A NEW METHOD OF COMPARING TWO SELF-INDUCTANCES.

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DURING a careful study of the more recent methods of measuring and comparing inductance and capacity I have developed the following method of comparing two self-inductances which gives very accurate results. Its freedom from the effects of other inductances permits its use for the comparison of very small values, and of values which are very different.

If l_1 and l_2 are the two coils whose coefficients of self-inductance, L_1 and L_2 , are to be compared, connect the two in parallel having in series with one of them a variable non-inductive resistance. Apply to their common terminals an harmonic electromotive force. This will cause currents to flow through the coils which will lag behind the impressed electromotive force by the phase angles :

$$\varphi_1 = \tan^{-1} \frac{\omega L_1}{R_1}, \text{ and } \varphi_2 = \tan^{-1} \frac{\omega L_2}{R_2},$$

in which R_1 and R_2 are the total resistances of the two parallel branches including the coils l_1 and l_2 , the variable resistance in one branch, and the connecting wires ; while ω is equal to 2π times the frequency of the alternating current or $2\pi n$. This shows that when the two ratios $\frac{\omega L_1}{R_1}$ and $\frac{\omega L_2}{R_2}$ are equal the two currents in the parallel branch will be in the same phase. If, then, we can by any means produce and detect this condition of phase equality, we get at once the relation between the two inductances. It is simply, $\frac{\omega L_1}{R_1} = \frac{\omega L_2}{R_2}$, or $\frac{L_1}{L_2} = \frac{R_1}{R_2}$.

The two primary currents can be brought into phase equality by varying the non-inductive resistance in one of the branches. This condition can be detected in the following manner : Introduce a sec-

ondary circuit which is made up of two equal coils, s_1 and s_2 , of many turns, and a telephone, all connected in series. Place coil s_1 close and parallel to l_1 , coil s_2 close to l_2 . The currents in l_1 and l_2 then induce electromotive forces in s_1 and s_2 respectively. The geometric sum of these two secondary electromotive forces will cause a current through the telephone. Suppose, however, the coils s_1 and s_2 to be so connected that when the currents in l_1 and l_2 are in phase with each other, the induced electromotive forces, Es_1 and Es_2 will be opposed to each other in the secondary circuit. The resultant secondary electromotive force, Es , will then be equal to $Es_1 - Es_2$; or, if these be made equal, will be zero. This condition will be detected by the silence of the telephone.

Fig. 1.

This silence indicates two things: 1st, that the two primary currents, Cl_1 and Cl_2 , are in the same phase; 2d, that $Es_1 = Es_2$. Since the first condition is true, when $\frac{\omega L_1}{R_1} = \frac{\omega L_2}{R_2}$, it can be made true by varying R_1 or R_2 . The second condition may be secured by varying the relative position of one of the secondary coils, S_2 for instance, with respect to its primary. This will change the electromotive force induced in it, and if it be moved back and forth until a position is found where the hum in the telephone is a minimum or zero we know that it is so placed that $Es_1 = Es_2$. If this adjustment does not give silence but only a minimum sound in the telephone it shows us that the two electromotive forces, Es_1 and Es_2 , are not in exact opposition. This means that the two primary currents, Cl_1 and Cl_2 , are not in the same phase and so R_1 or R_2 must be changed. After the value of R_1 or R_2 , and the position of S_2 have been adjusted so as to give silence in the receiver, it is only necessary to know $\frac{R_1}{R_2}$ to get $\frac{L_1}{L_2}$.

The proper position of S_2 can be found so quickly that the work

of obtaining a balance is but little more tedious than in the Wheatstone's bridge. When using the telephone as a current detector, however, there is no means of knowing which way R_1 or R_2 must be varied to reduce the phase difference between C_1 and C_2 . This corresponds to the same weakness of the telephone when used in the Wheatstone's bridge.

Fig. 2.

Fig. 3.

The phase relations of the currents and electromotive forces just discussed may be seen upon the two phase diagrams, Fig. 2 and Fig. 3. In these diagrams the vectors represent by their length the maximum values of the currents and electromotive forces, while their directions indicate their respective phases. The instantaneous values of all the quantities at any instant is given by the lengths of the projections of their respective vectors upon a moving vector which rotates in the clockwise direction through the entire 360° with each cycle of the current. According to this the secondary electromotive force, E_{S1} , reaches its maximum one quarter of a cycle later than the current C_1 which induces it, as it should. The vector E_{S2} is shown reversed or 180° out of its true position. This corresponds to the physical condition of being connected in series with E_{S1} in such a way that they oppose each other. As may be

seen from the two diagrams, the resultant secondary electromotive force may have any phase whatever and varies with every change in the value of R_1 or R_2 , or in the position of the coil S_2 .

