VI.—The Theory of Circulants from 1880 to 1900.

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My last paper on the "Theory of Circulants" appeared four years ago (*Proc. Roy. Soc. Edin.*, xxxii, pp. 136-149), and brought up the history of the subject to 1880. I have now completed the research up to the close of the nineteenth century—the goal originally aimed at. It is considered that from the beginning of the present century the indexing of mathematical writings is sufficiently full and detailed to make it comparatively easy for a worker to follow up for himself the recent lines of development.

STUDNIČKA, F. J. (1880).

[Ueber eine neue Determinanteneigenschaft. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 50-54.]

The circulant dealt with, namely, $C(-1, 1, 1, 1, \dots, 1)$ is a case of Sylvester's of 1855 (*Hist.*, ii, p. 407).

SCOTT, R. F. (1880).

[Note on a determinant theorem of Mr Glaisher's. *Quart. Journ. of Math.*, xvii, pp. 129-132.]

Taking $C(a, b, c, d, e, f)$ as an example of an even-ordered circulant, and performing the operations

$$\begin{array}{lll} \text{col}_1 + \text{col}_4, & \text{col}_2 + \text{col}_5, & \text{col}_3 + \text{col}_6, \\ \text{row}_3 - \text{row}_1, & \text{row}_4 - \text{row}_2, & \text{row}_5 - \text{row}_3, \end{array}$$

Scott finds it equal to

$$\begin{vmatrix} a+d & b+e & c+f \\ c+f & a+d & b+e \\ b+e & c+f & a+d \end{vmatrix} \cdot \begin{vmatrix} a-d & b-e & c-f \\ f-c & a-d & b-e \\ e-b & f-c & a-d \end{vmatrix}.$$

The form of the second factor leads him to draw attention to what he calls a cyclically skew determinant, namely,

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ -a_n & a_1 & a_2 & \dots & a_{n-1} \\ -a_{n-1} & -a_n & a_1 & \dots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_2 & -a_3 & -a_4 & \dots & a_1 \end{vmatrix},$$

of which an equivalent is

$$\prod_{r=1}^{r=n} \left(a_1 + a_2 \omega_r + a_3 \omega_r^2 + \dots + a_n \omega_r^{n-1} \right)$$

where ω_r is any root of the equation

$$x^n + 1 = 0.$$

His theorem is thus to the effect that a circulant of the $(2m)^{\text{th}}$ order is expressible as the product of a circulant and a skew circulant each of the m^{th} order; and he gives an alternative proof of it dependent on the ordinary factorial expressions for the determinants in question. It is noted also that on account of the relation between the roots of the equations

$$x^{2n+1} + 1 = 0, \quad x^{2n+1} - 1 = 0,$$

the circulant of

$$a_1, -a_2, a_3, -a_4, \dots, -a_{2n}, a_{2n+1}$$

is equal to the skew circulant of

$$a_1, a_2, a_3, a_4, \dots, a_{2n}, a_{2n+1},$$

or, say, to

$$C(a_1, a_2, \dots, a_{2n+1}).$$

GLAISHER, J. W. L. (1880).

[On some algebraical expressions which are unaltered by certain substitutions. *Messenger of Math.*, x, pp. 60–63.]

In effect the main result of the paper is that if s stands for $a_1 + a_2 + \dots + a_n$, then

$$C(s - a_1, s - a_2, \dots, s - a_n) = (-1)^{n-1} (n-1) C(a_1, a_2, \dots, a_n),$$

and consequently that

$$\frac{C(s - a_1, s - a_2, \dots, s - a_n)}{(s - a_1) + (s - a_2) + \dots + (s - a_n)} = (-1)^{n-1} \frac{C(a_1, a_2, \dots, a_n)}{a_1 + a_2 + \dots + a_n},$$

a simple example of the latter being

$$\begin{vmatrix} 1 & c+a & a+b \\ 1 & b+c & c+a \\ 1 & a+b & b+c \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix},$$

or, in non-determinant notation,

$$\begin{aligned} (b+c)^2 + (c+a)^2 + (a+b)^2 - (c+a)(a+b) - (a+b)(b+c) - (b+c)(c+a) \\ = a^2 + b^2 + c^2 - bc - ca - ab. \end{aligned}$$

The result of making one or two other readily suggested substitutions for $a_1, a_2, \dots, a_n$ in $C(a_1, a_2, \dots, a_n)$ is also noted.

WEIHRAUCH, K. (1880).

[Ueber doppelt-orthosymmetrische Determinanten. *Zeitschrift f. Math. u. Phys.*, xxvi, pp. 64–70.]

Weihrauch establishes Spottiswoode's property by using Fürstenau's theorem of 1879; that is to say, he multiplies each element in the $(r, s)^{\text{th}}$ place by ω^{r-s} , where ω is any one of the n^{th} roots of unity, and then takes the sum of the rows.

More interesting is the fresh proposition that *the circulant of $a_1, a_2, \dots, a_n$, is equal to the sum of the coefficients of the equation whose roots are the n^{th} powers of the roots of the equation*

$$a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0.$$

By way of proof it is noted that in the case where n is 4 we have

$$C(a, b, c, d)$$

$= (a + b\omega_1 + c\omega_1^2 + d\omega_1^3)(a + b\omega_2 + c\omega_2^2 + d\omega_2^3)(a + b\omega_3 + c\omega_3^2 + d\omega_3^3)(a + b\omega_4 + c\omega_4^2 + d\omega_4^3)$,
so that, if ξ_1, ξ_2, ξ_3 be the roots of the equation

$$ax^3 + bx^2 + cx + d = 0,$$

there results

$$\begin{aligned} & C(a, b, c, d) \\ &= a(1 - \omega_1\xi_1)(1 - \omega_1\xi_2)(1 - \omega_1\xi_3) \\ & \quad \cdot a(1 - \omega_2\xi_1)(1 - \omega_2\xi_2)(1 - \omega_2\xi_3) \\ & \quad \cdot a(1 - \omega_3\xi_1)(1 - \omega_3\xi_2)(1 - \omega_3\xi_3) \\ & \quad \cdot a(1 - \omega_4\xi_1)(1 - \omega_4\xi_2)(1 - \omega_4\xi_3) \\ &= a^4 \cdot (1 - \omega_1\xi_1)(1 - \omega_2\xi_1)(1 - \omega_3\xi_1)(1 - \omega_4\xi_1) \\ & \quad \cdot (1 - \omega_1\xi_2)(1 - \omega_2\xi_2)(1 - \omega_3\xi_2)(1 - \omega_4\xi_2) \\ & \quad \cdot (1 - \omega_1\xi_3)(1 - \omega_2\xi_3)(1 - \omega_3\xi_3)(1 - \omega_4\xi_3) \\ &= a^4(1 - \xi_1^4)(1 - \xi_2^4)(1 - \xi_3^4), \end{aligned}$$

as desired.

Instead of this, however, Weihrauch in effect proposes to substitute a wider proposition, namely, that *the equation whose roots are the 4th powers of the roots of the equation $ax^3 + bx^2 + cx + d = 0$ is*

$$- \begin{vmatrix} a & b & c & d \\ b & c & d & ay \\ c & d & ay & by \\ d & ay & by & cy \end{vmatrix} = 0,$$

the previous result being thence obtained on putting $y=1$. To establish this $y^{\frac{1}{4}}$ is substituted for x in the given equation and rationalisation is

effected in Cayley's manner of 1852 (*Hist.*, ii, pp. 124-126), the full set of equations being

$$\left. \begin{aligned} ay^{\frac{1}{2}} + by^{\frac{1}{2}} + cy^{\frac{1}{2}} + d &= 0 \\ ay + by^{\frac{1}{2}} + cy^{\frac{1}{2}} + dy^{\frac{1}{2}} &= 0 \\ ay \cdot y^{\frac{1}{2}} + by + cy^{\frac{1}{2}} + dy^{\frac{1}{2}} &= 0 \\ ay \cdot y^{\frac{1}{2}} + by \cdot y^{\frac{1}{2}} + cy + dy^{\frac{1}{2}} &= 0 \end{aligned} \right\}$$

and the eliminants $y^{\frac{1}{2}}, y^{\frac{3}{2}}, y^{\frac{5}{2}}$.

