440	0·0141	51	7,493	106
49	30,055·5	459	0·0153	46	7,387	113

E. H ^M Table. WOOLHOUSE'S GRADUATION				F. H ^M Table. NEW ADJUSTMENT					
(1) Age <i>x</i>	(2) <i>l_x</i>	(3) <i>d_x</i>	(4) <i>q_x</i>	(1) Age <i>x</i>	(2) <i>l_x</i>	(3) <i>d_x</i>	(4) <i>q_x</i>	(5)	(6)
0	...	...	...	0	14,137	2,380	0·0017 + 0·1667 = 0·1684		
1	...	...	...	1	11,757	679	0·0018 + 0·0560 = 0·0578		
2	...	...	...	2	11,078	359	0·0018 + 0·0306 = 0·0324		
3	...	...	...	3	10,719	220	0·0019 + 0·0186 = 0·0205		
4	...	...	...	4	10,499	147	0·0020 + 0·0120 = 0·0140		
5	...	...	...	5	10,352	105	0·0021 + 0·0080 = 0·0101		
6	...	...	...	6	10,247	79	0·0022 + 0·0055 = 0·0077		
7	...	...	...	7	10,168	71	0·0023 + 0·0048 = 0·0071		
8	...	...	...	8	10,097	52	0·0024 + 0·0028 = 0·0052		
9	...	...	...	9	10,045	45	0·0025 + 0·0020 = 0·0045		
10	10,000	49	0·0049	10	10,000	41	0·0026 + 0·0015 = 0·0041		
11	9,951	40	0·0040	11	9,959	37	0·0027 + 0·0011 = 0·0038		
12	9,911	33	0·0033	12	9,922	37	0·0028 + 0·0008 = 0·0036		
13	9,878	28	0·0029	13	9,885	35	0·0029 + 0·0007 = 0·0036		
14	9,850	28	0·0028	14	9,850	35	0·0030 + 0·0005 = 0·0035		
15	9,822	28	0·0029	15	9,815	35	0·0032 + 0·0004 = 0·0036		
16	9,794	32	0·0032	16	9,780	35	0·0034 + 0·0003 = 0·0037		
17	9,762	37	0·0039	17	9,745	37	0·0035 + 0·0002 = 0·0037		
18	9,725	47	0·0048	18	9,708	37	0·0037 + 0·0002 = 0·0039		
19	9,678	56	0·0057	19	9,671	38	0·0038 + 0·0001 = 0·0039		
20	9,622	61	0·0063	20	9,633	40	0·0040 + 0·0001 = 0·0041		
21	9,561	64	0·0067	21	9,593	41	0·0042 + 0·0001 = 0·0043		
22	9,497	65	0·0068	22	9,552	43	0·0044 + 0·0001 = 0·0045		
23	9,432	64	0·0068	23	9,509	44	0·0046 + 0·0001 = 0·0047		
24	9,368	62	0·0066	24	9,465	46	0·0048 + 0·0001 = 0·0049		
25	9,306	62	0·0066	25	9,419	48	0·0051		
26	9,244	61	0·0067	26	9,371	50	0·0053		
27	9,183	64	0·0069	27	9,321	52	0·0055		
28	9,119	65	0·0072	28	9,269	54	0·0058		
29	9,054	67	0·0074	29	9,215	56	0·0061		
30	8,987	70	0·0077	30	9,159	59	0·0064		
31	8,917	70	0·0079	31	9,100	61	0·0067		
32	8,847	72	0·0081	32	9,039	64	0·0070		
33	8,775	73	0·0083	33	8,975	66	0·0074		
34	8,702	74	0·0085	34	8,909	69	0·0078		
35	8,628	76	0·0088	35	8,840	72	0·0082		
36	8,552	77	0·0091	36	8,768	75	0·0086		
37	8,475	81	0·0095	37	8,693	79	0·0090		
38	8,394	82	0·0098	38	8,614	81	0·0095		
39	8,312	84	0·0101	39	8,533	85	0·0100		
40	8,228	84	0·0103	40	8,448	88	0·0105		
41	8,144	86	0·0105	41	8,360	93	0·0110		
42	8,058	86	0·0107	42	8,267	96	0·0116		
43	7,972	89	0·0111	43	8,171	99	0·0122		
44	7,883	91	0·0116	44	8,072	104	0·0129		
45	7,792	95	0·0122	45	7,968	108	0·0136		
46	7,697	100	0·0129	46	7,860	112	0·0143		
47	7,597	104	0·0137	47	7,748	117	0·0151		
48	7,493	108	0·0144	48	7,631	121	0·0159		
49	7,385	112	0·0152	49	7,510	126	0·0168		