Careful tests were made of the method in this form. With a fairly good telephone, whose resistance was much too low for the greatest sensitiveness, successive comparisons could be made to agree within one-half of 1 per cent. without difficulty.

In order that these results may not only agree among themselves, but may be correct, the apparatus must be so arranged that there shall be no appreciable inductive effects except those which are accounted for. To do this the two coils under comparison, I_1 and I_2 , which carry fairly large currents, must be so placed as to prevent mutual inductance between them. This was accomplished by placing them about two meters apart and in mutually perpendicular planes. Tests were made which showed this to be effectual. The lead wires from the source of the primary current to the point where the circuit divided were made up of twisted lamp cord and were kept as far as possible from the coils. The leads to each coil from this dividing point were placed about one centimeter apart and crossed over at frequent intervals. In comparing coils of low inductance it would be best to use lamp cord here also, as the effect of capacity would be very small compared with the inductive effect of the large current.

Any current which flows in the secondary circuit will, of course, react upon the primary currents, but when a balance is reached the secondary current disappears, and with it its disturbing influence. The secondary electromotive forces remain, however, and, together with the distributed capacity in the secondary coils, will set up a small local current in each coil. The amount of this hidden current is hard to estimate. It depends upon the capacity of the coil, which we can do no more than estimate, and its value is different at every point along the conductor. It can easily be shown to be very small however. Two pairs of coils were used at different times as secondaries, one pair of 300 turns each of No. 25 B. & S. wire and about 30 cm. in diameter; the other pair of 3000 turns each of No. 30 wire and about 10 cm. in diameter. The latter were made up of 50 layers of 60 turns each. If we were to con-

sider each layer as a plate of a condenser with alternate plates connected together the calculated capacity would be about .012 microfarad. In the coil, however, the difference in potential between layers is only about one-fiftieth of the potential across the terminals of the coil so the effective capacity will be something like .0002 microfarad. Suppose the total electromotive force induced in the coil to be as great as 50 volts or one volt per layer, and assume that the resistance and self-induction of the coil does not prevent it from becoming fully charged at each impulse of the current. Then, if the current in the primary passes through sixty cycles per second the local current in each secondary coil will be :

$$I = \frac{Q}{t} = \frac{EC}{t} = 120 \times 1. \times .000000002 \text{ or about } 24. \times 10^{-9}$$

ampères. If the coils of 300 turns each were used the local current would be much less than this. If this estimate be correct it shows that the reactive effect of these local currents upon the primary coils is entirely negligible. Further, it is easily seen that both primary coils are affected in the same way so that if the primaries were equal the error would be zero. Tests were made with both pairs of secondary coils upon the same primaries and the fact that no difference in the results due to the change if secondaries could be detected gives experimental evidence that the effect of the local currents is not measurable.

The next step in the development of the method was to replace the telephone by the electro-dynamometer. This required considerable change in details, but the principle remains the same, and the results showed a marked improvement in accuracy. The ordinary dynamometer having both coils connected in series is not sensitive to small currents, so a special instrument was used, in which the fixed coil was made up of 600 turns of No. 25 wire, and the suspended coil of 380 turns of No. 36. The latter was suspended by a phosphor bronze strip of about three circular mils cross-section and ten inches in length. This suspension strip and a similar strip in a loose spiral below the coil formed the connecting wires to the suspended coil. These strips would carry about .01 ampère for a short time, while the fixed coils of the instrument would safely carry 1,000 times as great a current for the same length of time. No

metal was used in the instrument near the coils except the conductors themselves, because it was designed for use in testing Professor Rowland's phase angle methods. In the new method it is used only to detect small currents, and for this could be improved by adding more turns to the suspended coil and an iron core to the fixed coils.