It is also noted that the same procedure is effective when the roots of the new equation are to be the p^{th} powers of the roots of the given equation. Thus, p being 3, we substitute $y^{\frac{1}{3}}$ for x , and multiply twice by $y^{\frac{1}{3}}$, the set of equations now being

$$\left. \begin{aligned} ay + by^{\frac{2}{3}} + cy^{\frac{2}{3}} + d &= 0 \\ ay \cdot y^{\frac{1}{3}} + by + cy^{\frac{2}{3}} + dy^{\frac{1}{3}} &= 0 \\ ay \cdot y^{\frac{2}{3}} + by \cdot y^{\frac{1}{3}} + cy + dy^{\frac{2}{3}} &= 0 \end{aligned} \right\}$$

and the desired equation

$$- \begin{vmatrix} b & c & d+ay \\ c & d+ay & by \\ d+ay & by & cy \end{vmatrix} = 0.$$

Here, it may be remarked, the sum of the coefficients is still a circulant, namely, $C(d+a, b, c)$.

WEIHRAUCH, K. (1880).

[Werth einiger doppelt-orthosymmetrischer Determinanten. *Zeitschrift f. Math. u. Phys.*, xxvi, pp. 132-133.]

Three of the four results are Scott's of the year 1878, and the remaining one, namely,

$$C(1, 3, 6, \dots, \tfrac{1}{2}n(n+1)) = (-1)^{n-1} \frac{n^{n-2}(n+1)(n+2)}{3 \cdot 2^{n+1}} \left\{ (n+3)^n - (n+1)^n \right\},$$

resembles Scott's third.

MUIR, T. (1881).

[On new and recently discovered properties of certain symmetric determinants. *Quart. Journ. of Math.*, xviii, pp. 166-177.]

To start with, Glaisher's theorem regarding a circulant of the $(2n)^{\text{th}}$ order is shown to be readily obtained by multiplying together two slightly altered forms of the circulant. The first form is got by advancing the odd-numbered rows in order to the first n places, and then altering the signs in the even-numbered columns, the result being

$$(-1)^{\frac{1}{2}(3+1)^3} \cdot C(a, b, c, d, e, f) = \begin{vmatrix} a & -b & c & -d & e & -f \\ e & -f & a & -b & c & -d \\ c & -d & e & -f & a & -b \\ f & -a & b & -c & d & -e \\ d & -e & f & -a & b & -c \\ b & -c & d & -e & f & -a \end{vmatrix}.$$

The second form is got from the first by deleting the negative signs, reversing the order of the rows and then the order of the columns, the result being

$$(-1)^{\frac{1}{2}(3-1)^3} \cdot C(a, b, c, d, e, f) = \begin{vmatrix} a & f & e & d & c & b \\ c & b & a & f & e & d \\ e & d & c & b & a & f \\ b & a & f & e & d & c \\ d & c & b & a & f & e \\ f & e & d & c & b & a \end{vmatrix}.$$

From these two results by multiplication there is obtained

$$(-1)^3 \left\{ C(a, b, c, d, e, f) \right\}^2 = \begin{vmatrix} P & R & Q & . & . & . \\ Q & P & R & . & . & . \\ R & Q & P & . & . & . \\ . & . & . & -P & -R & -Q \\ . & . & . & -Q & -P & -R \\ . & . & . & -R & -Q & -P \end{vmatrix},$$

and therefore

$$C(a, b, c, d, e, f) = C(P, Q, R),$$

where

$$P = a^2 - bf + ce - d^2 + ec - fb = a^2 + 2ce - d^2 - 2bf,$$

$$Q = ae - bd + c^2 - db + ea - f^2 = c^2 + 2ea - f^2 - 2db,$$

$$R = ac - b^2 + ca - df + e^2 - fd = e^2 + 2ac - b^2 - 2fd.$$

In the second place, Scott's theorem regarding a circulant of the $(2n)^{\text{th}}$ order is shown to be immediately deducible from Zehfuss' theorem of 1862, the reason being that by reversing the order of the last n rows and thereafter the order of the last n columns the circulant becomes centrosymmetric. For example,

$$C(a, b, c, d, e, f) = \begin{vmatrix} a & b & c & f & e & d \\ f & a & b & e & d & c \\ e & f & a & d & c & b \\ b & c & d & a & f & e \\ c & d & e & b & a & f \\ d & e & f & c & b & a \end{vmatrix}$$

$$\begin{aligned}
&= \begin{vmatrix} a+d & b+e & c+f \\ c+f & a+d & b+e \\ b+e & c+f & a+d \end{vmatrix} \cdot \begin{vmatrix} a-d & b-e & c-f \\ f-c & a-d & b-e \\ e-b & f-c & a-d \end{vmatrix} \\
&= C(a+d, b+e, c+f) \cdot C'(a-d, b-e, c-f).
\end{aligned}$$

It is next found that when n is odd there is no difficulty in deducing Glaisher's theorem from Scott's. We have only to change the columns of $C(a+d, b+e, c+f)$ into rows, after the signs of the second row and second column of $C'(a-d, b-e, c-f)$, and then perform row-by-row multiplication, there being a series of identities like

$$(a+d, c+f, b+e) \times (a-d, e-b, c-f) = a^2 + 2ce - d^2 - 2bf$$

to be relied on.

Lastly, attention is drawn to the fact that *there is a second way of expressing a circulant of the $(4n+2)^{\text{th}}$ order as a circulant of the $(2n+1)^{\text{th}}$ order*, namely, by resolving it in accordance with Scott's theorem, substituting for the skew circulant an ordinary circulant, and then performing row-by-row multiplication. For example,

$$\begin{aligned}
C(a, b, c, d, e, f) &= \begin{vmatrix} a+f & b+e & c+d \\ c+d & a+f & b+e \\ b+e & c+d & a+f \end{vmatrix} \cdot \begin{vmatrix} a-f & e-b & c-d \\ c-d & a-f & e-b \\ e-b & c-d & a-f \end{vmatrix}, \\
&= \begin{vmatrix} L & M & N \\ N & L & M \\ M & N & L \end{vmatrix},
\end{aligned}$$

where

$$L = a^2 + e^2 + c^2 - b^2 - d^2 - f^2 = \sum a^2 - \sum b^2 \text{ say}$$

$$M = \sum ae - \sum bd + \sum ab - \sum ad$$

$$N = \sum ae - \sum bd - \sum ab + \sum ad,$$

the $\sum$ implying summation of the terms got by changing

and

$$\begin{aligned}
&a, e, c \text{ into } e, c, a, \\
&b, d, f \text{ into } d, f, b.
\end{aligned}$$

As companion results to Scott's are given (§§ 9, 10) the identities

$$\begin{vmatrix} a & b & c & d & e & f \\ c & a & b & e & f & d \\ b & c & a & f & d & e \\ e & d & f & a & c & b \\ d & f & e & b & a & c \\ f & e & d & c & b & a \end{vmatrix} = C(a+f, b+e, c+d) \cdot C(a-f, b-e, c-d),$$

$$\begin{vmatrix} a & b & c & d & e & f \\ -c & a & b & e & f & -d \\ -b & -c & a & f & -d & -e \\ -e & -d & f & a & -c & b \\ -d & f & e & b & a & -c \\ f & e & d & c & b & a \end{vmatrix} = C'(a+f, b+e, c+d) \cdot C'(a-f, b-e, c-d),$$

where the determinants on the left are no longer circulants but resemble circulants in having the elements of the first row repeated in each of the other rows. Both are centro-symmetric; and in addition the former of the two has the array of its first three rows made up of the array of $C(a, b, c)$ and the array of $(-1)^{\frac{1}{2}(3-1)^2}C(d, e, f)$, while in the other arrays of $C'(a, b, c)$ and $(-1)^{\frac{1}{2}(3-1)^2}C'(d, e, f)$ are similarly used.

MUIR, T. (1881).

[On the resolution of a certain determinant into quadratic factors.
Messenger of Math., xi, pp. 105-108.]