D. H^M Table of Mortality.

(1) Age	(2) Numbers exposed to Risk of Death	(3) Numbers Dying	(4) Probability of Dying in a Year = $(3) \div (2)$	(5) Relative Weights	Unadjusted	
					Numbers Living	Numbers Dying
50	28,855.5	476	0.0165	41	7,274	120
51	27,510.5	479	0.0174	37	7,154	124
52	26,208.5	446	0.0170	36	7,080	120
53	24,785	426	0.0172	33	6,910	119
54	23,426	444	0.0190	29	6,791	129
55	22,170.5	509	0.0230	23	6,662	153
56	20,746	479	0.0231	21	6,509	150
57	19,377.5	463	0.0239	19	6,359	152
58	18,116.5	455	0.0251	17	6,207	156
59	16,890.5	428	0.0253	16	6,051	153
60	15,672.5	488	0.0311	12	5,898	184
61	14,392.5	468	0.0325	10	5,714	186
62	13,261	459	0.0346	9	5,528	191
63	12,147.5	454	0.0374	8	5,337	200
64	11,021.5	443	0.0402	7	5,137	206
65	9,984.5	435	0.0436	5	4,931	215
66	9,009.5	421	0.0467	5	4,716	220
67	8,081	396	0.0490	4	4,496	220
68	7,214	399	0.0553	3	4,276	237
69	6,375.5	389	0.0610	3	4,039	246
70	5,622	315	0.0560	2	3,793	213
71	4,953	308	0.0622	2	3,580	222
72	4,378	349	0.0797	1	3,358	268
73	3,771.5	297	0.0788	1	3,090	243
74	3,228	340	0.1053	1	2,847	300
75	2,693	254	0.0943	1	2,547	241
76	2,253	240	0.1065	1	2,306	245
77	1,848.5	201	0.1087	0	2,061	224
78	1,531	188	0.1228	0	1,837	226
79	1,257	171	0.1360	0	1,611	219
80	995	140	0.1407	0	1,392	196
81	782	125	0.1599	0	1,196	191
82	609.5	105	0.1723	0	1,005	173
83	464	96	0.2069	0	832	172
84	339	61	0.1799	0	660	119
85	254.5	55	0.2161	0	541	117
86	184	40	0.2174	0	424	92
87	128.5	28	0.2179	0	332	72
88	91.5	26	0.2842	0	260	74
89	57.5	11	0.1913	0	186	36
90	43.5	10	0.2299	0	150	34
91	32	10	0.3125	0	116	36
92	20	9	0.4500	0	80	36
93	10.5	7	0.6667	0	44	29
94	4	0	0.0000		15	0
95	3.5	1	0.2857		15	5
96	2	2	1.0000		10	10
97					0	
98						

E. H^M Table. WOOLHOUSE'S GRADUATION				F. H^M Table. NEW ADJUSTMENT					
(1) Age x	(2) l_x	(3) d_x	(4) q_x	(1) Age x	(2) l_x	(3) d_x	(4) q_x	(5)	(6)
50	7,273	116	0·0160	50	7,384	131	0·0177		
51	7,157	120	0·0167	51	7,253	136	0·0187		
52	7,037	123	0·0175	52	7,117	140	0·0198		
53	6,914	129	0·0186	53	6,977	146	0·0209		
54	6,785	134	0·0197	54	6,831	151	0·0221		
55	6,651	140	0·0210	55	6,680	156	0·0234		
56	6,511	146	0·0225	56	6,524	162	0·0248		
57	6,365	152	0·0240	57	6,362	167	0·0262		
58	6,213	160	0·0256	58	6,195	172	0·0278		
59	6,053	166	0·0275	59	6,023	178	0·0295		
60	5,887	175	0·0297	60	5,845	183	0·0313		
61	5,712	183	0·0320	61	5,662	188	0·0332		
62	5,529	192	0·0346	62	5,474	193	0·0353		
63	5,337	200	0·0375	63	5,281	198	0·0375		
64	5,137	207	0·0404	64	5,083	203	0·0399		
65	4,930	214	0·0434	65	4,880	207	0·0425		
66	4,716	220	0·0466	66	4,673	211	0·0452		
67	4,496	224	0·0499	67	4,462	215	0·0482		
68	4,272	228	0·0532	68	4,247	219	0·0514		
69	4,044	232	0·0573	69	4,028	221	0·0549		
70	3,812	237	0·0622	70	3,807	223	0·0587		
71	3,575	243	0·0681	71	3,584	225	0·0628		
72	3,332	250	0·0749	72	3,359	226	0·0672		
73	3,082	255	0·0829	73	3,133	225	0·0720		
74	2,827	258	0·0912	74	2,908	225	0·0772		
75	2,569	253	0·0984	75	2,683	222	0·0829		
76	2,316	246	0·1064	76	2,461	220	0·0892		
77	2,070	237	0·1147	77	2,241	215	0·0960		
78	1,833	226	0·1232	78	2,026	210	0·1035		
79	1,607	214	0·1331	79	1,816	203	0·1117		
80	1,393	201	0·1447	80	1,613	194	0·1208		
81	1,192	189	0·1580	81	1,419	186	0·1309		
82	1,003	172	0·1714	82	1,233	175	0·1420		
83	831	154	0·1859	83	1,058	164	0·1544		
84	677	135	0·1989	84	894	150	0·1683		
85	542	114	0·2099	85	744	137	0·1838		
86	428	94	0·2197	86	607	122	0·2014		
87	334	77	0·2312	87	485	107	0·2212		
88	257	61	0·2393	88	378	92	0·2440		
89	196	50	0·2532	89	286	78	0·2701		
90	146	41	0·2795	90	208	62	0·3004		
91	105	33	0·3127	91	146	49	0·3361		
92	72	25	0·3513	92	97	37	0·3787		
93	47	20	0·4158	93	60	26	0·4305		
94	27	13	0·5073	94	34	17	0·4956		
95	14	9	0·6370	95	17	10	0·5813		
96	5	4	0·8163	96	7	5	0·7053		
97	1	1	1·0000	97	2	2	1·0000		
98	0			98	0				