The small secondary current, which is to be made zero, is passed through the suspended coil, and the heavy primary current through the fixed coils of the dynamometer. In this way the instrument was made sensitive to currents of about 10^{-8} ampères. The direction of the deflection of the suspended coil reverses when the phase of the current in either coil is changed by 180° , so the instrument will give an indication concerning which way the resistances of the primary branches must be changed to bring the balance nearer. This is another advantage over the telephone. One peculiarity of the dynamometer must be remembered, however. If the two currents in the two coils differ by 90° in phase there will be no deflection. It becomes necessary, then, to distinguish between a zero deflection due to zero secondary current and a zero deflection due to 90° phase difference between the primary and secondary currents. The phase relations between these two currents may be seen in the phase diagrams, Figs. 2 and 3. We see that without some other arrangement it is impossible to tell whether the dynamometer is indicating zero secondary current or 90° phase difference. This difficulty was met in the following way: A non-inductive shunt, R_n , was connected across the terminals of the fixed coils of the dynamometer, with a key in its circuit so that it could be cut out at will. When this is connected the main primary current is divided between the two paths, and since the coils have considerable inductance while the shunt has not, the current in the fixed coils will lag behind the total primary current by a considerable angle. The currents and electromotive forces of the coils, I_1 , I_2 , s_1 , s_2 , do not change their relative phases in any way when this shunt connection is made, so the result is that the difference in phase between the two dynamometer currents is altered. If it was 90° before the shunt was connected, it must be something more or less afterwards, and unless the secondary current is zero, there will be a deflection of the suspended

coil. This result could have been attained easily in other ways, but this seemed the simplest and proved very satisfactory.

Fig. 4.

The connections as finally used are shown in the large diagram, Fig. 4.

The theory so far given is dependent upon the assumption that the electromotive force impressed upon the primary circuit is an harmonic function of the time. Since, however, the ratio of inductance to resistance in the two branches of the primary circuit are made the same at the final balance, it is true that, no matter what electromotive force is applied to their common terminals the currents in the two branches will rise and fall together. The two secondary electromotive forces will then have the same wave form and phase, and can be made to neutralize each other completely. This was shown to be true experimentally by the fact that when a balance had been obtained by the use of the dynamometer and an alternating current, no sound would be heard in the telephone when an interrupted current was used. This extreme test is certainly ample verification. The question of the source of the electromotive force is now an easy one, for any alternating current of any frequency, wave form, or degree of unsteadiness may be used. The tests given herewith were made with current from a two-phase, 60-cycle, Westinghouse generator. No tests of the wave form were made but the

makers state that it differs from a sine curve by less than five per cent. at every point.

Let us consider the conditions under which the method will give the best results. In short, what is the relation between L_1 and R_1 , or between L_2 and R_2 , for maximum sensitiveness? The resultant electromotive force in the secondary circuit, E_s , also the secondary current, C_s , upon which the observations depend, varies, for small differences, with the angle of phase difference between Cl_1 and Cl_2 . For maximum sensitiveness a given small change in the value of R_1 or R_2 should produce a maximum change in this angle, θ , of phase difference; or, the rate of change of R_1 with respect to θ should be a minimum. θ is equal to $\varphi_1 - \varphi_2$, where φ_1 and φ_2 are the phase angles by which the currents Cl_1 and Cl_2 lag behind the primary electromotive force, and to produce a change in θ we change either φ_1 or φ_2 while the other is unaltered. It follows then that a given change in φ_1 will produce an equal change in θ . This being true we can say that, for maximum sensitiveness, the rate of change of R_1 with respect to φ_1 must be a minimum.

The relation between R_1 and φ_1 is,

$$\tan \varphi_1 = \frac{\omega L_1}{R_1}$$

or,

$$R_1 = \omega L_1 \cot \varphi_1$$

The rate of change of R_1 with respect to φ_1 is then :

$$\frac{dR_1}{d\varphi_1} = -\omega L_1 \csc^2 \varphi_1.$$

This rate will be a minimum when :

$$\frac{d^2 R_1}{d\varphi_1^2} = 0, \text{ or when } 2\omega L \cot \varphi_1 \csc^2 \varphi_1 = 0.$$

Substituting for the trigonometric functions their values in terms of ωL_1 and R_1 we have,

$$2 \cdot \omega L_1 \cdot \frac{R_1}{\omega L_1} \cdot \frac{R_1^2 + \omega^2 L_1^2}{\omega^2 L_1^2} = 0.$$

Since R_1 is the only variable included in the expression we see that the most accurate results will be attainable when the resistances R_1 and R_2 are as near zero as possible. In other words, the greater

the angle by which the primary currents lag behind the electromotive force the greater will be the accuracy of the results. The formulæ also show that the sensitiveness when the primary currents lag 45° is one-half of the theoretical maximum. This is attainable in the case of most ordinary coils with a frequency of sixty.