The theorem here established is that *the determinant whose every element is the sum of the corresponding elements of the circulants*

$$C(a_1, a_2, \dots, a_n), \quad (-1)^{\frac{1}{2}n(n-1)} \cdot C(b_1, b_2, \dots, b_n)$$

is divisible by

$$(a_1 + \omega a_2 + \dots + \omega^{n-1} a_n)(a_1 + \omega^{-1} a_2 + \dots + \omega^{-n+1} a_n) \\ - (b_1 + \omega b_2 + \dots + \omega^{n-1} b_n)(b_1 + \omega^{-1} b_2 + \dots + \omega^{-n+1} b_n),$$

ω being one of the imaginary n^{th} roots of unity.

When n is 5 the determinant in question is

$$\begin{vmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 & a_4 + b_4 & a_5 + b_5 \\ a_5 + b_2 & a_1 + b_3 & a_2 + b_4 & a_3 + b_5 & a_4 + b_1 \\ a_4 + b_3 & a_5 + b_4 & a_1 + b_5 & a_2 + b_1 & a_3 + b_2 \\ a_3 + b_4 & a_4 + b_5 & a_5 + b_1 & a_1 + b_2 & a_2 + b_3 \\ a_2 + b_5 & a_3 + b_1 & a_4 + b_2 & a_5 + b_3 & a_1 + b_4 \end{vmatrix},$$

and may be viewed as the eliminant of a set of linear homogeneous equations in x_1, x_2, x_3, x_4, x_5 . Using the multipliers $\omega^0, \omega^1, \omega^2, \omega^3, \omega^4$ with these equations and performing addition we obtain

$$\frac{a_1 + \omega^{-1} a_2 + \dots + \omega^{-4} a_5}{b_1 + \omega b_2 + \dots + \omega^4 b_5} = - \frac{x_1 + \omega x_5 + \omega^2 x_4 + \omega^3 x_3 + \omega^4 x_2}{x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5},$$

and therefore on putting ω^{-1} for ω

$$\frac{a_1 + \omega a_2 + \dots + \omega^4 a_5}{b_1 + \omega^{-1} b_2 + \dots + \omega^{-4} b_5} = - \frac{x_1 + \omega^4 x_5 + \omega^3 x_4 + \omega^2 x_3 + \omega x_2}{x_1 + \omega^4 x_2 + \omega^3 x_3 + \omega^2 x_4 + \omega x_5},$$

from which two equalities there results by multiplication

$$\frac{a_1 + \omega^{-1}a_2 + \dots + \omega^{-4}a_5}{b_1 + \omega b_2 + \dots + \omega^4b_5} \cdot \frac{a_1 + \omega a_2 + \dots + \omega^4a_5}{b_1 + \omega^{-1}b_2 + \dots + \omega^{-4}b_5} = 1.$$

This being independent of the ω 's furnishes a factor of the eliminant, and the factor thus reached is seen to be the quadratic factor specified in the enunciation of the theorem.

Note is taken that when n is odd there is the linear factor

$$a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n,$$

and that when n is even there is in addition to this factor

$$a_1 - a_2 + \dots - a_n + b_1 - b_2 + \dots - b_n.$$

Further, the multiplications indicated in the quadratic factor are actually performed, and ω made to disappear through substitutions like $2 \cos \frac{2}{5}\pi$ for $\omega + \omega^{-1}$.

MUIR, T. (1881).

[On circulants of odd order. *Quart. Journ. of Math.*, xviii, pp. 261-265.]

The main theorem here established is that the cofactor of $a_1 + a_2 + \dots + a_{2n+1}$ in $C(a_1, a_2, \dots, a_{2n+1})$ is expressible as a symmetric determinant of the n^{th} order whose distinct elements are the $\frac{1}{2}n(n+1)$ differences of the cyclic sums $\sum a_i^2, \sum a_1a_2, \sum a_1a_3, \dots, \sum a_1a_n$.

Calling these cyclic sums A_1, A_2, A_3, A_4 , in the case where n is 3, and multiplying $C(a_1, a_2, \dots, a_7)$ by itself in the form obtained by diminishing each row by the immediately preceding row, we find

$$(C_7)^2 = \begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_4 & A_3 & A_2 \\ A_2 - A_1 & A_1 - A_2 & A_2 - A_3 & A_3 - A_4 & . & A_4 - A_3 & A_3 - A_2 \\ A_3 - A_2 & A_2 - A_1 & A_1 - A_2 & A_2 - A_3 & A_3 - A_4 & . & A_4 - A_3 \\ A_4 - A_3 & A_3 - A_2 & A_2 - A_1 & A_1 - A_2 & A_2 - A_3 & A_3 - A_4 & . \\ . & A_4 - A_3 & A_3 - A_2 & A_2 - A_1 & A_1 - A_2 & A_2 - A_3 & A_3 - A_4 \\ A_3 - A_4 & . & A_4 - A_3 & A_3 - A_2 & A_2 - A_1 & A_1 - A_2 & A_2 - A_3 \\ A_2 - A_3 & A_3 - A_4 & . & A_4 - A_3 & A_3 - A_2 & A_2 - A_1 & A_1 - A_2 \end{vmatrix}.$$

Here the increasing of the first column by all the others shows that

$$A_1 + 2A_2 + 2A_3 + 2A_4, \text{ i.e. } (a_1 + a_2 + \dots + a_7)^2$$

is a factor, and the full result takes the form

$$(C_7)^2 = (a_1 + a_2 + \dots + a_7)^2 \begin{vmatrix} a & \beta & \gamma & . & -\gamma & -\beta \\ -a & a & \beta & \gamma & . & -\gamma \\ -\beta & -a & a & \beta & \gamma & . \\ -\gamma & -\beta & -a & a & \beta & \gamma \\ . & -\gamma & -\beta & -a & a & \beta \\ \gamma & . & -\gamma & -\beta & -a & a \end{vmatrix}$$

if for shortness' sake we write

$$\alpha, \beta, \gamma \text{ for } A_1 - A_2, \quad A_2 - A_3, \quad A_3 - A_4.$$

Multiplying this new determinant by

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & . \\ . & . & 1 & 1 & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & 1 & 1 & 1 & . \\ . & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

in succession we obtain

$$(C_7)^2 = (\Sigma a)^2 \begin{vmatrix} \alpha & \beta & \gamma & . & . & . \\ \beta & \alpha + \beta + \gamma & \beta + \gamma & . & . & . \\ \gamma & \beta + \gamma & \alpha + \beta & . & . & . \\ . & \gamma & \beta & \alpha & \beta & \gamma \\ -\gamma & . & \gamma & \beta & \alpha + \beta + \gamma & \beta + \gamma \\ -\beta & -\gamma & . & \gamma & \beta + \gamma & \alpha + \beta \end{vmatrix},$$

and thence

$$C_7 = \Sigma a \cdot \begin{vmatrix} A_1 - A_2 & A_2 - A_3 & A_3 - A_4 \\ A_2 - A_3 & A_1 - A_4 & A_2 - A_4 \\ A_3 - A_4 & A_2 - A_4 & A_1 - A_3 \end{vmatrix},$$

as desired.

The latter part of the proof is seen to turn on the transformability of a certain six-line persymmetric determinant into the square of a three-line axisymmetric determinant, and this property is stated to hold for a persymmetric determinant of the $(2n)^{\text{th}}$ order whose distinct elements are

$$-a_2, \dots, -a_n, 0, a_n, \dots, a_1, -a_1, \dots, -a_n, 0, a_n, \dots, a_3,$$

the elements of the n -line axisymmetric determinant being of the form

$$a_h + a_{h+1} + \dots + a_{h+k-1}.$$

TORELLI, G. (1882).

[Sui determinanti circolanti. *Rendic. . . . Accad. delle Sci. Fis. e Mat.* (Napoli), pp. 3-11.]

This paper, following avowedly on those of Glaisher and Scott and Muir, contains important generalisations.

The first theorem is to the effect that *any circulant of the $(rs)^{\text{th}}$ order is expressible as a circulant of the s^{th} order, and each of the elements of the latter is an aggregate of r -line minors, r^{s-1} in number, of the former.* Torelli's proof, which is the natural extension of Glaisher's, is stated in

perfectly general terms. We shall, for the sake of greater clearness, apply it only to the case where $rs=15$, the elements of the circulant being

$$a_1, a_2, a_3, \dots, a_{15}.$$

The fifteenth roots of 1 are the three third roots of each of the five fifth roots of 1, so that if $\gamma_1, \gamma_2, \gamma_3$ be the third roots of 1 and $\epsilon_1, \epsilon_2, \dots, \epsilon_5$ the fifth roots of 1, the fifteenth roots of 1 may be represented by

$$\gamma_1\epsilon_1^{\frac{1}{3}}, \gamma_2\epsilon_1^{\frac{1}{3}}, \gamma_3\epsilon_1^{\frac{1}{3}}; \gamma_1\epsilon_2^{\frac{1}{3}}, \gamma_2\epsilon_2^{\frac{1}{3}}, \gamma_3\epsilon_2^{\frac{1}{3}}; \dots; \gamma_1\epsilon_5^{\frac{1}{3}}, \gamma_2\epsilon_5^{\frac{1}{3}}, \gamma_3\epsilon_5^{\frac{1}{3}},$$

and the first of the fifteen linear factors of the given circulant by

$$a_1 + a_2(\gamma_1\epsilon_1^{\frac{1}{3}}) + a_3(\gamma_1\epsilon_1^{\frac{1}{3}})^2 + a_4(\gamma_1\epsilon_1^{\frac{1}{3}})^3 + \dots + a_{15}(\gamma_1\epsilon_1^{\frac{1}{3}})^{14}.$$

But this, by attending to the powers of $\gamma_1\epsilon_2^{\frac{1}{3}}$ and rearranging accordingly is equal to

$$(a_1 + a_4\epsilon_1 + a_7\epsilon_1^2 + a_{10}\epsilon_1^3 + a_{13}\epsilon_1^4) + (a_2 + a_5\epsilon_1 + \dots + a_{14}\epsilon_1^4)\gamma_1\epsilon_1^{\frac{1}{3}} + (a_3 + a_6\epsilon_1 + \dots + a_{15}\epsilon_1^4)^2\gamma_1\epsilon_1^{\frac{2}{3}},$$

and as the second and third factors differ from it merely in having γ_2, γ_3 respectively for γ_1 , the product of the three is the circulant

$$\begin{vmatrix} a_1 + a_4\epsilon_1 + \dots & (a_2 + a_5\epsilon_1 + \dots)\epsilon_1^{\frac{1}{3}} & (a_3 + a_6\epsilon_1 + \dots)\epsilon_1^{\frac{2}{3}} \\ (a_3 + a_6\epsilon_1 + \dots)\epsilon_1^{\frac{2}{3}} & a_1 + a_4\epsilon_1 + \dots & (a_2 + a_5\epsilon_1 + \dots)\epsilon_1^{\frac{1}{3}} \\ (a_2 + a_5\epsilon_1 + \dots)\epsilon_1^{\frac{1}{3}} & (a_3 + a_6\epsilon_1 + \dots)\epsilon_1^{\frac{2}{3}} & a_1 + a_4\epsilon_1 + \dots \end{vmatrix},$$

which, by multiplying the rows in order by $\epsilon_1^{\frac{1}{3}}, \epsilon_1^{\frac{2}{3}}, \epsilon_1^{\frac{1}{3}}$, and thereafter dividing the columns in order by the same, becomes

$$\begin{vmatrix} (a_1 + a_4\epsilon_1 + \dots) & a_2 + a_5\epsilon_1 + \dots & a_3 + a_6\epsilon_1 + \dots \\ (a_3 + a_6\epsilon_1 + \dots)\epsilon_1 & a_1 + a_4\epsilon_1 + \dots & a_2 + a_5\epsilon_1 + \dots \\ (a_2 + a_5\epsilon_1 + \dots)\epsilon_1 & (a_3 + a_6\epsilon_1 + \dots)\epsilon_1 & a_1 + a_4\epsilon_1 + \dots \end{vmatrix}.$$

Performing the multiplications by ϵ_1 , we see this to be equal to

$$\begin{vmatrix} a_1 + a_4\epsilon_1 + \dots + a_{13}\epsilon_1^4 & a_2 + a_5\epsilon_1 + \dots + a_{14}\epsilon_1^4 & a_3 + a_6\epsilon_1 + \dots + a_{15}\epsilon_1^4 \\ a_{15} + a_3\epsilon_1 + \dots + a_{12}\epsilon_1^4 & a_1 + a_4\epsilon_1 + \dots + a_{13}\epsilon_1^4 & a_2 + a_5\epsilon_1 + \dots + a_{14}\epsilon_1^4 \\ a_{14} + a_2\epsilon_1 + \dots + a_{11}\epsilon_1^4 & a_{15} + a_3\epsilon_1 + \dots + a_{12}\epsilon_1^4 & a_1 + a_4\epsilon_1 + \dots + a_{13}\epsilon_1^4 \end{vmatrix},$$

and therefore to be expressible as a sum of 5^3 determinants with monomial elements which can be grouped so as to take the form

$$P + Q\epsilon_1 + R\epsilon_1^2 + S\epsilon_1^3 + T\epsilon_1^4,$$

where each of the coefficients of the powers of ϵ_1 is a sum of 5^{3-1} such determinants. It is next observed that the fourth, fifth, sixth linear factors of the given circulant differ from the first, second, third merely in having ϵ_2 in place of ϵ_1 : their product therefore is

$$P + Q\epsilon_2 + R\epsilon_2^2 + S\epsilon_2^3 + T\epsilon_2^4:$$

and hence finally the product of the five triads is equal to

$$C(P, Q, R, S, T),$$

as predicated. Here r is 3 and s is 5.

Had we written the first linear factor of the given circulant in the form

$$a_1 + a_2(\epsilon_1\gamma_1^{\frac{1}{5}}) + a_3(\epsilon_1\gamma_1^{\frac{2}{5}}) + \dots + a_{15}(\epsilon_1\gamma_1^{\frac{14}{5}}),$$

we should have changed it at the outset into

$$(a_1 + a_6\gamma_1 + a_{11}\gamma_1^2) + (a_2 + a_7\gamma_1 + a_{12}\gamma_1^2)\epsilon_1\gamma_1^{\frac{1}{5}} + \dots + (a_5 + a_{10}\gamma_1 + a_{15}\gamma_1^2)\epsilon_1^4\gamma_1^{\frac{4}{5}},$$

and so have obtained for the product of the first five linear factors the circulant

$$C \left\{ (a_1 + a_6\gamma_1 + a_{11}\gamma_1^2), (a_2 + a_7\gamma_1 + a_{12}\gamma_1^2)\gamma_1^{\frac{1}{5}}, \dots, (a_5 + a_{10}\gamma_1 + a_{15}\gamma_1^2)\gamma_1^{\frac{4}{5}} \right\}.$$

This by multiplication of rows and division of columns we should have changed into

$$\begin{vmatrix} a_1 + a_6\gamma_1 + a_{11}\gamma_1^2 & a_2 + a_7\gamma_1 + a_{12}\gamma_1^2 & \dots & a_5 + a_{10}\gamma_1 + a_{15}\gamma_1^2 \\ a_{15} + a_5\gamma_1 + a_{10}\gamma_1^2 & a_1 + a_6\gamma_1 + a_{11}\gamma_1^2 & \dots & a_4 + a_9\gamma_1 + a_{14}\gamma_1^2 \\ a_{14} + a_4\gamma_1 + a_9\gamma_1^2 & a_{15} + a_5\gamma_1 + a_{10}\gamma_1^2 & \dots & a_3 + a_8\gamma_1 + a_{13}\gamma_1^2 \\ a_{13} + a_3\gamma_1 + a_8\gamma_1^2 & a_{14} + a_4\gamma_1 + a_9\gamma_1^2 & \dots & a_2 + a_7\gamma_1 + a_{12}\gamma_1^2 \\ a_{12} + a_2\gamma_1 + a_7\gamma_1^2 & a_{13} + a_3\gamma_1 + a_8\gamma_1^2 & \dots & a_1 + a_6\gamma_1 + a_{11}\gamma_1^2 \end{vmatrix},$$

from which we should have passed on to the form

$$X + Y\gamma_1 + Z\gamma_1^2.$$

Thereupon we should have seen the product of the next five linear factors to be

$$X + Y\gamma_2 + Z\gamma_2^2,$$

and the product of the last five to be

$$X + Y\gamma_3 + Z\gamma_3^2;$$

and therefore the product of the whole to be

$$C(X, Y, Z).$$

Here r is 5 and s is 3.