As may be seen from Fig. 4 the telephone and dynamometer were connected so that either could be used in the secondary circuit. In practice the telephone was used in the preliminary adjustments and the dynamometer for obtaining the balance. The postoffice box was connected to the central posts where the leads to the primary coils were joined and as soon as an adjustment was completed a lead wire of one coil, say l_1 , was disconnected and the resistance of l_2 measured; then l_2 was disconnected and l_1 reconnected and its resistance measured. In this way R_1 and R_2 were determined before any appreciable change could occur for care was always taken to avoid heating the coils. The postoffice box used was the best in the laboratory and can be depended upon to give results within $\frac{1}{10}\%$ of the correct values while the ratios given by it are considerably more accurate because the two resistances were measured under the same conditions and as far as possible the same coils in the box were used. The resistances given below have all been corrected for the resistance of the leads from the postoffice box to the common terminals of the coils.

The inductances of five different coils were compared. The coils will be designated, l_1 , l_2 , l_3 , l_4 , and l_5 , with inductances, L_1 , L_2 , L_3 , L_4 and L_5 , respectively. Coils l_1 and l_2 were wound in regular layers in a rectangular groove cut in a disk of plaster of paris, and their dimensions accurately determined so that their inductances could be calculated. Coil l_1 was made up of 70 turns in 7 layers of No. 20, B. & S. gauge, single silk-covered copper wire. Its mean diameter was 38.605 cm. Calculated inductance, $L_1 = .0049046$ henry.

Coil l_2 was made up of 8 turns in 2 layers of No. 15 B. & S gauge, double cotton-covered copper wire. Its mean diameter was 38.681 cm. Calculated inductance, $L_2 = .0000726$ henry. These inductances were calculated by the formula due to Maxwell and Stefan.¹ The value calculated for L_1 was assumed to be correct

¹ Stefan, Wied. Ann., V. 22, p. 107, Apr., 1884.

and made the basis for finding the values of the others from the ratios. Coils l_3 , l_4 and l_5 were of unknown dimensions and wound on wooden frames.

DATA. TABLE I.
COMPARISON OF COILS l_1 AND l_3 .

R_1	R_3	L_3	
3.1574	27.865	.043285	
3.1198	27.559	.043324	
3.1104	27.479	.043327	
3.0940	27.363	.043375	
3.0690	27.115	.043330	
Average =			.043328
Probable error = .000010 =			.02 + per cent.

TABLE II.
COMPARISON OF COILS l_1 AND l_4 .

R	R_4	L_4	
2.9785	30.828	.050762	
2.9900	30.985	.050825	
3.0212	31.209	.050662	
3.0380	31.459	.050790	
2.9911	30.978	.050793	
2.9068	30.038	.050682	
2.9542	30.594	.050790	
2.9357	30.427	.050830	
Average =			.050767
Probable error = .000014 =			.03 — per cent.

TABLE III.
COMPARISON OF COILS l_3 AND l_4 .

R_3	R_4	L_4	
26.855	31.450	.050741	$\frac{R_4 L_3}{R_3} = \frac{L_4}{L_3} \text{ where } L_3 = .043328$
26.905	31.547	.050802	
26.897	31.540	.050808	
26.873	31.427	.050665	
26.800	31.406	.050774	
27.023	31.647	.050740	
26.915	31.513	.050730	
26.950	31.520	.050675	
Average =			.0507445
Probable error = .000012 =			.02 + per cent.

TABLE IV.
COMPARISON OF COILS L_1 AND L_5 .

R_1	R_5	L_5	
3.1641	.26870	.00041650	
3.1800	.26911	.00041510	
3.1905	.27033	.00041555	
3.1847	.27009	.00041593	
3.1838	.26996	.00041585	
3.1775	.27013	.00041695	
3.2108	.27266	.00041650	
Average =			.00041605
Probable error =			.00000016 = .04 — per cent.

TABLE V.
COMPARISON OF COILS L_2 AND L_5 .

R_5	R_2	L_2	
.76554	.13177	.000071615	
.76513	.13182	.000071680	
.76579	.13185	.000071628	
Average =			.000071641

The values given for L_2 in Table V. are based on the value of L_5 found in Table IV. This depends upon the calculated value of L_1 , so that in the approximate agreement between the above value and the calculated value of L_2 we have an inaccurate check upon the correctness of both. Table III. gives a direct check upon the correctness of the ratios $\frac{L_1}{L_3}$, $\frac{L_1}{L_4}$ and $\frac{L_3}{L_4}$. From the data there given the ratio of L_3 to L_4 is obtained, and the values of L_4 given are based upon the value of L_3 obtained in Table I. The two values L_4 differ by 0.05%, but this represents the combined error in three independent sets of observations.

The method here outlined might also be applied to the comparison of two capacities by connecting one in series with each primary coil L_1 and L_2 . The results, however, would require correction for the inductances L_1 and L_2 , and, further, unless the two condensers were exactly similar in construction, the result would be in error, because of the unequal rates of absorption.

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