From this transformation of a circulant of composite order into one of lower order Torelli proceeds to deduce a circulant factor of the original. Thus, having reached $C(P, Q, R, S, T)$ as an equivalent of the fifteen-line circulant, we know that

$$P + Q + R + S + T$$

must be a factor of it, and, looking back at the three-line determinant whence $P, Q, \dots$ originated, we see that their sum equals

$$C(a_1 + a_4 + \dots + a_{13}, a_2 + a_5 + \dots + a_{15}, a_3 + a_6 + \dots + a_{15}),$$

which is the factor desired. In like manner from the existence of the form $C(X, Y, Z)$ there is obtained the factor

$$C(a_1 + a_6 + a_{11}, \quad a_2 + a_7 + a_{12}, \quad \dots, \quad a_5 + a_{10} + a_{15}).$$

The same result is established by operating directly on the original circulant; and this procedure has the advantage of giving at the same time a convenient expression for the cofactor. Thus by performing on $C(a_1, a_2, \dots, a_{15})$ the operations

$$\begin{aligned} &\text{row}_1 + \text{row}_4 + \text{row}_7 + \text{row}_{10} + \text{row}_{15} \\ &\text{row}_2 + \text{row}_5 + \dots \\ &\text{row}_3 + \text{row}_6 + \dots, \end{aligned}$$

and writing

$$\xi, \eta, \zeta \quad \text{for} \quad a_1 + a_4 + \dots, \quad a_2 + a_5 + \dots, \quad a_3 + a_6 + \dots,$$

we obtain

$$\begin{vmatrix} \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta \\ \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta \\ \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi \\ a_{13} & a_{14} & a_{15} & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_1 \end{vmatrix}.$$

By column-subtraction twelve corresponding elements in the first three rows can then be made 0, and the determinant resolved into two; for example, if we choose the last twelve elements there is obtained

$$C(\xi, \eta, \zeta) \cdot \begin{vmatrix} a_1 - a_{13} & a_2 - a_{14} & a_3 - a_{15} & a_4 - a_1 & \dots & a_{12} - a_9 \\ a_{15} - a_{12} & a_1 - a_{13} & a_2 - a_{14} & a_3 - a_{15} & \dots & a_{11} - a_8 \\ a_{14} - a_{11} & a_{15} - a_{12} & a_1 - a_{13} & a_2 - a_{14} & \dots & a_{10} - a_7 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_5 - a_2 & a_6 - a_3 & a_7 - a_4 & a_8 - a_5 & \dots & a_1 - a_{13} \end{vmatrix}.$$

Since 3 is prime to 5, the pair of circulant factors can themselves have no factor in common save $a_1 + a_2 + \dots + a_{15}$, and therefore

$$(a_1 + a_2 + \dots + a_{15})C(a_1, a_2, \dots, a_{15})$$

must be divisible by the product of the said pair. As an expression for the resulting quotient, which must be of the eighth degree in the original elements, Torelli obtains a determinant of the eighth order. He also extends the reasoning to a circulant having its order-number resolvable into more than two mutually prime integers: and finally makes application (§§ 8, 9) of his results to generalise Stern's cofactor theorem of 1871 and a theorem of Kummer's of 1851 on complex numbers.

It will suffice to add that throughout his paper Torelli makes due

reference to the corresponding results regarding *skew* circulants ("gobbi circolanti").

MUIR, T. (1882).

[A proposed general method for the solutions of equations. *Proceed. Philos. Soc. Glasgow*, xiii, p. 616: *Educ. Times*, xlv, p. 442: *Math. from Educ. Times* (2), v, pp. 106-107.]

The property proposed to be utilised is the resolvability of $C(x, a, b, c, \dots)$ into linear factors and the consequent solubility of the equation

$$C(x, a, b, c, \dots) = 0.$$

The application to the quartic $x^4 + px^2 + qx + r = 0$ is carried out by E. Nesbitt in the *Educ. Times*, and, in effect, to a specialised quintic by Cayley in the *Quart. Journ. of Math.*, xviii, pp. 154-167.

BHUT, A. B. (1882).

[Question 7072. *Educ. Times*, xxxv, p. 155: *Math. from Educ. Times*, xl, p. 48.]

The problem is to find x, y, z, u from the equations

$$\begin{vmatrix} x & y & z \\ u & x & y \\ z & u & x \end{vmatrix} = a^3, \quad \begin{vmatrix} y & z & u \\ x & y & z \\ u & x & y \end{vmatrix} = b^3, \quad \begin{vmatrix} z & u & x \\ y & z & u \\ x & y & z \end{vmatrix} = c^3, \quad \begin{vmatrix} u & x & y \\ z & u & x \\ y & z & u \end{vmatrix} = d^3,$$

and it is not observed that this is the same as to express x, y, z, u in terms of their complementary minors in $C(x, y, z, u)$. Calling these when signed X, Y, Z, U , we have from the theorem regarding a minor of the adjugate

$$\begin{vmatrix} X & Y & Z \\ U & X & Y \\ Z & U & X \end{vmatrix} = x \begin{vmatrix} x & y & z & u \\ u & x & y & z \\ z & u & x & y \\ y & z & u & x \end{vmatrix}^2,$$

and therefore

$$x = \begin{vmatrix} X & Y & Z \\ U & X & Y \\ Z & U & X \end{vmatrix} \div \begin{vmatrix} X & Y & Z & U \\ U & X & Y & Z \\ Z & U & X & Y \\ Y & Z & U & X \end{vmatrix}^{\frac{1}{2}}.$$

CESÀRO, E. (1883): NEUBERG, J. (1883).

[Question 245. *Mathesis*, iii, pp. 118-119; vi, pp. 60-62.]

[Question 273. *Mathesis*, iii, p. 192; viii, p. 215; x, pp. 117-119.]

These concern special circulants which have already been fully dealt with.

FORSYTH, A. R. (1884).

[On certain symmetric products involving prime roots of unity. *Messenger of Math.*, xiv, pp. 40-56.]

Forsyth begins in effect by equating two forms of the eliminant of

$$\left. \begin{aligned} x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots &\equiv (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n) = 0 \\ x^s - 1 &\equiv (x - \omega_1)(x - \omega_2) \dots (x - \omega_s) = 0 \end{aligned} \right\}$$

obtaining naturally

$$\prod_{x=\omega_1}^{x=\omega_s} (x^n - p_1 x^{n-1} + \dots) = (-1)^{n(s+1)} \cdot (1 - \alpha_1^s)(1 - \alpha_2^s) \dots (1 - \alpha_n^s),$$

and when $s=3$

$$C(\sigma, \tau, \nu) = (1 - \alpha_1^3)(1 - \alpha_2^3) \dots (1 - \alpha_n^3)$$

where

$$\sigma = 1 - p_3 + p_6 - p_9 + \dots$$

$$\tau = -p_1 + p_4 - p_7 + p_{10} - \dots$$

$$\nu = p_2 - p_5 + p_8 - p_{11} + \dots$$

Three paragraphs (pp. 43-46) are then devoted to the evaluation of circulants. The procedure adopted is essentially the same as Minozzi's of 1878. As illustrative examples of its effectiveness the five-line and seven-line circulants are taken, the result in the former case agreeing with Glaisher's, and in the latter being new.

MUIR, T. (1884).

[Note on the final expansion of circulants. *Messenger of Math.*, xiv, pp. 169-175.]

The mode of expansion referred to is that used by Minozzi and Forsyth, the main object being to correct, if necessary, the latter's expansion of $C(a_0, a_1, a_2, \dots, a_6)$, or, say, $C(0, 1, 2, 3, 4, 5, 6)$. It is first pointed out that the procedure is considerably shortened by using an additional property, namely, that when

$$Aa_0^{\alpha}a_1^{\beta}a_2^{\gamma}a_3^{\delta}a_4^{\epsilon}a_5^{\zeta}a_6^{\eta}$$

is a term of the expansion, then also is

$$Aa_0^{\alpha}a_m^{\beta}a_n^{\gamma}a_p^{\delta}a_q^{\epsilon}a_r^{\zeta}a_s^{\eta}$$

a term, where m, n, p, q, r, s are numbers less than 7, and such that

$$m, 2m, 3m, 4m, 5m, 6m \equiv m, n, p, q, r, s \pmod{7}.$$

Thus, having got the term $-27a_0^3a_1^2a_2a_3$ in the expansion of $C(0, 1, 2, 3, 4, 5, 6)$, we operate on the suffixes 1, 2, 3 by multiplying in succession by the numbers 2, 3, 4, 5, 6 and casting out the sevens, the result being

the new sets of suffixes 2, 4, 6; 3, 6, 2; 4, 1, 5; 5, 3, 1; 6, 5, 4: and so finally the complete set of terms

$$-27a_0^3(a_1^2a_2a_3 + a_2^2a_4a_6 + a_3^2a_6a_2 + a_4^2a_1a_5 + a_5^2a_3a_1 + a_6^2a_5a_4).$$

The "reason of the rule" is that

$$\begin{aligned} C(0, 1, 2, 3, 4, 5, 6) &= C(0, 2, 4, 6, 1, 3, 5), \\ &= C(0, 3, 6, 2, 5, 1, 4), \\ &= C(0, 4, 1, 5, 2, 6, 3), \\ &= C(0, 5, 3, 1, 6, 4, 2), \\ &= C(0, 6, 5, 4, 3, 2, 1); \end{aligned}$$

as we can show either by transposition of rows and columns, or by considering the circulant in its form as a product of seven linear factors and using the fact that, if ω be one of the imaginary seventh roots of unity, $\omega^2, \omega^3, \omega^4, \omega^5, \omega^6$ are the others.

A caution is then given that *two terms having apparently the same form have not necessarily the same coefficient*; and as an illustration it is shown that in Forsyth's expression half of the terms with the coefficient 35 should have the coefficient -14 and that the coefficient of his final term should be changed from -448 to -105, the whole expansion then being

$$\begin{aligned} &\sum a_0^7 \\ &-7 \sum a_0^5 (a_1a_6 + a_2a_5 + a_3a_4) \\ &+7 \sum a_0^4 (a_1^2a_5 + a_2^2a_3 + a_3^2a_1 + a_4^2a_6 + a_5^2a_4 + a_6^2a_2) \\ &+14 \sum a_0^4 (a_1a_2a_4 + a_3a_5a_6) \\ &-7 \sum a_0^3 (a_1^3a_6 + a_2^3a_5 + a_3^3a_4) \\ &-21 \sum a_0^3 (a_1^2a_2a_3 + a_2^2a_4a_6 + a_3^2a_6a_2 + a_4^2a_1a_5 + a_5^2a_3a_1 + a_6^2a_5a_4) \\ &+14 \sum a_0^3 (a_1^2a_6^2 + a_2^2a_5^2 + a_3^2a_4^2) \\ &+7 \sum a_0^3 (a_1a_2a_5a_6 + a_2a_4a_3a_5 + a_3a_6a_1a_4) \\ &-7 \sum a_0^3 (a_1^2a_2^2a_4^2 + a_2^2a_6^2a_5^2) \\ &+35 \sum a_0^2 (a_1^2a_3a_4a_5 + a_2^2a_6a_1a_3 + a_3^2a_2a_6a_1) \\ &-14 \sum a_0^2 (a_1^2a_2a_4a_6 + a_2^2a_4a_1a_5 + a_3^2a_6a_5a_4) \\ &-105a_0a_1a_2a_3a_4a_5a_6; \end{aligned}$$

or, with the help of the fresh property, in the still more condensed form

$$\sum_{\mu=0}^{\mu=6} \sum_{\nu=1}^{\nu=6} \left(a_{\mu}^7 - 7a_{\mu}^5 a_{\mu+\nu} a_{\mu+6\nu} + 7a_{\mu}^4 a_{\mu+\nu}^2 a_{\mu+5\nu} + \dots \right),$$

where every suffix greater than 6 is to be made less than 7 by subtracting a multiple of 7 from it and duplicates are to be neglected.

An entirely different mode of saving labour in the computation rests on the fact that all the terms which contain an element in the first power can be got by evaluating a zero-axial determinant of the next lower order. For example, since every term of $C(a_0, a_1, a_2, a_3, a_4)$ is of this kind except $\sum a_0^5$, we evaluate

$$5 \begin{vmatrix} . & a_1 & a_2 & a_3 \\ a_4 & . & a_1 & a_2 \\ a_3 & a_4 & . & a_1 \\ a_2 & a_3 & a_4 & . \end{vmatrix},$$

and thus finding the cofactor of a_0 to be

$$-5(a_1^3 a_2 + a_2^3 a_4 + a_3^3 a_1 + a_4^3 a_2) + 5(a_1^2 a_4^2 + a_2^2 a_3^2) - 5a_1 a_2 a_3 a_4$$

we obtain the full expansion

$$\sum a_0^5 - 5 \sum a_0^3 (a_1 a_4 + a_2 a_3) + 5 \sum a_0^2 (a_1^2 a_3 + a_2^2 a_4) - 5a_0 a_1 a_2 a_3 a_4.$$

In the case of the seven-line continuant the saving is almost quite as great, since in addition to $\sum a_0^7$ the term $14 \sum a_0^3 a_1^2 a_6^2$ is the only one that does not involve a first power of an element.

MUIR, T. (1885).

[Detached theorems on circulants. *Trans. Roy. Soc. Edin.*, xxxii, pp. 639–643.]

The first of the theorems is that *if in a circulant the element in the place (p, q) be by the nature of the circulant the same as the element in the place (r, s), the complementary minor of the former element is to that of the latter as $(-1)^{r+s} : (-1)^{p+q}$* . But more than this is intended to be brought out, namely, that by cyclical transposition of rows and columns the element in the place (r, s) may be made to take the place (p, q) and the determinant be in outward form exactly the same as before. The second theorem is Stern's of 1871, although not so credited. The mode of establishing it is to alter the first column of $C(a, b, c, d, e)$ so as to make it possible to remove $a + b\omega + c\omega^2 + d\omega^3 + e\omega^4$ as a factor, and leave

$$1, \omega, \omega^2, \omega^3, \omega^4$$

for the elements of the column. The third theorem is that *if the linear factors of a circulant, say of the fifth order, be $\alpha, \beta, \gamma, \dots$, and the complementary minors of the elements of the first row be $A, -B, C, -\dots$ then*

$$C(\alpha\beta\gamma\delta, \alpha\beta\gamma\epsilon, \alpha\beta\delta\epsilon, \alpha\gamma\delta\epsilon, \beta\gamma\delta\epsilon) = 5^5 ABCDE.$$

This is proved by using the preceding theorem five times over in the form

$$\begin{aligned} A + \omega E + \omega^2 D + \omega^3 C + \omega^4 B &= \beta\gamma\delta\epsilon, \\ A + \omega^2 E + \omega^4 D + \omega C + \omega^3 B &= \gamma\delta\epsilon\alpha, \\ &\dots \end{aligned}$$

and thence obtaining the linear factors of

$$C(\alpha\beta\gamma\delta, \alpha\beta\gamma\epsilon, \alpha\beta\delta\epsilon, \alpha\gamma\delta\epsilon, \beta\gamma\delta\epsilon)$$

in the form

$$5A, 5B, 5C, 5D, 5E.$$

The fourth theorem is that *if the elements of the first row of a circulant of the n^{th} order be multiplied by $\omega^n, \omega^{n-1}, \dots, \omega$ respectively, the elements of the second row by $\omega^{n-1}, \omega^{n-2}, \dots, \omega, \omega^n$ respectively, the elements of the third row by $\omega^{n-2}, \omega^{n-3}, \dots, \omega, \omega^n, \omega^{n-1}$ respectively, and so on, where ω is any n^{th} root of unity, the circulant is unaltered in value. Here for proof we have only to multiply the rows in order by $\omega^0, \omega^1, \omega^2, \omega^3, \omega^4$, and then divide the columns by $\omega^5, \omega^4, \omega^3, \omega^2, \omega$. The fifth theorem concerns skew circulants, being the analogue of the same author's theorem of 1881 regarding ordinary circulants.*

In order to throw light on the law of the coefficients of the final expansion of $C(a, b, c, d, e)$ the determinant

$$\begin{vmatrix} a\alpha & b\beta & c\gamma & d\delta & e\epsilon \\ e\beta & a\gamma & b\delta & c\epsilon & d\alpha \\ d\gamma & e\delta & a\epsilon & b\alpha & c\beta \\ c\delta & d\epsilon & e\alpha & a\beta & b\gamma \\ b\epsilon & c\alpha & d\beta & e\gamma & a\delta \end{vmatrix}$$

is framed by attaching to each element of $C(a, b, c, d, e)$ the corresponding element of $(-1)^{3+2+1}C(\alpha, \beta, \gamma, \delta, \epsilon)$. This, which degenerates into the ordinary five-line circulant on putting $a=b=c=d=e=1$ or $\alpha=\beta=\gamma=\delta=\epsilon=1$, is found to be equal to

$$\begin{aligned} &\sum \alpha^5 \cdot \alpha\beta\gamma\delta\epsilon - \sum \alpha^3\beta\epsilon \cdot \sum \alpha^3\gamma\delta + \sum \alpha^2b^2d \cdot \sum \alpha^2\beta\gamma^2 \\ &+ \sum \alpha^5 \cdot abcde - \sum \alpha^3\beta\epsilon \cdot \sum \alpha^3cd + \sum \alpha^2\beta^2d \cdot \sum \alpha^2bc^2 - 10abcde \cdot \alpha\beta\gamma\delta\epsilon \end{aligned}$$

where all the coefficients of the ordinary circulant except the last are in a manner "sifted" into their unit constituents.

ROBINSON, L. W. (1889): THYAGARAGAIYAR, V. R. (1899).

[Question 10254. *Educ. Times*, xlii, p. 332: *Math. from Educ. Times*, liv, p. 54.]

[Question 14240. *Educ. Times*, lii, p. 270: *Math. from Educ. Times*, lxxiii, p. 56.]

Already known results.

ROGERS, L. J. (1891).

[Question 10992. *Educ. Times*, xliv, p. 199.]

This concerns the first primary minor of $C(a, b, c, \dots)$ when $a+b+c+\dots=0$, the problem being to resolve the said minor into linear factors. No solution ever appeared. Probably the proposer had in his mind special circulants of the form

$$C(a-b, b-c, c-d, d-a),$$

the first primary minor of which is

$$\begin{vmatrix} a-b & b-c & c-d \\ d-a & a-b & b-c \\ c-d & d-a & a-b \end{vmatrix} \text{ and } \therefore = \begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

so that its factors are factors of $C(a, b, c, d)$.

WOODALL, H. J. (1894).

[Question 12426. *Educ. Times*, xlvii, p. 305: *Math. from Educ. Times*, lxiii, pp. 35-36.]

The real point of interest here concerns the product of two circulants. If the circulants be zero-axial, the ordinary multiplication-theorem, though of course producing a circulant, does not produce one that is zero-axial; so that, strictly speaking, the product is in this case not of the same form as its factors. It is shown, however, that what is not true of the zero-axial circulant is true of the zero-axial circulant divided by the sum of its elements. Thus, by row-multiplication

$$\begin{vmatrix} . & a & b \\ b & . & a \\ a & b & . \end{vmatrix} \cdot \begin{vmatrix} . & x & y \\ y & . & x \\ x & y & . \end{vmatrix} = \begin{vmatrix} ax+by & bx & ay \\ ay & ax+by & bx \\ bx & ay & ax+by \end{vmatrix},$$

and consequently on removing the factors $a+b$, $x+y$, $(a+b)(x+y)$ from the three determinants respectively we have

$$\begin{vmatrix} 1 & 1 & 1 \\ b & . & a \\ a & b & . \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ y & . & x \\ x & y & . \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ ay & ax+by & bx \\ bx & ay & ax+by \end{vmatrix},$$

where, by diminishing in the determinant on the right the second and third rows by $ax+by$ times the first, there results

$$\begin{vmatrix} 1 & 1 & 1 \\ ay-ax-by & . & bx-ax-by \\ bx-ax-by & ay-ax-by & . \end{vmatrix}.$$

In other words,

$$(a^2-ab+b^2)(x^2-xy+y^2)=P^2-PQ+Q^2,$$

if P, Q be taken equal to $bx-ax-by, ay-ax-by$ respectively.

MUIR, T. (1896).

[On the resolution of circulants into rational factors. *Proc. Roy. Soc. Edin.*, xxi, pp. 369-382.]

Viewing the circulant $C(a_1, a_2, \dots, a_n)$ as the eliminant of the equations

$$\left. \begin{aligned} a_1x^{n-1}+a_2x^{n-2}+\dots+a_n &= 0 \\ x^n-1 &= 0 \end{aligned} \right\},$$

the author concludes that *corresponding to every rational factor of x^n-1 there must be a rational factor of the circulant*; and his object is to determine such factors and to present them, when found, in the most convenient forms. The means employed for the purpose is elimination.

Thus, to begin with, Catalan's factor corresponding to

$$(x^n-1)/(x-1)$$

is obtained by deducing from the equations

$$\left. \begin{aligned} ax^4+bx^3+cx^2+dx+e &= 0 \\ x^4+x^3+x^2+x+1 &= 0 \end{aligned} \right\}$$

the cubic

$$(b-a)x^3+(c-a)x^2+(d-a)x+(e-a)=0,$$

thence by cyclical substitution three other equations, and finally

$$\begin{vmatrix} b-a & c-a & d-a & e-a \\ c-b & d-b & e-b & a-b \\ d-c & e-c & a-c & b-c \\ e-d & a-d & b-d & c-d \end{vmatrix},$$

which being inherently persymmetric is more conveniently written

$$P(b-c, c-d, d-e, e-a, a-b, b-c, c-d).$$

The process is then extended to find the factor corresponding to

$$(x^{2m}-1)/(x^2-1);$$

in other words, to find the cofactor of

$$(a_1 + a_2 + \dots + a_{2m})(a_1 - a_2 + a_3 - \dots - a_{2m}) \text{ in } C(a_1, a_2, \dots, a_{2m}).$$

Thus, in the case where $2m=8$, from the equation

$$ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h = 0$$

the seventh and sixth powers of x are removed with the help of the other equation

$$x^6 + x^4 + x^2 + 1 = 0,$$

the result being

$$(c-a)x^5 + (d-b)x^4 + (e-a)x^3 + (f-b)x^2 + (g-a)x + (h-b) = 0,$$

whence by cyclical substitution and elimination we obtain

$$\begin{vmatrix} c-a & d-b & e-a & f-b & g-a & h-b \\ d-b & e-c & f-b & g-c & h-b & a-c \\ e-c & f-d & g-c & h-d & a-c & b-d \\ f-d & g-e & h-d & a-e & b-d & c-e \\ g-e & h-f & a-e & b-f & c-e & d-f \\ h-f & a-g & b-f & c-g & d-f & e-g \end{vmatrix},$$

which, again, can be expressed persymmetrically, namely,

$$P(c-e, d-f, e-g, f-h, g-a, h-b, a-c, b-d, c-e, d-f, e-g).$$

In the next place, the factor of $C(a_1, a_2, a_3, a_4, a_5, a_6)$ corresponding to the factor x^2+x+1 of x^6-1 is obtained. We simply, as before, remove in order from the equation

$$a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6 = 0,$$

with the help of the equation

$$x^2 + x + 1 = 0,$$

the fifth, fourth, third, second powers of x , with the result

$$(a_5 - a_4 + a_2 - a_1)x + (a_6 - a_4 + a_3 - a_1) = 0,$$

whence the eliminant

$$\begin{vmatrix} a_5 - a_4 + a_2 - a_1 & a_6 - a_4 + a_3 - a_1 \\ a_6 - a_5 + a_3 - a_2 & a_1 - a_5 + a_4 - a_2 \end{vmatrix},$$

which, as before, can be expressed persymmetrically.

Proceeding in this way, the prime factors of the circulants up to and including that of the tenth order are calculated, and are tabulated for reference. By reason of the persymmetry and the notation for it, the table occupies only one page; and it is pointed out how it might have been made still shorter by using the fact that when once the first element of any of the persymmetric determinants is obtained, the others follow

at once in cyclical procession. For example, if we desire to have the factor of $C(a_1, a_2, \dots, a_{10})$ corresponding to the factor $x^4 + x^3 + x^2 + x + 1$ of $x^{10} - 1$, and know from the table that the first element of it is $a_7 - a_8 + a_2 - a_3$, we have the whole of it immediately, namely,

$$\begin{vmatrix} a_7 - a_8 + a_2 - a_3 & a_8 - a_9 + a_3 - a_4 & a_9 - a_{10} + a_4 - a_5 & a_{10} - a_1 + a_5 - a_6 \\ a_8 - a_9 + a_3 - a_4 & a_9 - a_{10} + a_4 - a_5 & a_{10} - a_1 + a_5 - a_6 & a_1 - a_2 + a_6 - a_7 \\ a_9 - a_{10} + a_4 - a_5 & a_{10} - a_1 + a_5 - a_6 & a_1 - a_2 + a_6 - a_7 & a_2 - a_3 + a_7 - a_8 \\ a_{10} - a_1 + a_5 - a_6 & a_1 - a_2 + a_6 - a_7 & a_2 - a_3 + a_7 - a_8 & a_3 - a_4 + a_8 - a_9 \end{vmatrix}$$

The process is seen to be general, and to consist in deducing from the equation of higher degree, by means of the other equation, an equation of lower degree than the latter, and in then using cyclical substitution and elimination.

The next part of the paper, however, makes it clear that the same results can be obtained, when known, by operating on the circulants themselves in accordance with the laws of determinants. Thus, from the eight-line circulant it is shown how to remove the second linear factor after the first, and how then to resolve the remaining six-line determinant into the two previously found persymmetric determinants of the second and fourth orders respectively.

Finally, the results are applied to the special case of circulants whose elements, when the first is left aside, read forward the same as backward: and it is shown that, when from such circulants the rational linear factor or factors are removed, the remaining factor is a complete square. This is possible from the fact that the persymmetric cofactors above obtained are then zero-axial and skew. Thus it is found that

$$\begin{aligned} & C(a_1, a_2, a_3, a_4, a_4, a_3, a_2) \\ &= (a_1 + 2a_2 + 2a_3 + 2a_4) \begin{vmatrix} a_3 - a_4 & a_2 - a_3 & a_1 - a_2 & a_2 - a_1 & a_3 - a_2 \\ & a_3 - a_4 & a_2 - a_3 & a_1 - a_2 & a_2 - a_1 \\ & & a_3 - a_4 & a_2 - a_3 & a_1 - a_2 \\ & & & a_3 - a_4 & a_2 - a_3 \\ & & & & a_3 - a_4 \end{vmatrix}^2 \end{aligned}$$

and that

$$\begin{aligned} & C(a_1, a_2, a_3, a_4, a_5, a_4, a_3, a_2) \\ &= (a_1 + 2a_2 + 2a_3 + 2a_4 + a_5)(a_1 - 2a_2 + 2a_3 - 2a_4 + a_5) \\ &\quad \cdot \begin{vmatrix} a_3 - a_1 & a_4 - a_2 & a_5 - a_3 & & a_3 - a_5 \\ & a_3 - a_1 & a_4 - a_2 & a_5 - a_3 & \\ & & a_3 - a_1 & a_4 - a_2 & a_5 - a_3 \\ & & & a_3 - a_1 & a_4 - a_2 \\ & & & & a_3 - a_1 \end{vmatrix}^2. \end{aligned}$$

Further, it is shown that such circulants being centrosymmetric can be resolved into factors in quite a different way, the repeated factor now appearing not as a pfaffian but as a determinant; so that by the equatement of the two resolutions we obtain equality between the two determinants

$$\begin{vmatrix} a_1 - a_3 & a_2 - a_4 & a_3 - a_4 \\ a_2 - a_4 & a_1 - a_4 & a_2 - a_3 \\ a_3 - a_4 & a_2 - a_3 & a_1 - a_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 - a_3 & a_2 - a_4 & a_3 - a_5 \\ a_2 - a_4 & a_1 - a_5 & a_2 - a_4 \\ a_3 - a_5 & a_2 - a_4 & a_1 - a_3 \end{vmatrix}$$

and the above-given pfaffians respectively.

BICKMORE, C. E. (1898).

[Question 13748. *Educ. Times*, li, p. 40: *Math. from Educ. Times*, lxix, p. 85.]

The result here recorded is that the skew circulant of the fourth order, $C'(x, y, z, w)$, whose final expansion is

$$x^4 + y^4 + z^4 + w^4 + 2(x^2z^2 + y^2w^2) + 4(x^2yw - y^2zx - z^2wy + w^2xz),$$

is expressible in each of the forms

$$a^2 + b^2, \quad c^2 + 2d^2, \quad e^2 - 2f^2$$

where a, b, c, d, e, f are definite quadratic functions of x, y, z, w , namely,

$$\begin{aligned} x^2 + 2yw - z^2, & \quad y^2 - 2zx - w^2, \\ x^2 - y^2 + z^2 - w^2, & \quad xy - yz + zw + wx, \\ x^2 + y^2 + z^2 + w^2, & \quad xy + yz + zw - wx. \end{aligned}$$

This is reached by taking the four linear factors

$$x + \zeta y + \zeta^2 z + \zeta^3 w, \quad x + \zeta^3 y - \zeta^2 z + \zeta w, \quad x - \zeta y + \zeta^2 z - \zeta^3 w, \quad x - \zeta^3 y - \zeta^2 z - \zeta w$$

of the circulant and multiplying them in the three different ways as a pair of pairs. For example, the product of the first and second factors being

$$x^2 - y^2 + z^2 - w^2 + (\zeta + \zeta^3)(xy - yz + zw + wx),$$

and the product of the third and fourth being

$$x^2 - y^2 + z^2 - w^2 - (\zeta + \zeta^3)(xy - yz + zw + wx),$$

the product of the four is got in the form

$$(x^2 - y^2 + z^2 - w^2)^2 + 2(xy - yz + zw + wx)^2.$$

We note for ourselves that by the multiplication-theorem we obtain

$$\begin{vmatrix} x & y & z & w \\ -w & x & y & z \\ -z & -w & x & y \\ -y & -z & -w & x \end{vmatrix}^2 = \begin{vmatrix} e & f & . & -f \\ f & e & f & . \\ . & f & e & f \\ -f & . & f & e \end{vmatrix} \\ = \begin{vmatrix} e & f & . & . \\ 2f & e & . & . \\ . & f & e & 2f \\ -f & . & f & e \end{vmatrix} = (e^2 - 2f^2)^2;$$

also

$$\begin{vmatrix} x & y & z & w \\ -w & x & y & z \\ -z & -w & x & y \\ -y & -z & -w & x \end{vmatrix} \cdot \begin{vmatrix} x & -y & z & -w \\ -w & -x & y & -z \\ -z & w & x & -y \\ -y & z & -w & -x \end{vmatrix} = \begin{vmatrix} c & -d & . & -d \\ -d & -c & d & . \\ . & d & c & -d \\ -d & . & -d & -c \end{vmatrix},$$

i.e.

$$C'(x, y, z, w) \cdot (-1)^2 C'(x, y, z, w) = (c^2 + 2d^2)^2;$$

and

$$\begin{vmatrix} x & y & z & w \\ -w & x & y & z \\ -z & -w & x & y \\ -y & -z & -w & x \end{vmatrix} \cdot \begin{vmatrix} x & w & -z & y \\ -w & z & -y & x \\ -z & y & -x & -w \\ -y & x & w & -z \end{vmatrix} = \begin{vmatrix} a & . & b & . \\ . & -b & . & a \\ b & . & -a & . \\ . & a & . & b \end{vmatrix},$$

i.e.

$$C'(x, y, z, w) \cdot (-1)^2 C'(x, y, z, w) = (a^2 + b^2)^2.$$

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